Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients

Harm Derksen and Jerzy Weyman

1. Introduction

Let $Q$ be a quiver without oriented cycles. Let $\alpha$ be a dimension vector for $Q$. We denote by $\text{SI}(Q, \alpha)$ the ring of semi-invariants of the set of $\alpha$-dimensional representations of $Q$ over a fixed algebraically closed field $K$.

In this paper we prove some results about the set

$$\Sigma(Q, \alpha) = \{ \sigma \mid \text{SI}(Q, \alpha)_\sigma \neq 0 \}.$$ 

$\Sigma(Q, \alpha)$ is defined in the space of all weights by one homogeneous linear equation and by a finite set of homogeneous linear inequalities. In particular the set $\Sigma(Q, \alpha)$ is saturated, i.e., if $n\sigma \in \Sigma(Q, \alpha)$, then also $\sigma \in \Sigma(Q, \alpha)$.

These results, when applied to a special quiver $Q = T_{n,n,n}$ and to a special dimension vector, show that the $\text{GL}_n$-module $V_\lambda$ appears in $V_\mu \otimes V_\nu$ if and only if the partitions $\lambda$, $\mu$ and $\nu$ satisfy an explicit set of inequalities. This gives new proofs of the results of Klyachko ([7, 3]) and Knutson and Tao ([8]).

The proof is based on another general result about semi-invariants of quivers (Theorem 1). In the paper [10], Schofield defined a semi-invariant $c_W$ for each indecomposable representation $W$ of $Q$. We show that the semi-invariants of this type span each weight space in $\text{SI}(Q, \alpha)$. This seems to be a fundamental fact, connecting semi-invariants and modules in a direct way. Given this fact, the results on sets of weights follow at once from the results in another paper of Schofield [11].

2. The results

A quiver $Q$ is a pair $Q = (Q_0, Q_1)$ consisting of the set of vertices $Q_0$ and the set of arrows $Q_1$. Each arrow $a$ has its head $ha$ and tail $ta$, both in $Q_0$:

$$ta \xrightarrow{a} ha.$$ 

We fix an algebraically closed field $K$. A representation (or a module) $V$ of $Q$ is a family of finite dimensional vector spaces $\{ V(x) \mid x \in Q_0 \}$ and of linear maps...
$V(a) : V(ta) \to V(ha)$. The dimension vector of a representation $V$ is the function $d(V) : Q_0 \to \mathbb{Z}_{\geq 0}$ defined by $d(V)(x) := \dim V(x)$. The dimension vectors lie in the space $\Gamma$ of integer-valued functions on $Q_0$. A morphism $\phi : V \to V'$ of two representations is a collection of linear maps $\phi(x) : V(x) \to V'(x)$, $x \in Q_0$, such that for each $a \in Q_1$ we have $\phi(ha)V(a) = V'(a)\phi(ta)$. We denote the linear space of morphisms from $V$ to $V'$ by $\text{Hom}(V, V')$.

A path $p$ in $Q$ is a sequence of arrows $p = a_1, \ldots, a_n$ such that $ha_i = ta_{i+1}$ $(1 \leq i \leq n-1)$. We define $tp = ta_1, hp = ha_n$. We also have the trivial path $e(x)$ from $x$ to $x$. If $V$ is a representation and $p = a_1, \ldots, a_n$, then we define $V(p) := V(a_n)V(a_{n-1}) \cdots V(a_1)$. We assume throughout the paper that $Q$ has no oriented cycles, i.e., there are no paths $p = a_1, \ldots, a_n$ such that $ta_1 = ha_n$.

For representations $V$ and $W$ of $Q$ there is a canonical exact sequence (1)

$$0 \to \text{Hom}_Q(V, W) \xrightarrow{i} \bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) \xrightarrow{d_W^V} \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha)) \xrightarrow{p} \text{Ext}_Q(V, W) \to 0.$$  

The map $i$ is the obvious inclusion, the map $d_W^V$ is given by

$$\{f(x)\}_{x \in Q_0} \mapsto \{f(ha)V(a) - W(a)f(ta)\}_{a \in Q_1},$$

and the map $p$ constructs an extension of the representations $V$ and $W$ by adding the maps $V(ta) \to W(ha)$ to the direct sum representation $V \oplus W$.

For $\alpha, \beta \in \Gamma$ we define the Euler inner product

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

It follows from (1) that $\langle d(V), d(W) \rangle = \dim_K \text{Hom}_Q(V, W) - \dim_K \text{Ext}_Q(V, W)$.

For a dimension vector $\alpha$ we denote by

$$\text{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \text{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)})$$

the vector space of $\alpha$-dimensional representations of $Q$. The group

$$\text{GL}(Q, \alpha) := \prod_{x \in Q_0} \text{GL}(\alpha(x))$$

and its subgroup

$$\text{SL}(Q, \alpha) = \prod_{x \in Q_0} \text{SL}(\alpha(x))$$

act on $\text{Rep}(Q, \alpha)$ in an obvious way. We are interested in the ring of semi-invariants

$$\text{SI}(Q, \alpha) := K[\text{Rep}(Q, \alpha)]_{\text{SL}(Q, \alpha)}.$$
The ring $SI(Q, \alpha)$ has a weight space decomposition

$$SI(Q, \alpha) = \bigoplus_{\sigma} SI(Q, \alpha)_\sigma$$

where $\sigma$ runs through the (one-dimensional irreducible) characters of $GL(Q, \alpha)$ and

$$SI(Q, \alpha)_\sigma = \{ f \in K[\text{Rep}(Q, \alpha)] \mid g(f) = \sigma(g)f \ \forall g \in GL(Q, \alpha) \}.$$ 

Suppose that $\sigma$ lies in the dual space $\Gamma^* := \text{Hom}(\Gamma, \mathbb{Z})$. For each dimension vector $\alpha$ we can associate to $\sigma$ a character of $GL(Q, \alpha)$ defined as

$$\prod_{x \in Q_0} \sigma^{d_x}(e_x)$$

where $d_x$ is the determinant function on $GL(\alpha(x))$ and $e_x$ is the dimension vector defined by

$$e_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

In this way we will identify characters with $\Gamma^*$. Sometimes, for convenience, we will write $\sigma(x)$ instead of $\sigma(e_x)$ (and treat $\sigma$ as an element of $\Gamma$).

