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ON THE EQUATION $\operatorname{div} Y = f$ AND APPLICATION TO CONTROL OF PHASES

JEAN BOURGAIN AND HAÏM BREZIS

1. Introduction

The purpose of this paper is to present new results concerning the equation

$$\operatorname{div} Y = f \quad \text{on } \mathbb{T}^d, \tag{1.1}$$

i.e., we work on $\mathbb{R}^d$ with 2π-periodic functions in all variables. In what follows we will always assume that $d \geq 2$ and that

$$\int_Q f = 0 \tag{1.2}$$

where $Q = (0, 2\pi)^d$. The notations $L^p, W^{1,p}$, etc. refer to $L^p(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d)$, etc. or to 2π-periodic functions in $L^p_{loc}(\mathbb{R}^d), W^{1,p}_{loc}(\mathbb{R}^d)$, etc. We denote by $L^p_\#$ the space of functions in L^p satisfying (1.2).

Clearly, (1.1) is an underdetermined problem which admits many solutions. A standard way of tackling (1.1) is to look for a vector field Y satisfying the *additional* condition

$$\operatorname{curl} Y = 0,$$

i.e., one looks for a *special* Y of the form

$$Y = \operatorname{grad} u.$$

Equation (1.1) then becomes

$$\Delta u = f \tag{1.3}$$

and the standard L^p-regularity theory yields a solution $u \in W^{2,p}$ when $f \in L^p_\#, 1 < p < \infty$. Consequently (1.1) has a solution $Y \in W^{1,p}$ for every $f \in L^p_\#, 1 < p < \infty$. More precisely, the operator $\operatorname{div} : W^{1,p} \to L^p_\#$ admits a right inverse which is a bounded linear operator $K : L^p_\# \to W^{1,p}$. Strictly speaking, we should write $Y \in (W^{1,p})^d$(= d-fold copy of $W^{1,p}$), $\operatorname{div} : (W^{1,p})^d \to L^p$, etc. But we will often omit the superscript d to alleviate notation.

Three *limiting* cases are of interest:

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Case 1: $\mathbf{p=1}$. It is well known that when $f \in L^1$ equation (1.3) does not necessarily admit a solution $u \in W^{2,1}$. However, one might still hope to have some solution Y of (1.1) in $W^{1,1}$ or at least in BV. This is not true: for some f's in L^1, equation (1.1) has no solution in BV and not even in $L^{d/(d-1)}$; see Section 2.1.

Case 2: $\mathbf{p = \infty}$. It is well known that when $f \in L^\infty$ equation (1.3) does not necessarily admit a solution $u \in W^{2,\infty}$. However, one might hope to find a solution Y of (1.1) in $W^{1,\infty}$. This is not true: McMullen [13] has shown that for some f's in L^∞ (even continuous f) equation (1.1) has no solution in $W^{1,\infty}$. This is proved using a duality argument and a "non-estimate" of Ornstein [16]; see Section 2.2.

Case 3: $\mathbf{p=d}$. This is the heart of our work. For every $f \in L^d_\#$, equation (1.3) admits a solution $u \in W^{2,d}$ and thus equation (1.1) admits a solution $Y = \operatorname{grad} u \in W^{1,d}$. Since $W^{1,d}$ is *not* contained in L^∞ (this is a limiting case for the Sobolev imbedding), we *cannot* assert that this Y belongs to L^∞. In fact, we give in Section 3 (Remark 7) an explicit $f \in L^d$ such that the corresponding $Y = \operatorname{grad} u$ does *not* belong to L^∞. However one might still hope that given any $f \in L^d_\#$ there is *some* $Y \in L^\infty$ solving (1.1). This is indeed true:

Proposition 1. *Given any $f \in L^d_\#$ there exists some $Y \in L^\infty$ solving (1.1) (in the sense of distributions) with*

$$\|Y\|_{L^\infty} \le C(d)\|f\|_{L^d}. \tag{1.4}$$

Remark 1. A more precise statement established in the course of the proof says that there exists $Y \in C^0$ satisfying (1.1) and (1.4).

The proof of Proposition 1 is quite elementary; see Section 3. It relies on the Sobolev-Nirenberg imbedding $W^{1,1} \subset L^{d/(d-1)}$ (and even $BV \subset L^{d/(d-1)}$) combined with duality, i.e., Hahn-Banach. As a consequence, the argument is *not constructive*, and Y is not obtained as above via a bounded linear operator acting on f. In fact, surprisingly, the operator div has no bounded right inverse in this setting:

Proposition 2. *There exists no bounded linear operator $K\colon L^d_\# \to L^\infty$ such that $\operatorname{div} Kf = f \quad \forall f \in L^d_\#$ (in the sense of distributions).*

Remark 2. Another way of formulating Proposition 2 is to say that the subspace $\{Y \in L^\infty;\ \operatorname{div} Y = 0\}$ admits no complement in the space $\{Y \in L^\infty;\ \operatorname{div} Y \in L^d\}$ equipped with its natural norm. Alternatively, the closed subspace $\{\operatorname{grad} u;\ u \in W^{1,1}\}$ has no complement in L^1; see Section 3.

To summarize: for every $f \in L^d_\#$, equation (1.1) admits

a) a solution $Y_1 \in W^{1,d}$,
b) a solution $Y_2 \in L^\infty$.

A natural question is whether there exists a solution Y of (1.1) in $L^\infty \cap W^{1,d}$. This is indeed one of our main results.

Theorem 1. *For every $f \in L^d_\#$ there exists a solution $Y \in L^\infty \cap W^{1,d}$ of (1.1) satisfying*

$$\|Y\|_{L^\infty} + \|Y\|_{W^{1,d}} \le C(d)\|f\|_{L^d}. \tag{1.5}$$

Despite the simplicity of this statement the argument is rather involved and a simpler proof would be desirable.

We will present two techniques to tackle Theorem 1.

First proof of Theorem 1 when $d = 2$ (see Section 4). It relies on Hahn-Banach (via duality) and thus it is *not* constructive. But it is rather elementary; the main ingredient is the new estimate (1.6) which is established by L^2-Fourier methods.

Lemma 1. *On $\mathbb{T}^2$ we have*

$$\|u - ⨍u\|_{L^2} \le C \|\operatorname{grad} u\|_{L^1+H^{-1}}, \quad \forall u \in L^2, \tag{1.6}$$

for some absolute constant C.

The main difficulty, in proving (1.6), stems from the fact that if we decompose

$$\operatorname{grad} u = h_1 + h_2$$

with $h_1 \in L^1$ and $h_2 \in H^{-1}$, then h_1 and h_2 need *not* be gradients themselves; it is only their sum which is a gradient.

The analogue of Lemma 1 for $d > 2$ is the estimate on $\mathbb{T}^d$,

$$\|u - ⨍u\|_{L^{d/(d-1)}} \le C(d) \|\operatorname{grad} u\|_{L^1+W^{-1,d/(d-1)}}. \tag{1.7}$$

We have no direct proof of (1.7). But it can be deduced by duality from the statement of Theorem 1 (and thus from the second proof presented in Section 7).

Second proof of Theorem 1, valid for all $d \ge 2$ (see Sections 5 and 6). We exhibit via a *constructive* (nonlinear) argument some explicit $Y \in W^{1,d} \cap L^\infty$ satisfying (1.1) and (1.5). The argument for $d = 2$ is simpler and we start with this case for expository reasons.

One should observe a certain analogy with the Fefferman-Stein [10] decomposition of BMO-functions and Uchiyama's [21] constructive proof. Indeed, returning to equation (1.1) and defining F by $|\xi|\hat{F}(\xi) = \hat{f}(\xi)$, we obtain that $F \in W^{1,d} \subset BMO$ and (1.1) becomes

$$F = \sum_{j=1}^{d} R_j Y_j \tag{1.8}$$

with $R_j = j^{th}$ Riesz transform $(\widehat{R_j\psi}(\xi) = \hat{\psi}(\xi)\frac{\xi_j}{|\xi|})$, $Y = (Y_1, \dots, Y_d)$.

The statement of Theorem 1 is that (1.8) has a solution $Y \in L^\infty \cap W^{1,d}$. Recall that according to Fefferman-Stein [10] *any* $F \in BMO$ has a decomposition of the form

$$F = Y_0 + \sum_{j=1}^{d} R_j Y_j \quad \text{with } Y_0, Y_1, \dots, Y_d \in L^\infty. \tag{1.9}$$

The proof of this decomposition is again by duality and nonconstructive. The later constructive approach from Uchiyama [21] gives a different proof of (1.9). If we assume moreover that $F \in W^{1,d}$, Uchiyama's argument gives that (1.9) has a solution $Y_0, Y_1, \dots, Y_d \in L^\infty \cap W^{1,d}$. The new result in this paper shows that, in fact, for $F \in W^{1,d}$, the Y_0-component is unnecessary and (1.8) holds for some $Y_1, \dots, Y_d \in L^\infty \cap W^{1,d}$.

It should be mentioned that to achieve our decomposition we do use significantly different methods from Uchiyama. This raises the question what are the function spaces X, $W^{1,d} \subset X \subset BMO$, such that every $F \in X$ has a decomposition

$$F = \sum_{j=1}^{d} R_j Y_j \tag{1.10}$$

where $Y_j \in L^\infty$ or (assuming the Riesz transforms bounded on X) the stronger property $Y_j \in L^\infty \cap X$.

Remark 3. Using Theorem 1 we will prove (in Sections 4 and 6) that a slightly stronger conclusion holds:

Theorem 1′. *For every $f \in L^d_\#$ there exists a solution $Y \in C^0 \cap W^{1,d}$ of (1.1) satisfying (1.5).*

The original motivation for studying (1.1) comes from the following question about lifting discussed in Bourgain-Brezis-Mironescu [3], [4], [5]. Consider the equation

$$g = e^{i\varphi} \quad \text{on } \mathbb{T}^d$$

where φ is a smooth real-valued function.

Question. Assuming g is controlled in $H^{1/2}$, what kind of estimate can we deduce for φ?

Here is a first easy consequence of Theorem 1.