Let us choose the dimension vectors $\alpha$ and $\beta$ in such way that $\langle \alpha, \beta \rangle = 0$. Then for every $V \in \text{Rep}(Q, \alpha)$ and $W \in \text{Rep}(Q, \beta)$ the matrix of $d_V^W$ will be a square matrix. Following [10] we can therefore define the semi-invariant $c$ of the action of $GL(Q, \alpha) \times GL(Q, \beta)$ on $\text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)$ by $c(V, W) := \det d_V^W$. The value of the determinant depends on the choices of bases, so $c$ is well-defined up to a scalar. Notice that the semi-invariant $c$ vanishes at the point $(V, W)$ if and only if $\text{Hom}_Q(V, W) \neq 0$ which is equivalent to $\text{Ext}_Q(V, W) \neq 0$. For a fixed $V$ the restriction of $c$ to $\{V\} \times \text{Rep}(Q, \beta)$ defines a semi-invariant $c_V$ in $SI(Q, \beta)$. Schofield proves ([10], Lemma 1.4) that the weight of $c_V$ equals $\langle \alpha, \cdot \rangle \in \Gamma^*$ which is defined as $\gamma \mapsto \langle \alpha, \gamma \rangle$. Similarly, for a fixed $W$ the restriction of $c$ to $\text{Rep}(Q, \alpha) \times \{W\}$ defines a semi-invariant $c_W$ in $SI(Q, \alpha)$ of weight $-\langle \cdot, \beta \rangle$ ([10], Lemma 1.4). If $V, V' \in \text{Rep}(Q, \alpha)$ and $V \cong V'$, then $V$ and $V'$ are in the same $GL(Q, \alpha)$-orbit, and $c_V$ and $c_{V'}$ are equal up to a constant scalar. Semi-invariants of the types $c_V$ and $c_W$ are well-defined up to a scalar. These semi-invariants have the following properties.

**Lemma 1.** Suppose that $V, V', V''$ and $W, W', W''$ are representations of $Q$ such that $\langle d(V), d(W) \rangle = 0$, and that there are exact sequences

$$0 \to V' \to V \to V'' \to 0, \quad 0 \to W' \to W \to W'' \to 0.$$ 

a) If $\langle d(V'), d(W) \rangle < 0$, then $c_V(W) = 0$;

b) If $\langle d(V'), d(W) \rangle = 0$, then $c_V(W) = c_{V'}(W)c_{V''}(W)$;

c) If $\langle d(V), d(W') \rangle > 0$, then $c_V(W) = 0$;

d) If $\langle d(V), d(W') \rangle = 0$, then $c_V(W) = c_V(W')c_V(W'')$. 


Proof. Consider the following commutative diagram with exact columns:

\[
\begin{array}{cccc}
0 & 0 & \\
\downarrow & \downarrow & \\
\bigoplus_{x \in Q_0} \text{Hom}(V''(x), W(x)) & d''_W & \bigoplus_{a \in Q_1} \text{Hom}(V''(ta), W(2a)) & \\
\downarrow & \downarrow & \\
\bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) & d'_W & \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(2a)) & \\
\downarrow & \downarrow & \\
\bigoplus_{x \in Q_0} \text{Hom}(V'(x), W(x)) & d'_W & \bigoplus_{a \in Q_1} \text{Hom}(V'(ta), W(2a)) & \\
\downarrow & \downarrow & \\
0 & 0 & \\
\end{array}
\]

If \( \langle d(V'), d(W) \rangle = 0 \), then \( d''_W \), \( d'_W \) and \( d''_W \) are all represented by square matrices. It follows that \( e^V(W) = e^{V'}(W)e^{V''}(W) \). So b) follows and d) goes similarly. If \( \langle d(V'), d(W) \rangle < 0 \), then \( d'_W \) cannot be surjective, hence \( d''_W \) is not surjective. Now a) follows and c) goes similarly.

Our main result is that the semi-invariants of type \( c^V \) (resp. \( c^W \)) span all the weight spaces in the rings \( \text{SI}(Q, \alpha) \).

**Theorem 1.** Let \( Q \) be a quiver with oriented cycles and let \( \beta \) be a dimension vector. The ring of semi-invariants \( \text{SI}(Q, \beta) \) is a \( K \)-linear span of semi-invariants \( c^V \) with \( \langle d(V), \beta \rangle = 0 \). The analogous result is true for the semi-invariants \( c^W \).

After this paper was submitted we learned about the paper [12] where among other things the authors give another proof of Theorem 1 in Section 4.

We will prove Theorem 1 in Section 4.

**Remark 1.** If \( V = V_1 \oplus V_2 \) is decomposable, then by Lemma 1 we have \( e^V = 0 \) if \( \langle d(V_1), \beta \rangle \neq 0 \), and \( e^V = e^{V_1}e^{V_2} \) if \( \langle d(V_1), \beta \rangle = 0 \).

The algebra \( \text{SI}(Q, \beta) \) is generated by all \( e^V \) where \( V \) is indecomposable. Generators of \( \text{SI}(Q, \beta) \) therefore can be found in the degrees \( \langle \alpha, \cdot \rangle \) such that a general representation of dimension \( \alpha \) is indecomposable. By [5] this is equivalent to \( \alpha \) being a Schur root.

**Remark 2.** If \( \text{Rep}(Q, \beta) \) has a dense \( \text{GL}(Q, \beta) \)-orbit, then Schofield showed in [10] that the invariants of type \( c^V \) with \( V \) indecomposable generate \( \text{SI}(Q, \beta) \) (which is a polynomial ring in this case).

Theorem 1 has the following remarkable consequence.

**Corollary 1** (Reciprocity Property). Let \( \alpha, \beta \) be two dimension vectors for the quiver \( Q \). Assume that \( \langle \alpha, \beta \rangle = 0 \). Then

\[
\dim_K \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim_K \text{SI}(Q, \alpha)_{\langle \cdot, \beta \rangle}.
\]

**Proof.** Let \( V_1, \ldots, V_s \) be the modules of dimension \( \alpha \) such that \( e^{V_1}, \ldots, e^{V_s} \) form a basis of \( \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} \). These are linearly independent polynomials on \( \text{Rep}(Q, \beta) \) so there exist \( s \) representations \( W_1, \ldots, W_s \) in \( \text{Rep}(Q, \beta) \) such that \( \det(e^{V_i}(W_j))_{1 \leq i, j \leq s} \)
is not zero. But $c^V(W_j) = c_{V_j}(V_i)$ and this means that the semi-invariants $c_{W_1}, \ldots, c_{W_s}$ are linearly independent. This proves that
\[ \dim_K \text{SI}(Q, \beta)(\alpha, \cdot) \leq \dim_K \text{SI}(Q, \alpha)(\cdot, \beta). \]
The other inequality is proven in exactly the same way.

In the remainder of this section we investigate the consequences of Theorem 1. First we recall the main results of [11]. They can be summarized as follows.

We say that for two dimension vectors $\alpha, \beta$ the space $\text{Hom}_Q(\alpha, \beta)$ (respectively $\text{Ext}_Q(\alpha, \beta)$) vanishes generically if and only if for general representations $V, W$ of dimensions $\alpha, \beta$ respectively we have $\text{Hom}_Q(V, W) = 0$ (resp. $\text{Ext}_Q(V, W) = 0$). We also write $\alpha \hookrightarrow \beta$ if a general representation of dimension $\beta$ has a subrepresentation of dimension $\alpha$.

**Theorem 2** (Schofield). Let $\alpha$ and $\beta$ be two dimension vectors for the quiver $Q$.

a) $\text{Ext}_Q(\alpha, \beta)$ vanishes generically if and only if $\alpha \hookrightarrow \alpha + \beta$;

b) $\text{Ext}_Q(\alpha, \beta)$ does not vanish generically if and only if $\beta' \hookrightarrow \beta$ and $(\alpha, \beta - \beta') < 0$ for some dimension vector $\beta'$.

Part a) is proven in Section 3 of [11], and part b) is proven in Section 5.

**Remark 3.** Suppose that $V$ and $W$ are general modules of dimension $\alpha$ and $\beta$ respectively, such that $\langle \alpha, \beta \rangle = 0$. The condition in b) is equivalent to $\exists \beta' \hookrightarrow \beta$ such that $\langle \alpha, \beta' \rangle > 0$. If $c^V(W) = 0$, then $W$ must have a submodule $W'$ such that $\langle \alpha, d(W') \rangle > 0$. This means that the converse of Lemma 1c) is true for general $V$ and $W$.

**Theorem 3.** Let $Q$ be a quiver without oriented cycles and let $\beta$ be a dimension vector. The semigroup $\Sigma(Q, \beta)$ is the set of all $\sigma \in \Gamma$ such that $\sigma(\beta) = 0$ and $\sigma(\beta') \leq 0$ for all $\beta'$ such that $\beta' \hookrightarrow \beta$. Thus this condition is provided by one linear homogeneous equality and finitely many linear homogeneous inequalities. In particular the set $\Sigma(Q, \beta)$ is saturated in the lattice $\Gamma$.

**Proof.** Suppose that $\sigma \in \Gamma^*$. We can write $\sigma = \langle \alpha, \cdot \rangle$ with $\alpha \in \Gamma$.

We will first assume that $\alpha$ is a dimension vector, i.e., $\alpha(x) \geq 0$ for all $x \in Q_0$. It follows from Theorem 1 that $\text{SI}(Q, \beta)(\alpha, \cdot)$ is non-zero if and only if there exists a representation $V$ of dimension $\alpha$ such that $c^V$ is not zero, which is equivalent to $\sigma(\beta) = \langle \alpha, \beta \rangle = 0$ and $\text{Ext}_Q(\alpha, \beta)$ vanishing generically. By part b) of Theorem 2, $\text{Ext}_Q(\alpha, \beta)$ vanishes generically if and only if for all $\beta'$ such that $\beta' \hookrightarrow \beta$ we have $\sigma(\beta') \geq 0$. This means that for all $\beta'$ such that $\beta' \hookrightarrow \beta$ we have $\sigma(\beta') = \langle \alpha, \beta' \rangle \leq 0$. We conclude that $\text{SI}(Q, \beta)_{\alpha} \neq 0$ if and only if $\sigma(\beta) = 0$ and $\sigma(\beta') \leq 0$ for all $\beta' \hookrightarrow \beta$.

If $\alpha$ is not a dimension vector, then $\text{SI}(Q, \beta)_{n, \alpha} = 0$ for all integers $n > 0$. Suppose that $W \in \text{Rep}(Q, \beta)$. From [15] it follows that either $\sigma(d(W')) \neq 0$ or there exists a submodule $W'$ of $W$ such that $\sigma(d(W')) > 0$. If $W$ is in general position, then we obtain $\sigma(\beta) \neq 0$ or $\sigma(\beta') > 0$ for some $\beta' \hookrightarrow \beta$ (see also Remark 5).

**Remark 4.** Schofield in [11] gives an algorithm allowing one to determine the set of inequalities in Theorem 2 inductively. This algorithm is not very efficient.

**Remark 5.** A module $W \in \text{Rep}(Q, \beta)$ is called $\sigma$-stable if and only if there exist an $n > 0$ and an $f \in \text{SI}(Q, \beta)_{n, \alpha}$ such that $f(W) \neq 0$. King proved in [15] that a module
$W \in \text{Rep}(Q, \beta)$ is $\sigma$-stable if and only if $\sigma(W') \leq 0$ for all submodules $W'$ of $W$. Applied to a general representation $W$ of dimension $\beta$ this gives us the equivalence:

$$\exists n > 0 \text{ SI}(Q, \beta)_{n\sigma} \neq 0 \Leftrightarrow \sigma(\beta) = 0 \text{ and } \forall \beta' \beta' \hookrightarrow \beta \text{ we have } \sigma(\beta') \leq 0.$$ 

This shows that the saturation of $\Sigma(Q, \beta)$ is given by linear inequalities but it does not show that $\Sigma(Q, \beta)$ is saturated.