Corollary 1. *We have*

$$\|\varphi - ⨍\varphi\|_{L^{d/(d-1)}} \leq C(d)(1 + \|g\|_{H^{1/2}})\|g\|_{H^{1/2}}. \tag{1.11}$$

Proof. Write

$$\operatorname{grad} g = ie^{i\varphi} \operatorname{grad} \varphi$$

and thus

$$\operatorname{grad} \varphi = -i\bar{g}(\operatorname{grad} g). \tag{1.12}$$

Multiplying by Y gives

$$\int_Q \varphi \operatorname{div} Y = \int_Q i\bar{g}Y \cdot \operatorname{grad} g. \tag{1.13}$$

Given $f \in L^d$ we obtain from Theorem 1 some Y satisfying (1.1) (with f replaced by $f - ⨍ f$) and (1.5). Thus we have

$$|\int (\varphi - ⨍\varphi) f| \leq \|g\|_{H^{1/2}}(\|\bar{g}Y\|_{H^{1/2}}). \tag{1.14}$$

But

$$\begin{aligned} \|\bar{g}Y\|_{H^{1/2}} &\leq \|g\|_{H^{1/2}}\|Y\|_{L^\infty} + \|g\|_{L^\infty}\|Y\|_{H^{1/2}} \\ (\text{by } (1.5)) &\leq C(\|g\|_{H^{1/2}}\|f\|_{L^d} + \|f\|_{L^d}) \end{aligned} \tag{1.15}$$

where we have used the obvious fact that $\|Y\|_{H^{1/2}} \leq C\|Y\|_{W^{1,d}}$. Combining (1.14) and (1.15) yields (1.11). □

Remark 4. Estimate (1.11) cannot be improved, replacing the norm $\| \ \|_{L^{d/(d-1)}}$ by $\| \ \|_{L^p}, p > d/(d-1)$. This may be seen by choosing $g = e^{i\varphi}$ with $\varphi(x) = (|x|^2 + \varepsilon^2)^{-\alpha/2}$ with $\alpha < d-1, \alpha$ close to $(d-1)$ and ε close to 0 (the same example has already been used in Bourgain-Brezis-Mironescu [3], Lemma 5). There is however a better estimate than (1.11), namely

Theorem 4. *Let* φ *be a smooth real-valued function on* $\mathbb{T}^d$ *and set* $g = e^{i\varphi}$, *then*

$$\|\varphi\|_{H^{1/2}+W^{1,1}} \leq C(d)(1 + \|g\|_{H^{1/2}})\|g\|_{H^{1/2}}.$$

Theorem 4 has been announced in Bourgain-Brezis-Mironescu [4] (Theorem 3) and is proved in Section 8. Our proof of Theorem 4 is a direct estimate based on paraproducts. In view of the preceding argument one may wonder whether Theorem 4 can be proved by solving a divergence equation. After duality the required statement would be

$$\|u - {\textstyle\fint} u\|_{H^{1/2}+W^{1,1}} \leq C\|\operatorname{grad} u\|_{H^{-1/2}+L^1} \tag{1.16}$$

but we do not know whether (1.16) holds.

We now turn to the question of coupling equation (1.1) with the Dirichlet condition

$$Y = 0 \quad \text{on } \partial Q. \tag{1.17}$$

This question was addressed (in various forms) by a few authors; see e.g. Arnold–Scott–Vogelius [2], Duvaut–Lions [9] (Theorem 3.2), X. Wang [22], Temam [20] (Proposition 1.2(ii) and Lemma 2.4) and the references therein to Magenes–Stampacchia [12] and Nečas [14]. Our aim is to establish the analogue of Theorem 1′ under the Dirichlet condition. We start with the following known fact (see e.g. Arnold–Scott–Vogelius [2] for $d = 2$).

Theorem 2. *Given* $f \in L^p_\#(Q), 1 < p < \infty$, *there exists some* $Y \in W_0^{1,p}(Q)$ *satisfying (1.1) with*

$$\|Y\|_{W^{1,p}} \leq C(p)\|f\|_{L^p}. \tag{1.18}$$

Moreover Y *can be chosen, depending linearly on* f.

The operator and the estimate do not depend on p assuming we stay away from the end points.

For the convenience of the reader we include a new proof; our technique is extremely elementary and can be adapted to establish, for the limiting case $p = d$,

Theorem 3. *Given* $f \in L^d_\#(Q)$ *there exists some* $Y \in C^0(\bar{Q}) \cap W_0^{1,d}(Q)$ *satisfying (1.1) with*

$$\|Y\|_{L^\infty} + \|Y\|_{W^{1,d}} \leq C\|f\|_{L^d}.$$

Theorem 3 is stronger than Theorem 1′. However it will be deduced from Theorem 1′. There are variants of Theorems 2 and 3 when Q is replaced by a Lipschitz domain in $\mathbb{R}^d$ (see Section 7.2).

The plan of the paper is the following:

1. Introduction.
2. The cases $f \in L^p$ with $p = 1$ and $p = \infty$.
3. Proofs of Propositions 1 and 2 and related questions.
4. Proof of Theorem 1 when $d = 2$ via duality.
5. Proof of Theorem 1 when $d = 2$ (explicit construction).
6. Proof of Theorem 1 when $d > 2$ (explicit construction).
7. The equation div $Y = f$ with Dirichlet condition. Proof of Theorems 2 and 3.
8. Estimation of the phase in $H^{1/2} + W^{1,1}$. Proof of Theorem 4.

2. THE CASES $f \in L^p$ WITH $p = 1$ AND $p = \infty$

We consider here equation (1.1) with $f \in L^p_\#$ and ask whether there exists a solution $Y \in W^{1,p}$ of (1.1) when $p = 1$ and $p = \infty$. As we have already mentioned in the Introduction the answer is negative. Here is the proof.

2.1. The case $p = 1$**.** Assume by contradiction that for every $f \in L^1_\#$ there is some $Y \in W^{1,1}$ satisfying (1.1). It follows that the linear operator

$$Tu = \text{ div } u \text{ from } E = W^{1,1} \text{ into } F = L^1_\#$$

is bounded and surjective. By the open mapping principle there is a constant C such that for every $f \in F$ there exists a solution $Y \in E$ of (1.1) satisfying

$$\|Y\|_{W^{1,1}} \leq C\|f\|_{L^1}.$$

We now use a duality argument which occurs frequently in the rest of the paper. We will deduce that $W^{1,d} \subset L^\infty$ with continuous injection, and since this is false, we infer that for some f's in F there is no $Y \in W^{1,1}$ satisfying (1.1).

Let $u \in W^{1,d}$ and set

$$\operatorname{grad} u = h \in L^d. \tag{2.1}$$

Given any $f \in L^1$, let $Y \in W^{1,1}$ be such that

$$\text{div } Y = f - ⨍ f$$

and

$$\|Y\|_{W^{1,1}} \leq C\|f - ⨍ f\|_{L^1}.$$

Taking the scalar product of (2.1) with Y and integrating yields

$$\int_Q (u - ⨍_Q u) f = -\int_Q hY.$$

Consequently

$$|\int_Q (u - ⨍_Q u) f| \leq \|h\|_{L^d} \|Y\|_{L^{d/(d-1)}}. \tag{2.2}$$

By the Sobolev-Nirenberg imbedding we have $W^{1,1} \subset L^{d/(d-1)}$ and thus

$$\|Y\|_{L^{d/(d-1)}} \leq C\|Y\|_{W^{1,1}} \leq C\|f\|_{L^1}. \tag{2.3}$$

Combining (2.2) and (2.3) we deduce that $(u - ⨍_Q u) \in L^\infty$ with

$$\|u - ⨍_Q u\|_{L^\infty} \leq C\|\operatorname{grad} u\|_{L^d}.$$

Impossible.

Remark 5. The same argument shows that equation (1.1) with $f \in L^1_\#$ need not have a solution Y in the sense of distributions with $Y \in L^{d/(d-1)}$. (Note, however, that the solution Y given via (1.3) belongs to L^p, $\forall p < d/(d-1)$, and even to weak-$L^{d/(d-1)}$). It suffices to follow the above argument with $E = W^{1,1}$ replaced by

$$\widetilde{E} = \{Y \in L^{d/(d-1)};\ \text{div } Y \in L^1\}$$

equipped with its natural norm.

2.2. The case $p = \infty$. This case has been settled negatively by McMullen [13] (the interest in this kind of problem grew out of the study of the equation $\det(\nabla\varphi) = f$ with φ bi-Lipschitz and also from a question of Gromov [11] on separated nets; see Dacorogna-Moser [18], Ye [24], Rivière-Ye [17],[18], Burago-Kleiner [7]).

For the convenience of the reader we sketch a proof when $d = 2$, which is essentially similar to the one of McMullen [13]. We argue by contradiction as above. Then, for every $f \in L^\infty$ there is a $Y \in W^{1,\infty}$ satisfying

$$\text{div } Y = f - ⨍ f$$

and

$$\|Y\|_{W^{1,\infty}} \leq C\|f\|_{L^\infty}.$$

Let ψ be a smooth function on $\mathbb{T}^2$ and set $g = \psi_{x_1x_2}$. Write

$$\int g_{x_1}Y_1 + g_{x_2}Y_2 = -\int gf = -\int \psi_{x_1x_1}Y_{1x_2} + \psi_{x_2x_2}Y_{2x_1}.$$

Consequently

$$\left|\int gf\right| \leq C(\|\psi_{x_1x_1}\|_{L^1} + \|\psi_{x_2x_2}\|_{L^1})\|f\|_{L^\infty}$$

and thus

$$\|g\|_{L^1} = \|\psi_{x_1x_2}\|_{L^1} \leq C(\|\psi_{x_1x_1}\|_{L^1} + \|\psi_{x_2x_2}\|_{L^1}).$$

This contradicts a celebrated "non-inequality" of Ornstein [16] and completes the proof.

Remark 6. The same argument shows that equation (1.1) with $f \in C^0$ and $⨍ f = 0$ need not have a solution $Y \in W^{1,\infty}$.