**Remark 6.** In Theorem 3 instead of considering all $\beta'$ with $\beta' \hookrightarrow \beta$ we only need to consider those $\beta'$ such that the general representation of dimension $\beta'$ is indecomposable, which is equivalent to $\beta'$ being a Schur root. Still, the set of inequalities obtained in this way may not be a minimal set of inequalities as we will see in the next example.

**Example 1.** Let $Q$ be the quiver

```
1
\downarrow
4 \rightarrow 5 \leftarrow 2
\uparrow
3
```

and let $\beta$ be the dimension vector

```
1
1 2 1
1
```

For a general representation $V$ of $Q$ with dimension vector $\beta$, the dimension vectors of indecomposable submodules are:

```
0 1 1 1 1
1 2 1 1 2 0 1 2 1 0 2 1
1 1 1 0 1
1 0 0
0 1 0 0 1 1 0 1 0 1 1 0
0 0 1
0 0
0
0
```

Let $\sigma$ be the weight given by $\sigma(\alpha) = \sum_{i=1}^{5} a_i \alpha(i)$, in other words

$$\sigma = a_1 \ a_2 \ a_3 \ a_4 \ a_5 .$$

We investigate when $\text{SI}(Q, \beta)_{\sigma} \neq 0$. First of all we must have $\sigma(\beta) = 0$, so $a_1 + a_2 + a_3 + a_4 + 2a_5 = 0$. In particular $a_1 + a_2 + a_3 + a_4$ must be even. The
indecomposable submodules listed above correspond to the inequalities (using $a_5 = -(a_1 + a_2 + a_3 + a_4)/2$):

\begin{align}
  a_1 &\geq 0, \quad a_2 \geq 0, \quad a_3 \geq 0, \quad a_4 \geq 0, \nonumber \\
  a_1 &\leq a_2 + a_3 + a_4, \quad a_2 \leq a_1 + a_3 + a_4, \quad a_3 \leq a_1 + a_2 + a_4, \quad a_4 \leq a_1 + a_2 + a_3, \nonumber \\
  a_1 + a_2 + a_3 + a_4 &\geq 0. \nonumber 
\end{align}

The last inequality is redundant.

In the next section we will see how semi-invariants can be interpreted in terms of tensor products of modules of the general linear group. This particular example shows that for a 2-dimensional vector space $U$, the tensor product of symmetric powers $S_{a_1}(U) \otimes S_{a_2}(U) \otimes S_{a_3}(U) \otimes S_{a_4}(U)$ contains a non-trivial SL($U$)-invariant subspace if and only if $a_1 + a_2 + a_3 + a_4$ is even and the inequalities (2) hold. In this case, the inequalities are obvious from the Clebsch-Gordan formula.

3. Application to Littlewood-Richardson coefficients

Let us apply Theorem 3 in the following special case. Let us define the quiver $Q = T_{n,n,n}$ as follows:

\[
\begin{array}{cccccccc}
  x_1 & \rightarrow & x_2 & \rightarrow & \ldots & \rightarrow & x_{n-1} & \rightarrow & u \\
  \uparrow & & & & & & & & \uparrow \\
  z_{n-1} & & & & & & & & y_{n-1} \\
  \uparrow & & & & & & & & \uparrow \\
  & \vdots & & & & & & \vdots & \\
  & & & & & & & & z_2 \\
  \uparrow & & & & & & & & \uparrow \\
  & & & & & & & & z_1 \\
  & & & & & & & & y_1 \\
  & & & & & & & & y_2 \\
  & & & & & & & & y_{n-2} \\
  & & & & & & & & \ldots \\
  \downarrow & & & & & & & & \downarrow \\
  & & & & & & u & \rightarrow & x_1 \\
\end{array}
\]

Let us choose the dimension vector $\beta(x_i) = \beta(y_i) = \beta(z_i) = i$ for $i = 1, \ldots, n - 1$, $\beta(u) = n$. The following proposition is a direct application of Cauchy’s formula and is a standard calculation in representation theory.

**Proposition 1.** The weight space $SI(T_{n,n,n}, \beta)_\sigma$ is isomorphic to the space of SL($U$)-invariants in the triple tensor product $S_\lambda(U) \otimes S_\mu(U) \otimes S_\nu(U)$ of Schur functors on $U$, where $U$ is the vector space of dimension $n$, and $\lambda, \mu, \nu$ are partitions whose conjugate partitions are given as follows:

\begin{align}
  \lambda' &= ((n-1)^{\sigma(x_{n-1})}, (n-2)^{\sigma(x_{n-2})}, \ldots, 1^{\sigma(x_1)}), \\
  \mu' &= ((n-1)^{\sigma(y_{n-1})}, (n-2)^{\sigma(y_{n-2})}, \ldots, 1^{\sigma(y_1)}), \\
  \nu' &= ((n-1)^{\sigma(z_{n-1})}, (n-2)^{\sigma(z_{n-2})}, \ldots, 1^{\sigma(z_1)}).
\end{align}

Here $\sigma(q)$ is defined as $\sigma(e_q)$ where the dimension vector $e_q$ is given by $e_q(q) = 1$ and $e_q(p) = 0$ if $p \neq q$.