3. PROOFS OF PROPOSITIONS 1 AND 2 AND RELATED QUESTIONS

Proof of Proposition 1. Recall the Sobolev-Nirenberg imbedding $W^{1,1} \subset L^{d/(d-1)}$ and, more generally, $BV \subset L^{d/(d-1)}$ with

$$\|u - ⨍ u\|_{L^{d/(d-1)}} \leq C(d)\|\operatorname{grad} u\|_{\mathcal{M}} \quad \forall u \in BV, \tag{3.1}$$

where $\mathcal{M}$ denotes the space of measures. Set

$$E = C^0, \quad F = L^d_\#$$

and consider the unbounded linear operator $A = D(A) \subset E \to F$, defined by

$$D(A) = \{Y \in E;\ \text{div } Y \in L^d\}, \quad AY = \text{ div } Y,$$

so that A is densely defined and has closed graph. Clearly we have

$$E^* = \mathcal{M}, \quad F^* = L^{d/(d-1)}_\#,$$

$$D(A^*) = F^* \cap BV,\ A^*u = \operatorname{grad} u.$$

By (3.1) we have

$$\|u\|_{F^*} \leq C(d)\|A^*u\|_{E^*} \quad \forall u \in D(A^*).$$

It follows from the closed-range theorem (see e.g. Brezis [6], Section II.7) that A is surjective. More precisely, we claim that for any $f \in F$ there is some $Y \in E$ satisfying (1.1) and

$$\|Y\|_{L^\infty} \leq 2C(d)\|f\|_{L^d},$$

where $C(d)$ is the constant in (3.1). $\square$

Indeed, let $f \in F$ with $\|f\|_{L^d} = 1$ and consider the two convex sets

$$B = \{Y \in E;\ \|Y\|_E < 2C(d)\}$$

and

$$L = \{Y \in E;\ \operatorname{div} Y = f\}.$$

We have to prove that $B \cap L \neq \emptyset$. Suppose not, and $B \cap L = \emptyset$. Then, by Hahn-Banach there exists $\mu \in E^*, \mu \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\langle \mu, Y\rangle \leq \alpha \quad \forall Y \in B \tag{3.2}$$

and

$$\langle \mu, Y\rangle \geq \alpha \quad \forall Y \in L. \tag{3.3}$$

From (3.2) we have $\|\mu\| \leq \alpha/2C(d)$ and from (3.3) we deduce, in particular, that $\langle \mu, Z\rangle = 0 \quad \forall Z \in N(A)$. It follows that $\mu \in N(A)^\perp = R(A^*)$. Hence there exists some $u \in F^* \cap BV$ such that grad $u = \mu$. Applying (3.1) we see that

$$\|u\|_{L^{d/(d-1)}} \leq C(d)\|\mu\| \leq \alpha/2. \tag{3.4}$$

On the other hand, by (3.3), $\forall Y \in L$,

$$\alpha \leq \langle \mu, Y\rangle = \langle \operatorname{grad} u, Y\rangle = -\int u \operatorname{div} Y = -\int uf \leq \|u\|_{L^{d/(d-1)}} \leq \alpha/2.$$

This is impossible since $\alpha > 0$ (because $\mu \neq 0$).

Remark 7. The special solution of (1.1) given by $Y = \operatorname{grad} u$, where u is the solution of (1.3), belongs to $W^{1,d}$ when $f \in L^d$; however, in general, it does not belong to L^∞. Here is an example due to L. Nirenberg. Using $(x_1, x_2, \ldots, x_d)$ as coordinates in $\mathbb{R}^d$ consider the function

$$u = x_1 |\log r|^\alpha \zeta$$

where ζ is a smooth cut-off function with support near 0 and $0 < \alpha < (d-1)/d$. Note that $Y = \operatorname{grad} u$ does not belong to L^∞ while

$$|\Delta u| \leq \frac{C}{r} |\log r|^{\alpha - 1},$$

so that $\Delta u \in L^d$.

We now turn to the proof of Proposition 2, i.e., the non-existence of a bounded right inverse $K : L^d_\# \to L^\infty$ for the operator div. We present two proofs. The first is the simplest: after a standard averaging trick we obtain a bounded multiplier $L^d \to L^\infty$ and we reach a contradiction by a direct summability consideration. The second proof is related to Remark 2: the existence of K would yield a factorization of the identity map I: $W^{1,1} \to L^{d/(d-1)}$ through the Banach space L^1; however no such factorization exists by a general argument from the geometry of Banach spaces.

First proof of Proposition 2. Assume $K : L^d_\# \to L^\infty$ is a bounded operator satisfying $\operatorname{div} K = I$ on $L^d_\#$. Then the averaged operator

$$\widetilde{K} = \int_{\mathbb{T}^d} \tau_{-x} K \tau_x dx,$$

where $\tau_x f(y) = f(y + x)$, still satisfies

$$\operatorname{div} \widetilde{K} = I \quad \text{ on } L^d. \tag{3.5}$$

On the other hand, $\widetilde{K}$ is clearly a multiplier

$$\widetilde{K}(e^{in\cdot x}) = (\lambda_1(n), \lambda_2(n), \dots, \lambda_d(n))e^{in\cdot x}$$

which is bounded from L^d into L^∞ and hence from L^1 into $L^{d'}$ where $d' = d/(d-1)$. By (3.5) we have

$$\sum_{j=1}^{d} n_j \lambda_j(n) = 1 \quad \forall n \in \mathbb{Z}^d$$

so that

$$|\lambda(n)|^2 = \sum_{j=1}^{d} |\lambda_j(n)|^2 \geq 1/|n|^2 \quad \forall n. \tag{3.6}$$

Consider the multiplier

$$M(e^{in\cdot x}) = \frac{1}{|n|^{\frac{d}{2}-1}} e^{in\cdot x}, \quad n \neq 0.$$

Then M is bounded from $L^{d'}$ into L^2. Hence $M\widetilde{K}$ is a bounded multiplier from L^1 into L^2. Thus

$$\sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} \frac{|\lambda_j(n)|^2}{|n|^{d-2}} < \infty, \quad \forall j.$$

Summing over $j = 1, 2, \dots, d$, and using (3.6) we deduce

$$\sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} \frac{1}{|n|^d} < \infty.$$

A contradiction. □

Second proof of Proposition 2. Assuming the existence of $K : L^d_\# \to L^\infty$ we obtain a factorization of the identity map $I : W^{1,1} \to L^{d'}$ as

$$I = K^* \circ \text{grad}$$

which, in particular, gives a factorization of I through the Banach space L^1. We claim that there in no such factorization, as a consequence of Grothendieck's theorem on absolutely summing operators. Both the result and the method are well known and we briefly recall them (see Wojtaszczyk [23] for details). First take $d = 2$. Then $I : W^{1,1} \to L^2$ and we consider the operator $I \circ D$ where $D : L^2 \to W^{1,1}$ is defined by

$$D(e^{in\cdot x}) = \frac{1}{\sqrt{1+|n|^2}} e^{in\cdot x}.$$

Thus D is clearly bounded as an operator into H^1, hence into $W^{1,1}$. Since I is assumed to factor through L^1, so does $I \circ D$:

$$\begin{tikzcd}
L^2 \arrow[rr, "I \circ D"] \arrow[dr, "A"'] & & L^2 \\
& L^1 \arrow[ur, "B"'] &
\end{tikzcd}$$

□

Next, recall Grothendieck's theorem that any bounded operator $B : L^1 \to L^2$ is 1-summing, i.e.,

$$\pi_1(B) \equiv \sup \left\{ \sum \|Bx_i\|; (x_i) \subset L^1 \text{ and } \max_{x^* \in L^\infty, \|x^*\| \leq 1} \sum |\langle x_i, x^* \rangle| \leq 1 \right\} \leq K_G \|B\|,$$

where K_G is Grothendieck's constant.

From the usual ideal properties, we obtain

$$\begin{aligned}\left(\sum_{n \in \mathbb{Z}^2} \frac{1}{1+|n|^2} \right)^{1/2} &= \|I \circ D\|_{HS} = \pi_2(I \circ D) \leq \pi_1(I \circ D) \\ &= \pi_1(B \circ A) \leq \|A\| \pi_1(B) \leq K_G \|A\| \, \|B\| < \infty,\end{aligned}$$

which in an obvious contradiction.

For $d > 2$, we have $I : W^{1,1} \to L^{d'}$ and we consider the multiplication operator $M : L^{d'} \to L^2$ given by $M(e^{in \cdot x}) = (1+|n|)^{1-\frac{d}{2}} e^{in \cdot x}$. Hence, considering now $M \circ I \circ D : L^2 \to L^2$ factoring through L^1, we obtain a contradiction again:

$$\left(\sum \frac{1}{(1+|n|)^{d-2}(1+|n|^2)} \right)^{1/2} = \|M \circ I \circ D\|_{HS} = \pi_2(M \circ I \circ D) \leq \pi_1(M \circ I \circ D) < \infty.$$

Proof of Remark 2. Consider the Banach space

$$E = \{Y \in L^\infty; \text{ div } Y \in L^d\}$$

equipped with its natural norm $\|Y\|_{L^\infty} + \| \text{ div } Y\|_{L^d}$. Then

$$N = \{Y \in L^\infty; \text{ div } Y = 0\}$$

is a closed subspace of E which admits no complement in E. Indeed, set

$$F = L^d_\#$$

and consider the bounded linear operator $T : E \to F$ defined by $TY = \text{ div } Y$. By Proposition 1, T is surjective. If $N = N(T)$ admits a complement in E, then T has a bounded right inverse, i.e., an operator $S : F \to E$ such that

$$\text{div } (Sf) = f \quad \forall f \in F$$

(see e.g. Brezis [6], Théorème II.10). But this is impossible by Proposition 2.

Similarly, the subspace

$$R = \{ \text{grad}\, u; \; u \in W^{1,1}\}$$

of L^1 is closed and admits no complement in L^1. Indeed, consider the spaces $E = \{u \in W^{1,1}; \int u = 0\}, F = L^1$ and the operator $T = \text{grad}$, a bounded linear injective operator from E into F. If $R = R(T)$ admits a complement in F, then

T has a bounded left inverse $S : F \to E$ (see e.g. Brezis [6], Théorème II.11). In particular, $S : F \to L^{d/(d-1)}_{\#}$ satisfies

$$S(\operatorname{grad} u) = u, \quad \forall u \in W^{1,1} \text{ with } \int u = 0.$$

Then $S^* : L^d_{\#} \to L^\infty$ satisfies

$$\operatorname{div}\,(S^* f) = f, \quad \forall f \in L^d_{\#},$$

and this is again impossible by Proposition 2. □

4. Proof of Theorem 1 when $d = 2$ via duality

We now return to the periodic setting and we will prove the slightly stronger form of Theorem 1,

Theorem 1′ (for $d = 2$). *For every $f \in L^2_{\#}$ there exists a solution $Y \in C^0 \cap H^1$ of (1.1) with*

$$\|Y\|_{L^\infty} + \|Y\|_{H^1} \le C\|f\|_{L^2} \tag{4.1}$$

for some absolute constant C.

Theorem 1′ is proved by duality from

Lemma 2. *On $\mathbb{T}^2$ we have*

$$\|u - {\textstyle\int} u\|_{L^2} \le C\|\operatorname{grad} u\|_{L^1 + H^{-1}}, \forall u \in L^2 \tag{4.2}$$

where C is an absolute constant.

Assuming the lemma we turn to the

Proof of Theorem 1′. First observe that

$$L^1 + H^{-1} \subset \mathcal{M} + H^{-1}$$

and that

$$\|\cdots\|_{L^1+H^{-1}} = \|\cdots\|_{\mathcal{M}+H^{-1}} \text{ on } L^1 + H^{-1} \tag{4.3}$$

(this may be easily seen using regularization by convolution).