**Proof.** Let us denote by $a_i$ (resp. $b_i, c_i$) the arrow in $T_{n,n,n}$ with $ta_i = x_i, ha_i = x_{i+1}$ (resp. $tb_i = y_i, hb_i = y_{i+1}, tc_i = z_i, hc_i = z_{i+1}$) for $1 \leq i \leq n - 1$. The space $\text{Rep}(T_{n,n,n}, \beta)$ can be identified with

\[
\bigoplus_{1 \leq i \leq n-1} (\text{Hom}(V(x_i), V(x_{i+1})) \oplus \text{Hom}(V(y_i), V(y_{i+1})) \oplus \text{Hom}(V(z_i), V(z_{i+1})))
\]

where we write $x_n = y_n = z_n = u$. 

The Cauchy formula [4, §A.1] gives the decomposition of $K[\text{Rep}(T_{n,n,n}, \beta)]$ as a direct sum over the $3(n - 1)$-tuples of partitions
\[
(\alpha^i)_{1 \leq i \leq n-1}, (\beta^i)_{1 \leq i \leq n-1}, (\gamma_i)_{1 \leq i \leq n-1}
\]
of the summands
\[
\bigotimes_{1 \leq i \leq n-1} (S_\alpha V(x_i) \otimes S_\alpha V(x_{i+1}^*) \otimes S_\beta V(y_i) \otimes S_\beta V(y_{i+1}^*) \otimes S_\gamma V(z_i) \otimes S_\gamma V(z_{i+1}^*)).
\]
Let us denote $H = \prod_{1 \leq i \leq n-1} (\text{SL}(V(x_i)) \times \text{SL}(V(y_i)) \times \text{SL}(V(z_i)))$. Then it follows from the Littlewood-Richardson Rule [4, §A.1] that the summand corresponding to the $3(n - 1)$-tuple
\[
(\alpha^i)_{1 \leq i \leq n-1}, (\beta^i)_{1 \leq i \leq n-1}, (\gamma_i)_{1 \leq i \leq n-1}
\]
contains an $H$-invariant if and only if we have for each $i$, $1 \leq i \leq n - 1$,
\[
(\alpha^i)' = ((i)\sigma(x_i), (i-1)\sigma(x_{i-1}), \ldots, 1\sigma(x_1)),
\]
\[
(\beta^i)' = ((i)\sigma(y_i), (i-1)\sigma(y_{i-1}), \ldots, 1\sigma(y_1)),
\]
\[
(\gamma_i)' = ((i)\sigma(z_i), (i-1)\sigma(z_{i-1}), \ldots, 1\sigma(z_1))
\]
for some non-negative numbers $\sigma(x_i), \sigma(y_i), \sigma(z_i)$. Moreover, if these conditions are satisfied, then the space of $H$-invariants is isomorphic to
\[
S_{\alpha} V(u)^* \otimes S_{\beta} V(u)^* \otimes S_{\gamma} V(u)^*.
\]
Therefore the space of $\text{SL}(T_{n,n,n}, \beta)$-semi-invariants can be identified with the space of $\text{SL}(V(u))$-invariants in the above triple tensor product. \qed

**Corollary 2.** The set of triples of partitions $(\lambda, \mu, \nu)$ such that the space of $\text{SL}(U)$-invariants in $S_{\lambda}(U) \otimes S_{\mu}(U) \otimes S_{\nu}(U)$ is non-zero, in the space of triples of weights is given by a finite set of linear homogeneous inequalities in the parts of $\lambda, \mu, \nu$ and the condition that $|\lambda| + |\mu| + |\nu|$ is divisible by $n := \dim U$.

**Proof.** Let $\sigma \in \Gamma$ be given by (8) and let $\sigma(\beta) = 0$. All components of $\sigma$ are integers only if $|\lambda| + |\mu| + |\nu|$ is divisible by $n$, because
\[
0 = \sigma(\beta) = n\sigma(u) + \sum_{i=1}^{n-1} i (\sigma(x_i) + \sigma(y_i) + \sigma(z_i)) = n\sigma(u) + |\lambda| + |\mu| + |\nu|.
\]
By Theorem 3 and Proposition 1 those $(\lambda, \mu, \nu)$ for which $\text{SL}(T_{n,n,n}, \beta)_\sigma \neq 0$ are given by $\sigma(\beta) = 0$ and a finite set of homogeneous linear inequalities in $\sigma(x_i), \sigma(y_i), \sigma(z_i), 1 \leq i \leq n - 1$. These inequalities can be written as inequalities in the parts of $\lambda, \mu$ and $\nu$. \qed

4. **The Proof of Theorem 1**

We define $[x, y]$ to be the vector space with the basis formed by paths from $x$ to $y$. We assumed that $Q$ has no oriented cycles, so the spaces $[x, y]$ are finite dimensional.

The indecomposable projective representations are in a bijection with $Q_0$. The indecomposable projective corresponding to $x$ is defined by
\[
P_x(y) = [x, y], \quad P_x(a) = a \circ [x, ta] \to [x, ha],
\]
where $P_x(a)$ is given by the composition $p \mapsto a \circ p$. We have $\text{Hom}_Q(P_x, V) = V(x)$.

In particular $\text{Hom}_Q(P_x, P_y) = [y, x]$.

We choose a numbering $Q_0 = \{x_1, \ldots, x_n\}$ of vertices of $Q$ such that for every $\alpha \in Q_1$ with $t\alpha = x_i, h\alpha = x_j$, we have $i < j$. Let $b_{i,j}$ be the number of arrows $\alpha \in Q_1$ with $t\alpha = x_i, h\alpha = x_j$. Let $p_{i,j} = \dim [x_i, x_j]$ be the number of paths $p$ in $Q$ such that $tp = x_i, hp = x_j$.

The relations between the $\alpha(x_j)$ and $\sigma(x_i)$ are as follows:

\begin{align*}
\sigma(x_j) &= \alpha(x_j) - \sum_{i<j} b_{i,j} \alpha(x_i), \\
\alpha(x_j) &= \sigma(x_j) + \sum_{i<j} p_{i,j} \sigma(x_i).
\end{align*}

We define the $m$-arrow quiver $\Theta_m$ as a quiver with two vertices $x_+$ and $x_-$, and $m$ arrows $a_1, \ldots, a_m$ with $ta_i = x_-, ha_i = x_+$ for $i = 1, \ldots, m$. We define the weight $\tau$ given by $\tau(x_+) = 1, \tau(x_-) = -1$. The dimension vector $\theta(n)$ is defined by $\theta(n)(x_+) = \theta(n)(x_-) = n$.