Let $E = C^0 \cap H^1, F = L^2_{\#}$ and consider the bounded operator $T : E \to F$ defined by $TY = \operatorname{div} Y$. Clearly, $T^* : F^* = F \to E^* = \mathcal{M} + H^{-1}$ is given by $T^* u = \operatorname{grad} u$. By Lemma 2 we have

$$\|u\|_{F^*} \le C\|T^* u\|_{E^*} \quad \forall u \in F^*,$$

and therefore T is surjective from E onto F. Estimate (4.1) follows from the open mapping principle or one could argue directly using (4.2) and Hahn-Banach as in the proof of Proposition 1. □

Proof of Lemma 2. Assume

$$u \in L^2_{\#}, \tag{4.4}$$

$$\partial_x u = F_1 + h_1,\ \partial_y u = F_2 + h_2 \tag{4.5}$$

and

$$\|F_1\|_{L^1} + \|F_2\|_{L^1} + \|h_1\|_{H^{-1}} + \|h_2\|_{H^{-1}} \le 1. \tag{4.6}$$

We have to prove that

$$\|u\|_{L^2} \leq C. \tag{4.7}$$

□

The main ingredient is

Lemma 3. *Under assumptions (4.4)–(4.6) we have*

$$\sum_{n_1,n_2\in\mathbb{Z}} \frac{n_1^2 n_2^2}{(n_1^2+n_2^2)^2}\ |\hat{u}(n_1,n_2)|^2 \leq C(\|u\|_{L^2}+1). \tag{4.8}$$

Assuming Lemma 3 we may now complete the proof of Lemma 2. Define

$$u'(x',y') = u(x'+y',x'-y') = \sum_{n_1,n_2} \hat{u}(n_1,n_2)\, e^{i[(n_1+n_2)x'+(n_1-n_2)y']} \tag{4.9}$$

so that

$$\widehat{u'}(n_1+n_2,n_1-n_2) = \hat{u}(n_1,n_2) \tag{4.10}$$

and

$$\begin{aligned}\partial_{x'}u'(x',y') &= \partial_x u(x'+y',x'-y') + \partial_y u(x'+y',x'-y')\\ &= (F_1+F_2)(x'+y',x'-y') + (h_1+h_2)(x'+y',x'-y')\\ &\in L^1 + H^{-1}\end{aligned}$$

and similarly for $\partial_{y'}u'$.

From (4.8) and (4.10) we obtain

$$\begin{aligned}&\sum_{n_1,n_2} \frac{(n_1+n_2)^2(n_1-n_2)^2}{4(n_1^2+n_2^2)^2}|\hat{u}(n_1,n_2)|^2 = \sum_{n_1',n_2'} \frac{(n_1')^2(n_2')^2}{\big((n_1')^2+(n_2')^2\big)^2}|\widehat{u'}(n_1',n_2')|^2\\ &\quad \leq C(\|u'\|_{L^2}+1) = C(\|u\|_{L^2}+1).\end{aligned} \tag{4.11}$$

Addition of (4.8) and (4.11) implies that

$$\|u\|_{L^2}^2 = \sum_{n_1,n_2} |\hat{u}(n_1,n_2)|^2 \leq C(\|u\|_{L^2}+1)$$

and the desired estimate (4.7) follows.

We now turn to the

Proof of Lemma 3. We have

$$\begin{aligned}\sum_{n\neq 0} \frac{n_1^2 n_2^2}{(n_1^2+n_2^2)^2}|\hat{u}(n)|^2 &= \frac{1}{i}n\sum \frac{n_1 n_2^2}{(n_1^2+n_2^2)^2}\ \widehat{\partial_x u}(n)\hat{u}(-n)\\ &\overset{\text{by (4.5)}}{=} \frac{1}{i}\sum \frac{n_1 n_2^2}{(n_1^2+n_2^2)^2}\ \hat{F}_1(n)\hat{u}(-n) + \frac{1}{i}\sum \frac{n_1 n_2^2}{(n_1^2+n_2^2)^2}\hat{h}_1(n)\hat{u}(-n)\\ &= (4.12) + (4.13).\end{aligned}$$

Estimate

$$|(4.13)| \leq \sum_{n_1,n_2} \frac{|\hat{h}_1(n)|}{\sqrt{n_1^2+n_2^2}}\ |\hat{u}(-n)| \leq \|h_1\|_{H^{-1}}\|u\|_{L^2}. \tag{4.14}$$

Write

$$
\begin{aligned}
(4.12) \quad &= \sum \frac{n_1 n_2}{(n_1^2+n_2^2)^2}\, \hat{F}_1(n)\widehat{\partial_y u}(-n)\\
&= \sum \frac{n_1 n_2}{(n_1^2+n_2^2)^2}\hat{F}_1(n)\hat{F}_2(-n) + \sum \frac{n_1 n_2}{(n_1^2+n_2^2)^2}\hat{F}_1(n)\hat{h}_2(-n)\\
&= (4.15) + (4.16).
\end{aligned}
$$

Estimate

$$
\begin{aligned}
|(4.16)| &\leq \sum \frac{|n_1|\,|n_2|}{(n_1^2+n_2^2)^2}(|\widehat{\partial_x u}(n)| + |\hat{h}_1(n)|)\,|\hat{h}_2(-n)|\\
&\leq \sum \frac{n_1^2|n_2|}{(n_1^2+n_2^2)^2}\,|\hat{u}(n)|\,|\hat{h}_2(-n)| + \sum \frac{|\hat{h}_1(n)|}{\sqrt{n_1^2+n_2^2}}\,\frac{|\hat{h}_2(-n)|}{\sqrt{n_1^2+n_2^2}} \qquad (4.17)\\
&\leq \|f\|_{L^2}\,\|h_2\|_{H^{-1}} + \|h_1\|_{H^{-1}}\,\|h_2\|_{H^{-1}}.
\end{aligned}
$$

□

Estimation of (4.15). This is the key point. Since $\|F_1\|_{L^1} \leq 1, \|F_2\|_{L^1} \leq 1$, it suffices (by convexity) to replace $\widehat{F}_i(n)$ by

$$
\widehat{F_1}(n) = e^{in\cdot a}, \qquad \widehat{F_2}(n) = e^{in\cdot b} \tag{4.18}
$$

for some $a, b \in \mathbb{T}^2$ (this amounts to replacing F_1, F_2 by the Dirac measures δ_a, δ_b, respectively).

Thus we obtain

$$
\begin{aligned}
\sum_{n_1,n_2\in\mathbb{Z}} \frac{n_1 n_2}{(n_1^2+n_2^2)^2}\hat{F}_1(n)\hat{F}_2(-n) &= \sum \frac{n_1 n_2}{(n_1^2+n_2^2)^2} e^{i[n_1(a_1-b_1)+n_2(a_2-b_2)]}\\
&= -\sum \frac{n_1 n_2}{(n_1^2+n_2)^2}\sin n_1(a_1-b_1)\sin n_2(a_2-b_2) \qquad (4.19)
\end{aligned}
$$

by parity considerations.

Claim. *For all* $\theta_1, \theta_2 \in \mathbb{T}$

$$
\left| \sum_{n_1,n_2} \frac{n_1 n_2}{(n_1^2+n_2^2)^2} \sin n_1\theta_1 \sin n_2\theta_2 \right| \leq C. \tag{4.20}
$$

From the claim, we conclude that $|(4.15)|, |(4.19)| \leq C$ and, recalling also (4.14), (4.17), inequality (4.8) follows.

Proof of the Claim. Splitting $\mathbb{Z}$ in dyadic intervals, we obtain

$$
\sum_{k_1,k_2\geq 0} \left| \sum_{n_1\sim 2^{k_1}, n_2\sim 2^{k_2}} \frac{n_1 n_2}{(n_1^2+n_2^2)^2} \sin n_1\theta_1 \sin n_2\theta_2 \right|. \tag{4.21}
$$

Recall the inequality

$$
\left| \sum_{n\in I} \sin n\theta \right| \lesssim 4^k|\theta| \wedge \frac{1}{|\theta|} \tag{4.22}
$$

if $\theta \in \mathbb{T}$ and $I \subset [2^{k-1}, 2^k]$ is an interval (where $\wedge$ denotes min).

From (4.22), assuming $k_1 \geq k_2$, we have

$$\left| \sum_{n_1 \sim 2^{k_1}, n_2 \sim 2^{k_2}} \frac{n_1 n_2}{(n_1^2+n_2^2)^2} \sin n_1\theta_1 \sin n_2\theta_2 \right| \leq$$

$$\left(4^{k_1}|\theta_1| \wedge \frac{1}{|\theta_1|}\right)\left(4^{k_2}|\theta_2| \wedge \frac{1}{|\theta_2|}\right) \left\| \left\{ \frac{n_1 n_2}{(n_1^2+n_2^2)^2} \right\} \right\|_{\ell^\infty(n_1 \sim 2^{k_1}) \hat{\otimes} \ell^\infty(n_2 \sim 2^{k_2})} \tag{4.23}$$

where $\ell^\infty(I)\hat{\otimes}\ell^\infty(J)$ denotes the usual projective tensor product. Thus the last factor in (4.23) may be bounded by

$$\left\| \partial^2_{n_1 n_2} \frac{n_1 n_2}{(n_1^2+n_2^2)^2} \right\|_{\ell^1(n_1 \sim 2^{k_1}, n_2 \sim 2^{k_2})} \leq C \left\| \frac{1}{(n_1^2+n_2^2)^2} \right\|_{\ell^1(n_1 \sim 2^{k_1}, n_2 \sim 2^{k_2})} \leq C \frac{2^{k_2}}{8^{k_1}}. \tag{4.24}$$

Substitution of (4.23), (4.24) in (4.21) gives the bound

$$\begin{aligned}(4.20), (4.21) &\leq C \sum_{k_1 \geq k_2 \geq 0} 4^{k_2 - k_1} \left(2^{k_1}|\theta_1| \wedge \frac{1}{2^{k_1}|\theta_1|}\right)\left(2^{k_2}|\theta_2| \wedge \frac{1}{2^{k_2}|\theta_2|}\right) \\ &\lesssim C \prod_{i=1}^{2} \left[\sum_{k \in \mathbb{Z}_+} \left(2^k|\theta_i| \wedge \frac{1}{2^k|\theta_i|}\right)\right] \leq C.\end{aligned}$$

This completes the proof of the Claim and of Theorem 1$'$ for $d = 2$. □

5. PROOF OF THEOREM 1 WHEN $d = 2$ (EXPLICIT CONSTRUCTION)

Our aim is to construct $Y \in L^\infty \cap H^1$ such that

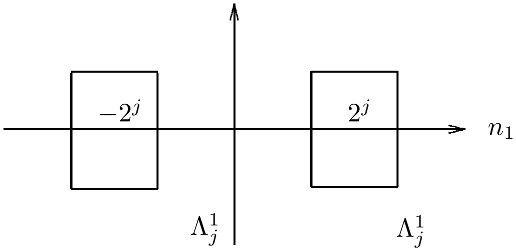
$$\operatorname{div} Y = f \in L^2_\#(\mathbb{T}^2). \tag{5.1}$$

Write

$$\mathbb{Z}^2 = \bigcup_{j \geq 0} (\Lambda_j^1 \cup \Lambda_j^2)$$

where

$$\begin{aligned}\Lambda_j^1 &= [2^{j-1} < |n_1| \leq 2^j; |n_2| \leq 2^j] \\ \Lambda_j^2 &= [2^j < |n_2| \leq 2^{j+1}; |n_1| \leq 2^j].\end{aligned}$$



Let

$$\Lambda^\alpha = \bigcup_j \Lambda_j^\alpha \qquad (\alpha = 1, 2).$$

Decompose

$$f = f^1 + f^2 \text{ where } f^\alpha = P_{\Lambda^\alpha} f \equiv \sum_{n \in \Lambda^\alpha} \hat{f}(n) e^{in\cdot x}.$$

Claim. Let $\delta > 0$ be small enough and $\|f\|_2 \le \delta$. Then there are Y_1, Y_2 such that

$$\|Y_\alpha\|_{L^\infty \cap H^1} \le 1 \tag{5.2}$$

and

$$\|\partial_\alpha Y_\alpha - f^\alpha\|_2 \le \delta^{4/3} \qquad (\alpha = 1, 2). \tag{5.3}$$

Thus if $\|f\|_2 = \delta$, then

$$\|f - \partial_1 Y_1 - \partial_2 Y_2\|_2 \le \delta^{1/3}\|f\|_2$$

and iteration of this gives (5.1).