The idea of the proof of Theorem 1 is to reduce the calculation to the weight space $\text{SI}(\Theta_m, \theta(n))$. The method comes from Classical Invariant Theory with a slight adjustment to accommodate the definition of semi-invariants $c^V$.

**Proof of Theorem 1.** Let us fix $Q$, $\beta$ and a weight $\sigma$. We proceed in three steps. In the first step, we reduce the theorem to the case that $Q$ is a quiver with exactly one source $x_-$ and one sink $x_+$, and $\sigma(x_-) = 1, \sigma(x_+) = -1$ and $\sigma$ is zero on all other vertices. In the second step we reduce to the case that there are no vertices $x$ with $\sigma(x) = 0$. The only case left is the quiver $\Theta_m$ with weight $\tau$. In Step 3 we will prove the theorem in this case.

**Step 1.** Construct a quiver $Q(\sigma)$ as follows:

\begin{align*}
Q(\sigma)_0 &= Q_0 \cup x_- \cup x_+, \\
Q(\sigma)_1 &= Q_1 \cup Q_- \cup Q_+
\end{align*}

where $Q_-$ consists of the set of arrows from $x_-$ to $x_i$, with $\sigma(x_i)$ arrows going to the vertex $x_i$ for which $\sigma(x_i) > 0$ and no arrows going to other vertices. The set $Q_+$ consists of the set of arrows from $x_i$ to $x_+$, with $-\sigma(x_i)$ arrows going from the vertex $x_i$ for which $\sigma(x_i) < 0$ and no arrows going from other vertices to $x_+$.

**Example 2.** Let $Q$ be the quiver

\[
\begin{array}{c}
\bullet & \xrightarrow{x_1} & \bullet \\
| & \downarrow & | \\
\bullet & \xrightarrow{x_2} & \bullet \\
| & \downarrow & | \\
\bullet & \xrightarrow{x_3} & \bullet
\end{array}
\]

Let $\sigma = (1, 1, -2)$. Then the quiver $Q(\sigma)$ is

\[
\begin{array}{c}
\bullet & \xrightarrow{x_1} & \bullet \\
| & \downarrow & | \\
\bullet & \xrightarrow{x_2} & \bullet \\
| & \downarrow & | \\
\bullet & \xrightarrow{x_3} & \bullet \\
\end{array}
\]
We will write $Q = Q(\sigma)$. Define the weight $\sigma$ of $Q$ by $\sigma(x_-) = 1$, $\sigma(x_i) = 0$, $\sigma(x_+)$ $=-1$. The dimension vector $\beta = \beta(\sigma)$ is defined by $\beta(x_i) = \beta(x_+)$, $\beta(x_-) = \sum_{i\in\sigma(x_+)>0} \sigma(x_i)\beta(x_i)$, $\beta(x_+)$ $= \sum_{i\in\sigma(x_+)\leq 0} -\sigma(x_i)\beta(x_i)$. Suppose that $W \in \text{Rep}(Q, \beta)$. The matrices of all maps $W(a)$ with $a \in Q_-$ form a square matrix. Let $D^-(W)$ be the determinant of this block matrix. Let $D^+(W)$ be the determinant of all $W(a)$ with $a \in Q_+$. Then the correspondence $c \rightarrow D^- c D^+$ gives the isomorphism of weight spaces $SI(Q, \sigma) \rightarrow SI(Q, \beta)$. Let $\sigma$ be the dimension vector of $Q$ such that $\sigma = \langle x^r, \cdot \rangle$. Let $\tau$ be a representation of $Q$ with dimension vector $\sigma$ and let $\tau^\circ$ be the corresponding non-zero semi-invariant on $SI(Q, \beta)$.

**Proposition 2.** The factor $c$ in the decomposition $c^\tau = D^- c D^+$ is of the form $c^\nu$ for some $\nu \in \text{Rep}(Q, \alpha)$.

**Proof.** Notice that the weight of $D^-$ is equal to $\langle \gamma_-, \cdot \rangle$ where

\[
\gamma_-(x_-) = 1, \quad \gamma_-(x_j) = \gamma_-(x_+) = 0.
\]

Similarly, by $\sigma$, the weight of $D^+$ equals $\langle \gamma_+, \cdot \rangle$ where

\[
\gamma_+(x_-) = 0, \quad \gamma_+(x_j) = - \sum_{i < j} p_{i,j} \sigma(x_i),
\]

\[
\gamma_+(x_+) = -1 + \sum_{\sigma(x_i) < 0} p_{i,j} \sigma(x_i) \sigma(x_j).
\]

It is easy to see that $\langle \gamma_-, \beta \rangle = \langle \gamma_+, \beta \rangle = 0$.

Let $\nu \in \text{Rep}(Q, \sigma)$. Then $\nu$ has an obvious submodule $\nu_1 = \nu \mid_{\sigma(x_-)}$. We have an exact sequence

\[
0 \rightarrow \nu_1 \rightarrow \nu \rightarrow \nu_2 \rightarrow 0
\]

with the dimension of $\nu_2$ equal to $\gamma_-$.

Let $M$ be the module defined by the exact sequence

\[
0 \rightarrow P_{x_+} \overset{i}{\rightarrow} \bigoplus_{b, \sigma(b) = x_+} P_b \rightarrow M \rightarrow 0,
\]

where the morphism $i$ from $P_{x_+}$ to a copy $P_b$ maps the trivial path $e(x_+)$ to the path $b$. The dimension vector of $M$ is $\gamma_+$, and $\nu^M$ is the determinant $D^+$. Consider the map

\[
\sum_{b} \nu_1(b) \equiv \bigoplus_{b, \sigma(b) = x_+} \nu_1(tb) \rightarrow \nu_1(x_+).
\]

The dimension of the kernel is at least 1. Let $(s_b)_{b, \sigma(b) = x_+}$ with $s_b \in \nu_1(tb)$ be a non-trivial element in the kernel. We can now define a map $\bigoplus_{b, \sigma(b) = x_+} P_b \rightarrow \nu_1$ by sending the generator $e(tb) \in P_b(tb)$ to $s_b$ for all $b$. Because $(s_b)_{b, \sigma(b) = x_+}$ lies in the kernel, this actually defines a morphism $M \rightarrow \nu_1$. Let $\nu_3$ be the image of this morphism.