The construction of Y_1, Y_2 is explicit but *nonlinear* (see Proposition 2).

Take $\alpha = 1$ and denote f^1 by f, Λ_j^1 by Λ_j.

Define

$$\begin{aligned} f_j &= P_{\Lambda_j} f, \\ c_j &= \|f_j\|_2, \\ F_j &= D_{x_1}^{-1} f_j \equiv \sum \frac{1}{n_1} \hat{f}_j(n) e^{in\cdot x}. \end{aligned}$$

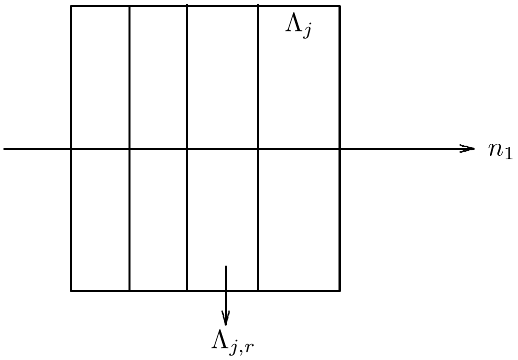
Hence

$$\left(\sum c_j^2\right)^{1/2} = \|f\|_2,$$

$$\|F_j\|_\infty \le \sum_{n \in \Lambda_j} \frac{1}{|n_1|} |\hat{f}(n)| \lesssim 2^{-j} |\Lambda_j|^{1/2} \|f_j\|_2 \lesssim c_j. \tag{5.4}$$

Fix $\varepsilon > 0$ a small constant and partition

$$\Lambda_j = \bigcup_{r < \frac{1}{\varepsilon} + 1} \Lambda_{j,r}$$



in stripes $\Lambda_{j,r}$ such that

$$|\mathrm{Proj}_{n_1} \Lambda_{j,r}| \sim \varepsilon 2^j. \tag{5.5}$$

Define first

$$\tilde{F}_j(x) = \sum_r \left| \sum_{n \in \Lambda_{j,r}} \frac{1}{n_1} \hat{f}_j(n) e^{in\cdot x} \right|. \tag{5.6}$$

Thus

$$|F_j(x)| \le |\tilde{F}_j(x)| \lesssim c_j. \tag{5.7}$$

From Cauchy-Schwarz

$$\|\tilde{F}_j\|_2 \le \varepsilon^{-1/2}\|F_j\|_2 \lesssim \varepsilon^{-1/2}2^{-j}c_j. \tag{5.8}$$

Observe that if $\operatorname{Proj}_{n_1}\Lambda_{j,r} = [a_r, b_r], b_r - a_r \sim \varepsilon 2^j$, then

$$|\partial_1 \tilde{F}_j| \le \sum_r \Big| \sum_{n\in\Lambda_{j,r}} \frac{n_1 - a_r}{n_1} \hat{f}_j(n) e^{in\cdot x} \Big|$$

where

$$\left| \frac{n_1 - a_r}{n_1} \right| < \varepsilon.$$

Therefore

$$\|\partial_1 \tilde{F}_j\|_2 \lesssim \sum_r \varepsilon \|P_{\Lambda_{j,r}} f\|_2 \lesssim \varepsilon^{1/2}\|P_{\Lambda_j} f\|_2 = \varepsilon^{1/2} c_j \tag{5.9}$$

(this is the purpose of the construction of $\tilde{F}_j$).

We also need to make an appropriate localization of the Fourier transform of $\tilde{F}_j$. Denote

$$K_N(y) = \sum_{|n|<N} \frac{N - |n|}{N} e^{iny},$$

the usual Féjer kernel on $\mathbb{T}$. It is easy to see that if

$$P(y) = \sum_{|n|<N} \hat{P}(n) e^{iny}$$

is a trigonometric polynomial, then

$$|P| \le 3(|P| * K_N). \tag{5.10}$$

Using this fact in the variables x_1, x_2, we see that

$$|F_j| \le \tilde{F}_j \le G_j \tag{5.11}$$

denoting

$$G_j = 9\tilde{F}_j * (K_{N_1} \otimes K_{N_2}) \tag{5.12}$$

where each $\Delta_{j,r}$ is an $N_1 \times N_2$ rectangle, $N_1 \sim \varepsilon 2^j, N_2 \sim 2^j$.

Thus, by construction

$$\operatorname{supp} \hat{G}_j \subset [-N_1, N_1] \times [-N_2, N_2] \subset [|n| \le 2^j] \tag{5.13}$$

and inequalities (5.7), (5.8), (5.9) remain preserved.

Therefore,

$$\|G_j\|_\infty \le 9\|\tilde{F}_j\|_\infty \lesssim c_j \qquad (0 < \delta < 1), \tag{5.14}$$

$$\|G_j\|_2 \lesssim \varepsilon^{-1/2}2^{-j}c_j, \tag{5.15}$$

$$\|\partial_1 G_j\|_2 \lesssim \varepsilon^{1/2}c_j, \tag{5.16}$$

$$\|\nabla G_j\|_2 \lesssim \varepsilon^{-1/2}c_j. \tag{5.17}$$

Assume that $\{f_j \mid j \le K\}$ is a finite sequence (which is no restriction).

Define

$$\begin{aligned} Y_1 &= F_K + F_{K-1}(1-G_K) \\ &\quad + F_{K-2}(1-G_{K-1})(1-G_K) + \cdots \\ &= \sum_{j\le K} F_j \prod_{k>j}(1-G_k). \end{aligned} \tag{5.18}$$

Thus from (5.11)

$$\begin{aligned} |Y_1| \le |F_K| &+ (1-|F_K|)|F_{K-1}| \\ &+ (1-|F_K|)(1-|F_{K-1}|)|F_{K-2}| + \cdots \le 1. \end{aligned}$$

One may also rewrite (5.18) as

$$Y_1 = \sum F_j - \sum G_j H_j \tag{5.19}$$

with

$$\begin{aligned} H_j &= F_{j-1} + F_{j-2}(1-G_{j-1}) \\ &\quad + F_{j-3}(1-G_{j-2})(1-G_{j-1}) + \cdots \\ &= \sum_{k<j} F_k \prod_{k<k'<j}(1-G_{k'}). \end{aligned} \tag{5.20}$$

Clearly

$$|H_j| < 1.$$

By construction

$$\partial_1 Y_1 = \sum f_j - \sum \partial_1(G_j H_j). \tag{5.21}$$

Next, we estimate the second term in (5.21) that will appear as an error term.

Observe that since $\operatorname{supp}\hat{F}_j \subset [|n| \sim 2^j]$ and (5.13), also

$$\operatorname{supp}\hat{H}_j \subset [|n| \lesssim 2^j]. \tag{5.22}$$

Denote P_k Fourier projection operators on $[|n| \sim 2^k]$ such that $Id = \sum_{k\ge 0} P_k$.

From the preceding, we may thus ensure that

$$G_j H_j = \sum_{k\le j} P_k(G_j H_j). \tag{5.23}$$

Estimate then

$$\Big\| \sum_j \partial_1(G_j H_j)\Big\|_2 \le \sum_{s\ge 0}\Big(\sum_j \|\partial_1 P_{j-s}(G_j H_j)\|_2^2\Big)^{1/2} \tag{5.24}$$

(since for fixed s, the P_{j-s} have disjoint ranges).

Returning to the parameter $0<\varepsilon<1$ introduced earlier, write

$$\varepsilon = 2^{-s_*}\ (s_* > 0) \tag{5.25}$$

and estimate (5.24) in the ranges

$$s > s_* \tag{5.26}$$

$$0 \le s \le s_*. \tag{5.27}$$

Contribution of (5.26). Since $|H_j| \leq 1$ and (5.15),

$$\begin{aligned} \|\partial_1 P_{j-s}(G_j H_j)\|_2 &\lesssim 2^{j-s}\|G_j H_j\|_2 \\ &\leq 2^{j-s}\|G_j\|_2 \leq \varepsilon^{-1/2}\, 2^{-s} c_j. \end{aligned} \tag{5.28}$$

Substitution in (5.24) gives the contribution

$$\sum_{s \geq s_*} 2^{-s}\varepsilon^{-1/2}\Big(\sum c_j^2\Big)^{1/2} < 2^{-s_*}\varepsilon^{-1/2}\|f\|_2 < \varepsilon^{1/2}\|f\|_2. \tag{5.29}$$

Contribution of (5.27). Estimate now

$$\begin{aligned} \|\partial_1 P_{j-s}(G_j H_j)\|_2 \leq \|\partial_1 (G_j H_j)\|_2 &\leq \|\partial_1 G_j\|_2 + \|G_j \partial_1 H_j\|_2 \\ &\leq \varepsilon^{1/2} c_j + \|G_j \partial_1 H_j\|_2 \end{aligned} \tag{5.30}$$

using (5.16).