Now $\nu_3$ is a submodule of $\nu_1$ and $\nu_3^\circ \neq 0$. By Lemma 1 a) we have $\langle d(\nu_3), \beta \rangle \geq 0$. We also have $\nu^M = D^+ \neq 0$. If we apply Lemma 1 a) to the kernel $N$ of
Because $D^+$ is an irreducible semi-invariant we must have $c^V = D^+$, $\gamma_+ = \dim V_3$ and $\overline{V}_3$ is isomorphic to $M$.

We have an exact sequence

$$0 \to V_3 \to V_1 \to V_4 \to 0.$$ 

Now it is clear by the multiplicative property that $c^V = c^V_2 c^V_3$ with the first factor being proportional to $D^-$ and the last one to $D^+$. Let us also define a submodule $\overline{V}_5 = V_4 \mid_{\{x_+\}}$, so we have an exact sequence

$$0 \to \overline{V}_5 \to \overline{V}_4 \to \overline{V}_6 \to 0.$$ 

Note that $\overline{V}_6$ has support within $Q$. The restriction of $\overline{V}_6$ to $Q$ will be denoted by $V$. We will prove that the restriction of $c^V$ to $\text{Rep}(Q, \beta)$ is $c^V$.

Extend $W \in \text{Rep}(Q, \beta)$ to the module $\overline{W}$ of dimension $\overline{\beta}$ by putting $\overline{W}(x_-) = \bigoplus_{a, t a = x} W(ha)$, $\overline{W}(x_+) = \bigoplus_{b, h b = x_+} W(tb)$, with the maps $\overline{W}(a)$ and $\overline{W}(b)$ being the components of the identity map. Define the canonical submodule $\overline{W}_1 = \overline{W} \mid_{\{x_+\}}$. We have an exact sequence

$$0 \to \overline{W}_1 \to \overline{W} \to \overline{W}_2 \to 0.$$ 

Define the submodule $\overline{W}_3 = \overline{W}_2 \mid_{Q \setminus \{x_-\}}$ of $\overline{W}_2$. Now we have an exact sequence

$$0 \to \overline{W}_3 \to \overline{W}_2 \to \overline{W}_4 \to 0.$$ 

The representation $\overline{W}_3$ has support within $Q$ and its restriction to $Q$ is just $W$.

We now have

$$c^V(\overline{W}) = c^V_4(\overline{W}) = c^V_4(\overline{W}_1)c^V_4(\overline{W}_3)c^V_4(\overline{W}_4) = c^V_4(\overline{W}_3)$$

because $c^V_4(\overline{W}_1)$ and $c^V_4(\overline{W}_4)$ are constant. Moreover,

$$c^V_4(\overline{W}_3) = c^V_4(\overline{W}_3)c^V_4(\overline{W}_3) = c^V_4(\overline{W}_3) = c^V(W)$$

because $c^V_4(\overline{W}_3)$ is constant. This concludes the proof of the proposition.

**Step 2.** Let $Q$, $\beta$, $\sigma$ be as above. Let $x \in Q_0$ be a vertex such that $\sigma(x) = 0$. Let $a_1, \ldots, a_s$ be the arrows in $Q_1$ with $ha_k = x$ ($k = 1, \ldots, s$) and let $b_1, \ldots, b_t$ be the arrows in $Q_1$ with $tb_l = x$ ($l = 1, \ldots, t$). Let $\overline{Q}$ be the quiver such that $\overline{Q}_0 = Q_0 \setminus \{x\}$ and $\overline{Q}_1 = (Q_1 \setminus \{a_1, \ldots, a_s, b_1, \ldots, b_t\}) \cup \{ba_{k, l} \mid 1 \leq k \leq s, 1 \leq l \leq t\}$, where $t(ba_{k, l}) = ta_k, h(ba_{k, l}) = hb_l$. Let $\overline{\beta}, \overline{\sigma}$ be the restrictions of $\beta, \sigma$ to $Q_0 \setminus \{x\}$.

The Fundamental Theorem of Invariant Theory (see [2] for a characteristic free version) says that every semi-invariant from $\text{SI}(Q, \overline{\beta})_\overline{\sigma}$ can be obtained from the semi-invariants from $\text{SI}(\overline{Q}, \overline{\beta})_\overline{\sigma}$ by substituting the actual compositions $b_ka_k$ for the arrows of type $ba_{k, l}$. Assuming Theorem 1 for $\text{SI}(\overline{Q}, \overline{\beta})_\overline{\sigma}$ to be true, we need to show that every semi-invariant $c^V$ from $\text{SI}(\overline{Q}, \overline{\beta})_\overline{\sigma}$ pulls back to a semi-invariant of type $c^V$. For a given representation $V$ of $\overline{Q}$ of dimension $\overline{\sigma}$ we define the representation $V = \text{ind} \overline{V}$ as follows. We notice that the condition $\sigma(x) = 0$ means that we expect $\dim V(x) = \sum_{k=1}^s \dim V(ta_k)$. 

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This means we put
\[ V(y) = \begin{cases} V(y) & \text{if } y \neq x, \\ \bigoplus_{k=1}^s V(ta_k) & \text{if } y = x. \end{cases} \]

We define the linear maps \( V(a) \) as follows:
\[ V(a) = \begin{cases} V(a) & \text{if } a \neq a_k, b_i, \\ i(a_k) & \text{if } a = a_k, \\ \sum_{k=1}^s V(ba_{k,i}) & \text{if } b = b_i, \end{cases} \]
where \( i(a_k) : V(ta_k) \to \bigoplus_{k=1}^s V(ta_k) \) is the injection on the \( k \)-th summand.