Recalling definition (5.20) of H_j, one easily verifies that

$$|\nabla H_j| \leq \sum_{k<j}(|\nabla F_k| + |\nabla G_k|). \tag{5.31}$$

Hence

$$\|\nabla H_j\|_\infty \leq \sum_{k<j} 2^k c_k \tag{5.32}$$

and from (5.15)

$$\|G_j \partial_1 H_j\|_2 \leq \varepsilon^{-1/2} c_j \Big(\sum_{k<j} 2^{-(j-k)} c_k\Big). \tag{5.33}$$

Substitution of (5.30), (5.33) in (5.24) gives the following bound on the contribution of (5.27):

$$\begin{aligned} & s_* \varepsilon^{1/2}\Big(\sum c_j^2\Big)^{1/2} + s_* \varepsilon^{-1/2}\Big[\sum_j c_j^2 \Big(\sum_{k<j} 2^{-(j-k)} c_k\Big)^2\Big]^{1/2} \\ & \leq \Big(\log\frac{1}{\varepsilon}\Big)\varepsilon^{1/2}\|f\|_2 + \Big(\log\frac{1}{\varepsilon}\Big)\varepsilon^{-1/2}\|f\|_2^2. \end{aligned} \tag{5.34}$$

Consequently, from (5.21), (5.29), (5.34),

$$\|f - \partial_1 Y_1\|_2 = \Big\|\sum_j \partial_1 (G_j H_j)\Big\|_2 \leq \log\frac{1}{\varepsilon}(\varepsilon^{1/2}\|f\|_2 + \varepsilon^{-1/2}\|f\|_2^2). \tag{5.35}$$

Under the assumption $\|f\|_2 \leq \delta$, letting $\varepsilon = \delta$ in (5.35), we obtain thus

$$\|f - \partial_1 Y_1\|_2 \leq \delta^{\frac{3}{2}-} \leq \delta^{\frac{4}{3}} \tag{5.36}$$

which is (5.3).

It remains to estimate $\|Y_1\|_{H^1} = \|\nabla Y_1\|_2$.

By (5.19)

$$\|\nabla Y_1\|_2 \leq \Big\|\sum_j \nabla F_j\Big\|_2 + \Big\|\sum \nabla(G_j H_j)\Big\|_2. \tag{5.37}$$

From the definition of F_j and since $\operatorname{supp} \hat{F}_j \subset \Lambda_j^1$, it follows that

$$\Big\| \sum_j \nabla F_j \Big\|_2 \sim \Big(\sum \|f_j\|_2^2 \Big)^{1/2} = \|f\|_2. \tag{5.38}$$

Estimate the second term in (5.37) as in (5.24),

$$\Big\| \sum_j \nabla (G_j H_j) \Big\|_2 \leq \sum_{s\geq 0} \Big(\sum_j \|\nabla P_{j-s}(G_j H_j)\|_2^2 \Big)^{1/2} \tag{5.39}$$

and

$$\|\nabla P_{j-s}(G_j H_j)\|_2 \lesssim 2^{j-s} \|G_j H_j\|_2 \leq \varepsilon^{-1/2} 2^{-s} c_j. \tag{5.40}$$

Thus

$$(5.39) \leq \varepsilon^{-1/2} \sum_{s\geq 0} 2^{-s} \Big(\sum_j c_j^2 \Big)^{1/2} \leq \varepsilon^{-1/2} \|f\|_2 \tag{5.41}$$

and

$$\|\nabla Y_1\|_2 \leq \delta^{-1/2} \|f\|_2 \leq \delta^{1/2}. \tag{5.42}$$

Since $\|Y_1\|_\infty \lesssim 1$, this establishes (5.2).

This proves the Claim and completes the proof of Theorem 1 for $d = 2$.

6. Proof of Theorem 1 when $d > 2$ (explicit construction)

Let $f \in L^d_{\#}(\mathbb{T}^d)$. Our aim is to construct a solution Y of $\operatorname{div} Y = f$ satisfying

$$\|Y\|_\infty \leq C \|f\|_d, \tag{6.1}$$

$$\|\nabla Y\|_d \leq C \|f\|_d. \tag{6.2}$$

We do this by standard modification of the previous L^2-argument with the Littlewood-Paley square function theory as main additional ingredient. Consider again a partition

$$\mathbb{Z}^d = \bigcup_{j\geq 0} (\Lambda_j^1 \cup \cdots \cup \Lambda_j^d)$$

of disjoint d-rectangles Λ_j^α of side length $\sim 2^j$.

We formulate the analogue of the Claim with Y_α satisfying bounds (6.1), (6.2). Letting $\alpha = 1, f = f^1$, define again

$$F_j = D_{x_1}^{-1} f_j \tag{6.3}$$

satisfying

$$\|F_j\|_\infty \lesssim (2^{j/d})^d \|F_j\|_d = 2^j \|D_{x_1}^{-1} f_j\|_d \sim \|f_j\|_d \equiv c_j. \tag{6.4}$$

Define $\tilde{F}_j$ and G_j as in (5.6), (5.12). Thus (5.11), (5.13) hold. Also

$$\begin{aligned}
\|G_j\|_\infty \lesssim \|\tilde{F}_j\|_\infty &\le \varepsilon^{-1/d'}\Big(\sum_{r<\frac{1}{\varepsilon}}\Big\|\sum_{n\in\Lambda_{j,r}}\frac{1}{n_1}\hat{f}_j(n)e^{inx}\Big\|_\infty^d\Big)^{1/d}\\
&\le \varepsilon^{-1/d'}\Big(\sum_{r<\frac{1}{\varepsilon}}\Big(2^{j\frac{d-1}{d}}(\varepsilon 2^j)^{\frac{1}{d}}\Big\|\sum_{n\in\Lambda_{j,r}}\frac{1}{n_1}\hat{f}_j(n)e^{in\cdot x}\Big\|_d\Big)^d\Big)^{1/d}\\
&\lesssim \varepsilon^{-1/d'+1/d}\Big(\sum_{r<\frac{1}{\varepsilon}}\Big\|\sum_{n\in\Lambda_{j,r}}\hat{f}_j(n)e^{inx}\Big\|_d^d\Big)^{\frac{1}{d}}\\
&\lesssim \varepsilon^{\frac{2}{d}-1}\|f_j\|_d = \varepsilon^{\frac{2}{d}-1}c_j \le \varepsilon^{\frac{2}{d}-1}\delta. \qquad (6.5)
\end{aligned}$$

(We assume that δ is small enough compared with ε to ensure, in particular, that $\varepsilon^{\frac{2}{d}-1}\delta \ll 1$.)

Repeat the construction from Section 5. In place of estimate (5.24) we now have

$$\Big\|\sum_j \partial_1(G_jH_j)\Big\|_d \le \sum_{s\ge 0}\Big\|\sum_j|\partial_1P_{j-s}(G_jH_j)|^2\Big)^{1/2}\Big\|_d \qquad (6.6)$$

and distinguish between the cases (5.26), (5.27).

Contribution of (5.26). Estimate

$$\begin{aligned}
&\Big\|\Big(\sum_j|\nabla P_{j-s}(G_jH_j)|^2\Big)^{1/2}\Big\|_d\\
&\qquad\lesssim \Big\|\Big(\sum_j 4^{j-s}|P_{j-s}(G_jH_j)|^2\Big)^{1/2}\Big\|_d\\
&\qquad\lesssim 2^{-s}\Big\|\Big(\sum_j 4^j|G_jH_j|^2\Big)^{1/2}\Big\|_d\\
&\qquad\lesssim 2^{-s}\Big\|\Big(\sum_j 4^j(\tilde{F}_j * K_j)^2\Big)^{1/2}\Big\|_d \qquad (6.7)
\end{aligned}$$

where K_j is a product of Féjer kernels

$$K_{N_1}\otimes K_{N_2}\otimes\cdots\otimes K_{N_d}, \qquad N_1\sim\varepsilon 2^j, \quad \text{and } N_2,\dots,N_d\sim 2^j.$$

Again from standard square function inequalities

$$(6.7) \lesssim 2^{-s}\Big\|\Big(\sum_j 4^j(\tilde{F}_j)^2\Big)^{1/2}\Big\|_d. \qquad (6.8)$$

Recalling the definition of $\tilde{F}_j$, estimate

$$(\tilde{F}_j)^2 \le \varepsilon^{-1}\sum_{r\le\varepsilon^{-1}}\Big|\sum_{n\in\Lambda^1_{j,r}}\frac{1}{n_1}\hat{f}(n)e^{inx}\Big|^2. \qquad (6.9)$$

Substituting in (6.8), this gives

$$\begin{aligned}&\varepsilon^{-1/2}2^{-s}\Big\|\Big(\sum_j\sum_{r<\varepsilon^{-1}}\Big|\sum_{n\in\Lambda^1_{j,r}}\frac{2^j}{n_1}\hat f(n)e^{inx}\Big|^2\Big)^{1/2}\Big\|_d\\ &\lesssim\varepsilon^{-1/2}\,2^{-s}\Big\|\Big(\sum_j\sum_{r<\varepsilon^{-1}}\Big|\sum_{n\in\Lambda^1_{j,r}}\hat f(n)e^{inx}\Big|^2\Big)^{1/2}\Big\|_d.\end{aligned}\tag{6.10}$$

We use here the fact that $|n_1|\sim|n|\sim 2^j$ for $n\in\Lambda^1_j$.

Recall also the definition of $\Lambda_{j,r}$ obtained by partitioning the n_1-variable in intervals of size $\varepsilon 2^j$.

At this stage, we use the following (1-variable) inequality due to Rubio de Francia [19], which generalizes the Littlewood-Paley inequality to arbitrary intervals.

Proposition 3. *Let $\{I_\alpha\}$ be disjoint intervals in $\mathbb{Z}$ and*

$$P_I f=\sum_{n\in I}\hat f(n)e^{inx}$$

the corresponding Fourier projection.

Then, for $2\le d<\infty$, there is the (one-sided) inequality

$$\Big\|\Big(\sum|P_{I_\alpha}f|^2\Big)^{1/2}\Big\|_d\le C\|f\|_d.\tag{6.11}$$

Since $\{\mathrm{Proj}_{n_1}\Lambda^1_{jr}\}$ are disjoint intervals in $\mathbb{Z}$, application of (6.11) in the x_1-variable implies that

$$(6.6)\lesssim\varepsilon^{-1/2}\,2^{-s}\|f\|_d.\tag{6.12}$$

Summation of (6.12) for $s\ge s_*$ gives then

$$(5.26)\text{-contribution }\le\varepsilon^{1/2}\|f\|_d.\tag{6.13}$$

Remark 8. We used the general Proposition 3 for convenience; the present case could in fact be treated by more elementary means.

Contribution of (5.27). Estimate

$$\begin{aligned}\Big\|\Big(\sum_j|\partial_1P_{j-s}(G_jH_j)|^2\Big)^{1/2}\Big\|_d&\lesssim\Big\|\Big(\sum_j|\partial_1(G_jH_j)|^2\Big)^{1/2}\Big\|_d\\ &\le\Big\|\Big(\sum_j|\partial_1G_j|^2\Big)^{1/2}\Big\|_d+\Big\|\Big(\sum_j|G_j(\partial_1H_j)|^2\Big)^{1/2}\Big\|_d=(6.14)+(6.15).\end{aligned}$$

Estimate (6.14) by

$$\Big\|\Big(\sum_j|\partial_1\tilde F_j|^2\Big)^{1/2}\Big\|_d.\tag{6.16}$$

We have that

$$|\partial_1 \tilde{F}_j| \leq \sum_{r<\varepsilon^{-1}} \Big| \sum_{n\in\Lambda^1_{j,r}} \frac{n_1 - a_{j,r}}{n_1} \hat{f}(n) e^{inx} \Big|$$
$$\leq \varepsilon^{-1/2} \Big(\sum_{r<\varepsilon^{-1}} \Big| \sum_{n\in\Lambda^1_{jr}} \frac{n_1 - a_{j,r}}{n_1} \hat{f}(n) e^{inx} \Big|^2 \Big)^{1/2}$$

where $\mathrm{Proj}_{n_1} \Lambda^1_{jr} = [a_{jr}, b_{jr}], b_{jr} - a_{jr} \sim \varepsilon 2^j$. Thus $\left|\frac{n_1 - a_{j,r}}{n_1}\right| \leq \varepsilon$.

We get therefore

$$(6.16) \leq \varepsilon^{-1/2} \cdot \varepsilon \Big\| \Big(\sum_j \sum_{r<\varepsilon^{-1}} \Big| \sum_{n\in\Lambda^1_{jr}} \hat{f}(n) e^{inx} \Big|^2 \Big)^{1/2} \Big\|_d$$
$$\lesssim \varepsilon^{1/2} \|f\|_d. \tag{6.17}$$

To estimate (6.15), use again inequality (5.31), together with (6.4), (6.5). Thus

$$\|\nabla H_j\|_\infty \leq \varepsilon^{\frac{2}{d}-1} \sum_{k<j} 2^k c_k < \varepsilon^{\frac{2}{d}-1}\, 2^j \|f\|_d. \tag{6.18}$$

Hence

$$(6.15) \leq \varepsilon^{\frac{2}{d}-1} \|f\|_d \Big\| \Big(\sum_j 4^j G_j^2 \Big)^{1/2} \Big\|_d$$
$$\leq \varepsilon^{\frac{2}{d}-1} \|f\|_d \Big\| \Big(\sum_j (2^j \tilde{F}_j)^2 \Big)^{1/2} \Big\|_d$$
$$\leq \varepsilon^{\frac{2}{d}-\frac{3}{2}} \|f\|_d^2 \tag{6.19}$$

applying again the (6.8)-bound using Proposition 3.

Thus the (5.27)-contribution is

$$\leq \varepsilon^{1/2} \log \frac{1}{\varepsilon} \|f\|_d + \varepsilon^{\frac{2}{d}-\frac{3}{2}} \log \frac{1}{\varepsilon} \|f\|_d^2. \tag{6.20}$$

Collecting estimates (6.13), (6.20), it follows that

$$\|f - \partial_1 Y\|_d = \Big\| \sum_j \partial_1 (G_j H_j) \Big\|_d$$
$$\leq \varepsilon^{1/2} \log \frac{1}{\varepsilon} \|f\|_d + \varepsilon^{\frac{2}{d}-\frac{3}{2}} \log \frac{1}{\varepsilon} \|f\|_d^2 \tag{6.21}$$

which is the analogue of (5.35). Assuming $\|f\|_d = \delta$, take $\varepsilon = \delta^{1/2}$ to obtain

$$\|f - \partial_1 Y\|_d \leq \delta^{1/5} \|f\|_d. \tag{6.22}$$

It remains to estimate

$$\|\nabla Y\|_d \leq \Big\| \sum \nabla F_j \Big\|_d + \Big\| \sum \nabla (G_j H_j) \Big\|_d = (6.23) + (6.24).$$

We have

$$(6.23) \sim \Big\| \Big(\sum |\nabla F_j|^2 \Big)^{1/2} \Big\|_d \sim \Big\| \Big(\sum |f_j|^2 \Big)^{1/2} \Big\|_d \lesssim \|f\|_d.$$

Estimate (6.24) as

$$\left\| \sum_{s\geq 0} \left(\sum_j |\nabla P_{j-s}(G_j H_j)|^2 \right)^{1/2} \right\|_d \lesssim \varepsilon^{-1/2} \|f\|_d \tag{6.25}$$

using (6.7)–(6.12).

This completes the argument.

We conclude this section with a

Proof of Theorem $1'$ *when* $d > 2$. The argument is somewhat bizarre: one uses duality twice! First, from Theorem 1 we easily deduce the estimate on $\mathbb{T}^d$

$$\|u - {\textstyle\fint} u\|_{L^{d/(d-1)}} \leq C(d) \|\operatorname{grad} u\|_{L^1 + W^{-1,d/(d-1)}}, \forall u \in L^{d/(d-1)}. \tag{6.26}$$

Next, we argue as in the beginning of Section 4. Observe that

$$L^1 + W^{-1,d/(d-1)} \subset \mathcal{M} + H^{-1}$$

and that

$$\|\cdots\|_{L^1+W^{-1,d/(d-1)}} = \|\cdots\|_{\mathcal{M}+W^{-1,d/(d-1)}} \quad \text{on } L^1 + W^{-1,d/(d-1)} \tag{6.27}$$

(this may be easily seen using regularization by convolution).

Let $E = C^0 \cap W^{1,d}, F = L^d_\#$ and consider the bounded operator $T : E \to F$ defined by $TY = \operatorname{div} Y$. Clearly $T^* : F^* \to E^* = \mathcal{M} + W^{-d,d/(d-1)}$ is given by $T^* u = \operatorname{grad} u$. By (6.26) and (6.27) we obtain

$$\|u\|_{F^*} \leq C \|T^* u\|_{E^*} \quad \forall u \in F^*$$

and therefore T is surjective from E onto F. Applying the open mapping principle (or use Hahn-Banach as in the proof of Proposition 1), we see that for every $f \in F$ there is some $Y \in E$ satisfying $TY = f$ and $\|Y\|_E \leq C\|f\|_F$. $\square$

Remark 9. Alternatively, one may approximate $f \in L^d_\#(\mathbb{T}^d)$ by trigonometric polynomials. If f is a trigonometric polynomial, we may clearly obtain Y as a trigonometric polynomial (after convolution). A standard limit procedure permits then to complete the argument.

7. THE EQUATION $\operatorname{div} Y = f$ WITH DIRICHLET CONDITION. PROOF OF THEOREMS 2 AND 3

So far we have studied problem (1.1) coupled with a periodic condition. We consider here problem (1.1) coupled with a Dirichlet condition. Usually one associates with (1.1) the "partial" Dirichlet condition

$$Y \cdot n = 0 \quad \text{on } \partial Q \tag{7.1}$$

(n is normal to ∂Q). It is quite standard that for every $f \in L^p_\#$, $1 < p < \infty$, there is some $Y \in W^{1,p}$ satisfying (1.1), (7.1) and

$$\|Y\|_{W^{1,p}} \leq C \|f\|_{L^p}.$$

Indeed, one may look for a *special* Y of the form $Y = \operatorname{grad} u$ and one is led to the Neumann problem

$$\begin{cases} \Delta u = f & \text{in } Q, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial Q, \end{cases} \tag{7.2}$$

which admits a solution $u \in W^{2,p}$ such that

$$\|u\|_{W^{2,p}} \leq C\|f\|_{L^p}.$$

It is also possible to couple problem (1.1) with the *full* Dirichlet condition

$$Y = 0 \qquad \text{on } \partial Q. \tag{7.3}$$

For simplicity we investigate first the case where the domain is a cube and then the case of a Lipschitz bounded domain.

7.1. The case of a cube. Let $Q = (0,1)^d$. Here is the first result:

Theorem 2. *Given $f \in L^p_\#(Q), 1 < p < \infty$, there exists some $Y \in W^{1,p}_0(Q)$ solving (1.1) with*

$$\|Y\|_{W^{1,p}} \leq C(p,d)\|f\|_{L^p},$$

where we use the standard notation

$$W^{1,p}_0(Q) = \{Y \in W^{1,p}(Q); Y = 0 \text{ on } \partial Q\}.$$

Moreover Y can be chosen, depending linearly on f.

We will make use of the following lemma (which is a special case of Theorem 2).

Lemma 4. *Given $f \in W^{1,p}_0(Q), 1 < p < \infty$, with $\int f = 0$, there exists $Y \in W^{1,p}_0(Q)$, such that*

$$\operatorname{div} Y = f$$

and

$$\|Y\|_{W^{1,p}(Q)} \leq C(d)\|f\|_{W^{1,p}(Q)}. \tag{7.4}$$

Moreover Y can be chosen, depending linearly on f.

Proof. Following a known construction (see Adams [1], p. 58 and Nirenberg [15]), we construct Y by induction on the dimension d. The assertion is obvious for $d = 1$. Assume that it holds in dimension $(d-1)$. Let $f \in W^{1,p}_0(Q_d)$, where $Q_d = (0,1)^d$, with $\int_{Q_d} f = 0$.

Set

$$g(x') = \int_0^1 f(x',t)dt, \text{ where } x' = (x_1, \dots, x_{d-1}) \in Q_{d-1}.$$

Clearly, $g \in W^{1,p}_0(Q_{d-1})$ with

$$\|g\|_{W^{1,p}(Q_{d-1})} \leq C\|f\|_{W^{1,p}(Q_d)}$$

and also $\int_{Q_{d-1}} g = 0$. By the induction assumption there is some $Z \in W^{1,p}_0(Q_{d-1})$ such that

$$\operatorname{div}_{x'} Z = g \qquad \text{on } Q_{d-1} \tag{7.5}$$

and

$$\|Z\|_{W^{1,p}(Q_{d-1})} \leq C\|g\|_{W^{1,p}(Q_{d-1})} \leq C\|f\|_{W^{1,p}(Q_d)}.$$

Fix a function $\zeta \in C^\infty_0(0,1)$ such that

$$\int_0^1 \zeta(t)dt = 1. \tag{7.6}$$

For $x = (x', x_d) \in Q_d$ set

$$h(x) = \int_0^{x_d} (f(x',t) - \zeta(t)g(x'))dt.$$

It is easy to see (using (7.6)) that $h \in W_0^{1,p}(Q_d)$ and

$$\|h\|_{W^{1,p}(Q_d)} \leq C\|f\|_{W^{1,p}(Q_d)}.$$

Moreover

$$\frac{\partial h}{\partial x_d}(x) = f(x) - \zeta(x_d)g(x').$$

Combining this with (7.5) yields

$$f(x) = \operatorname{div}_{x'}\big(\zeta(x_d)Z(x')\big) + \frac{\partial h}{\partial x_d}$$

i.e., the conclusion holds with

$$Y(x) = \big(\zeta(x_d)Z(x'), h(x)\big).$$

□

Proof of Theorem 2. For simplicity we assume that $d = 2$; the argument is similar for $d > 2$.