Then it is easy to check directly from the definition of semi-invariants \( c^V \) that if the representation \( W = \text{res} W \) of dimension \( \beta \) is a restriction of a representation \( W \) of \( Q \) of dimension \( \beta \), then \( c^V(W) = c^V(W) \).

Notice that the functor \( \text{ind} V \) is the left adjoint of the obvious restriction functor \( \text{res} : \text{Rep}(Q) \to \text{Rep}(Q) \), i.e., we have the natural isomorphisms
\[ \text{Hom}_Q(\text{ind} V, W) = \text{Hom}_Q(V, \text{res} W) \]
which explains why \( c^V(W) \) and \( c^V(W) \) vanish simultaneously.

**Step 3.** It remains to deal directly with the weight space \( \text{SI}(\Theta_m, \theta(n))_\tau \). Writing the representation \( W \) of dimension \( \theta(n) \) as an \( m \)-tuple of linear maps,
\[ W(a_1), \ldots, W(a_m) : W_+ \to W_-, \]
we can introduce the additional action of the group \( \text{GL}(m) \) acting on this space by taking linear combinations of the linear maps \( W(a_1), \ldots, W(a_m) \). Using the Cauchy formula (in its characteristic free version, see from [1]) we see that the space \( \text{SI}(\Theta_m, \theta(n)) \) of semi-invariants can be identified with \( \bigwedge^n W_+ \otimes \bigwedge^n W_-^* \otimes D_n(K^m) \).

Here \( D_n \) denotes the \( n \)-th divided power. Since the divided power \( D_n(K^m) \) is generated as a \( \text{GL}(m) \)-module by its highest weight vector (which corresponds to the semi-invariant \( \det W(a_1) \)) and the set of semi-invariants of the form \( c^V \) is preserved by the action of \( \text{GL}(m) \), it is enough to express \( \det W(a_1) \) as the semi-invariant of the form \( c^V \). Notice that \( \tau = (\alpha, \cdot) \) for the dimension vector \( \alpha = (1, m-1) \). Taking the module \( V \) to be the \( m \)-tuple of linear maps \( V(a_1), \ldots, V(a_m) : K \to K^{m-1} \) where \( V(a_1) = 0 \) and \( V(a_i) \) is the embedding sending 1 to the \( i \)-th basis vector, for \( i = 2, \ldots, m \), we check directly that \( c^V = \det W(a_1) \). This concludes the proof of Theorem [1].

We now will give another description for semi-invariants \( \text{SI}(Q, \beta)_\sigma \). Let \( \overline{Q} = Q(\sigma) \), \( \overline{\beta} \) and \( \overline{\sigma} \) be as in Step 1 of the proof of Theorem [1]. We know that \( \text{SI}(Q, \beta)_\sigma \cong \text{SI}((Q, \beta), \overline{\sigma}) \). Let \( \overline{\sigma} \) be a dimension vector of \( \overline{Q} \) such that \( (\overline{\sigma}, \cdot) = \overline{\sigma} \). Now \( \text{SI}(Q, \beta)_\sigma \) is generated by semi-invariants \( c^V \) with \( d(V) = \overline{\sigma} \). In fact we only need to take those \( c^V \) where \( V \) lies in a Zariski dense set of \( \text{Rep}(\overline{Q}, \overline{\sigma}) \). A general representation \( V \) of dimension \( \overline{\sigma} \) has the following projective resolution:
\[ 0 \to P_{x_+} \xrightarrow{d_V} P_{x_-} \to V \to 0 \]
with \( d_V \in \text{Hom}_Q(P_{x_+}, P_{x_-}) = [x_-, x_+] \). So \( d_V \) can be seen as some linear combination \( \sum_{i=1}^r \lambda_ip_i \) where \( p_i, \ldots, p_r \) are all paths from \( x_+ \) to \( x_- \). For any \( \overline{W} \in \text{Rep}(\overline{Q}, \overline{\beta}) \) we have the following exact sequence:
\[ 0 \to \text{Hom}_{\overline{Q}}(\overline{V}, \overline{W}) \to \text{Hom}_{\overline{Q}}(P_{x_+}, \overline{W}) \xrightarrow{d_{\overline{\sigma}}} \text{Hom}_{\overline{Q}}(P_{x_-}, \overline{W}) \to \text{Ext}_{\overline{Q}}(\overline{V}, \overline{W}) \to 0. \]
It is easy to see that $\det(\tilde{\mathbf{d}}(V)) = c^V(W) = c^V(W)$.

We have that
\[
\text{Hom}_Q(P_{x^+}, W) \cong W_{x^+} = \bigoplus_{\sigma(x_i) > 0} W(x_i)^{\sigma(x_i)},
\]
\[
\text{Hom}_Q(P_{x^-}, W) \cong W_{x^-} = \bigoplus_{\sigma(x_i) < 0} W(x_i)^{\sigma(x_i)},
\]
\[
\tilde{\mathbf{d}}(V) = \sum_i \lambda_i(V(p_i)).
\]

Let $F$ be a function from the set of paths from $x_+$ to $x_-$ to the set of non-negative integers. For each such $F$ we can define the semi-invariant $I_F$ as the coefficient of $\lambda_1^{F(p_1)} \lambda_2^{F(p_2)} \ldots \lambda_r^{F(p_r)}$ in $\det(\tilde{\mathbf{d}}(V))$.

**Corollary 3.** The space of semi-invariants $\text{SI}(Q, \beta)$ is spanned by semi-invariants of the form $I_F$.

A necessary condition for $I_F$ to be non-zero is
\[
\sum \sigma(x_i) = \sum \sigma(x_i) > 0, \quad \sum \sigma(x_i) < 0.
\]

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**References**


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02151

E-mail address: hderksen@math.mit.edu

Department of Mathematics, Northeastern University, Boston, Massachusetts 02115

E-mail address: weyman@neu.edu