Let

$$Q = \{(x,y) \in \mathbb{R}^2; \quad 0 < x < 1,\, 0 < y < 1\}.$$

Given $f \in L^p_{\#}(Q), 1 < p < \infty$, we will construct a solution $Y \in W_0^{1,p}(Q)$ of (1.1); moreover

$$\|Y\|_{W^{1,p}} \leq C_p\|f\|_{L^p} \tag{7.7}$$

and Y depends linearly on f. This is done in three steps. □

Step 1. Construct a solution $Y \in W^{1,p}(Q)$ of (1.1) satisfying (7.7) and

$$Y = 0 \ \text{ on the edge } \{(x,0); 0 < x < 1\}. \tag{7.8}$$

Proof. Set

$$\tilde{Q} = \{(x,y); 0 < x < 1, -2 < y < 1\}$$

and

$$\tilde{f} = \begin{cases} f & \text{in } Q, \\ 0 & \text{in } \tilde{Q}\backslash Q. \end{cases} \tag{7.9}$$

Let $Z \in W^{1,p}(\tilde{Q})$ be the solution of

$$\operatorname{div} Z = \tilde{f} \quad \text{in } \tilde{Q} \tag{7.10}$$

obtained via (7.2) (or via periodic conditions on $\tilde{Q}$).

The heart of the matter is the following construction. Write $Z = (Z_1, Z_2)$ and define $Y = (Y_1, Y_2)$ in Q, where

$$\begin{aligned} Y_1(x,y) &= Z_1(x,y) + 3Z_1(x,-y) - 4Z_1(x,-2y), \\ Y_2(x,y) &= Z_2(x,y) - 3Z_2(x,-y) + 2Z_2(x,-2y). \end{aligned} \tag{7.11}$$

(This type of "reflection" is reminiscent of standard extension techniques in $W^{m,p}$, $m \geq 2$; see e.g. Adams [1]).

It is easy to see using (7.9), (7,10) and (7.11) that

$$\operatorname{div}\ Y = f \quad \text{in } Q$$

while (7.8) is clear from the definition of Y.

It is important (for the next step) to observe that if we had started with the additional information

$$Z = 0 \quad \text{on the edge } \{(0,y); -2 < y < 1\} \text{ of } \tilde{Q},$$

then we could infer that Y *also vanishes* on the edge $\{(0,y); 0 < y < 1\}$ of Q. □

Step 2. Construct a solution $Y \in W^{1,p}(Q)$ of (1.1) satisfying (7.7) and
(7.12)
$Y = 0$ on the 2 adjacent edges $\{(x,0);\ 0 < x < 1\}$ and $\{(0,y);\ 0 < y < 1\}$.

Proof. Set

$$\hat{Q} = \{(x,y); -2 < x < 1, 0 < y < 1\}$$

and

$$\hat{f} = \begin{cases} f & \text{in } Q, \\ 0 & \text{in } \hat{Q}\backslash Q. \end{cases}$$

From Step 1 applied to $\hat{f}$ in $\hat{Q}$ we obtain a solution $\hat{Z}$ of

$$\operatorname{div} \hat{Z} = \hat{f} \quad \text{in } \hat{Q}$$

such that

$$\hat{Z} = 0 \quad \text{on the edge } \{(x,0); -2 < x < 1\} \text{ of } \hat{Q}.$$

Starting with $\hat{Z}$ (instead of Z) we repeat the construction of Step 1 changing the roles of x and y. We thus obtain a $Y \in W^{1,p}(Q)$ satisfying (1.1) in Q, (7.7) and (7.12). □

Step 3. Proof of Theorem 2 completed.

Consider a smooth partition of unity $(\theta_i), i = 1,2,3,4$, subordinate to the covering of Q consisting of the 4 discs of radius 1 centered at the 4 vertices. Let $Y_i \in W^{1,p}(Q)$ be the solution constructed in Step 2 relative to each vertex.

Set

$$Z = \sum_{i=1}^{4} \theta_i Y_i.$$

It is easy to see from this construction that $\theta_i Y_i \in W_0^{1,p}(Q)$, $\forall i$ and thus $Z \in W_0^{1,p}(Q)$. Moreover

$$\operatorname{div} Z = f + \sum_i \nabla\theta_i \cdot Y_i$$

and $\sum_i \nabla\theta_i \cdot Y_i \in W_0^{1,p}(Q)$. By Lemma 4 we may construct $X \in W_0^{1,p}(Q)$ satisfying

$$\operatorname{div} X = \sum_i \nabla\theta_i \cdot Y_i$$

and $Y = Z - X$ has all the desired properties in Theorem 2.

Next we have a variant of Theorem 1′ for the full Dirichlet condition.

Theorem 3. *Given $f \in L^d_{\#}(Q)$ there exists some $Y \in C^0(\bar{Q}) \cap W_0^{1,d}(Q)$ satisfying (1.1) with*

$$\|Y\|_{L^\infty} + \|Y\|_{W^{1,d}} \le C\|f\|_{L^d}.$$

Remark 10. Clearly, Theorem 3 implies Theorem 1′ since the function Y extended by periodicity belongs to $C^0(\mathbb{T}^d) \cap W^{1,d}(\mathbb{T}^d)$ and satisfies (1.1) on $\mathbb{T}^d$. However its proof relies heavily on Theorem 1′.

Proof of Theorem 3. Follow the same strategy as in the proof of Theorem 2. The only difference is that in Step 1 use Theorem $1'$ to obtain Z (instead of taking the special Z in the form of a gradient). Of course the dependence of Y on f is not linear anymore.

In Step 3 rely on the following variant of Lemma 4 (with an identical proof). $\square$

Lemma 4′. *Given* $f \in C^0(\bar{Q}) \cap W_0^{1,p}(Q), 1 < p < \infty$, *with* $\int f = 0$, *there exists* $Y \in C^0(\bar{Q}) \cap W_0^{1,p}(Q)$ *such that*

$$\operatorname{div} Y = f$$

and

$$\|Y\|_{L^\infty} + \|Y\|_{W^{1,p}} \leq C(\|f\|_{L^\infty} + \|f\|_{W^{1,p}}).$$

7.2. The case of Lipschitz domains. Let Ω be a Lipschitz, connected, bounded domain in $\mathbb{R}^d$. Recall that Ω is Lipschitz if there is a $\delta > 0$ such that for every point $p \in \partial\Omega$, $\partial\Omega \cap B_\delta(p)$ is the graph of a Lipschitz function (in an appropriate coordinate system varying with p).

We have the following variants of Theorems 2 and 3.

Theorem 2′. *Given any* $f \in L^p_{\#}(\Omega), 1 < p < \infty$, *there exists some* $Y \in W_0^{1,p}(\Omega)$ *solving (1.1) with*

$$\|Y\|_{W^{1,p}} \leq C(p, \Omega)\|f\|_{L^p}. \tag{7.13}$$

Moreover Y *can be chosen, depending linearly on* f.

Theorem 3′. *For every* $f \in L^d_{\#}(\Omega)$ *there exists some* $Y \in C^0(\bar{\Omega}) \cap W_0^{1,d}(\Omega)$ *solving (1.1) with*

$$\|Y\|_{L^\infty} + \|Y\|_{W^{1,d}} \leq C(p, \Omega)\|f\|_{L^d}. \tag{7.14}$$

The heart of the argument (for both theorems) is the following.

Lemma 5. *There is a bounded operator* $S : L^p(\Omega) \to W_0^{1,p}(\Omega)$ *such that*

$$f - \operatorname{div} Sf \in W_0^{1,p} \qquad \forall f \in L^p$$

and

$$\|f - \operatorname{div} Sf\|_{W^{1,p}} \leq C\|f\|_{L^p}. \tag{7.15}$$

The variant needed for the proof of Theorem $3'$ is

Lemma 5′. *There is a nonlinear map* $S : L^d(\Omega) \to C^0(\bar{\Omega}) \cap W_0^{1,d}(\Omega)$ *such that*

$$\|Sf\|_{L^\infty} + \|Sf\|_{W^{1,d}} \leq C\|f\|_{L^d} \tag{7.16}$$

and

$$\|f - \operatorname{div} Sf\|_{W^{1,d}} \leq C\|f\|_{L^d}. \tag{7.17}$$

The proof of Lemma 5 relies on the following construction. Let Q' be a cube of side δ in $\mathbb{R}^{d-1}$ and set

$$U = \{(x', y) \in Q' \times \mathbb{R}; \psi(x') < y < \psi(x') + \delta\}$$

where $\psi \in \operatorname{Lip}(Q')$.

Lemma 6. *Assume*

$$\|\nabla\psi\|_{L^\infty(Q')} \le \varepsilon_0(d) \text{ sufficiently small (depending only on } d). \tag{7.18}$$

Then, given any $g \in L^p(U)$ *there is some* $Z \in W^{1,p}(U)$ *satisfying*

$$\operatorname{div} Z = g \quad \text{in } U, \tag{7.19}$$

$$Z = 0 \text{ on } \{y = \psi(x');\ x' \in Q'\} \text{ and on the lateral boundary of } U, \tag{7.20}$$

with

$$\|Z\|_{W^{1,p}(U)} \le C(p,d)\|g\|_{L^p(U)}.$$

Moreover Z *can be chosen to depend linearly on* g.

Proof. For $x' \in Q'$ and $0 < y < \delta$ set

$$\tilde{g}(x',y) = g\big(x', y + \psi(x')\big).$$

Note that

$$\|\tilde{g}\|_{L^p(Q)} = \|g\|_{L^p(U)}$$

where $Q = Q' \times (0,\delta)$.

By Theorem 2 there exists $\tilde{Z} \in W^{1,p}(Q)$ such that

$$\begin{cases} \operatorname{div} \tilde{Z} = \tilde{g} & \text{in } Q, \\ \tilde{Z} = 0 & \text{on } \{(x',0);\ x' \in Q'\} \cup \big(\partial Q' \times (0,\delta)\big) \end{cases}$$

with

$$\|\tilde{Z}\|_{W^{1,p}(Q)} \le C(d)\|\tilde{g}\|_{L^p(Q)}. \tag{7.21}$$

Note that here $\int \tilde{g} = 0$ is not required since we may consider in $\hat{Q} = Q' \times (0, 2\delta)$ the function

$$\hat{g}(x',y) = \begin{cases} \tilde{g}(x',y) & \text{for } x' \in Q' \text{ and } 0 < y < \delta, \\ -\tilde{g}(x', y-\delta) & \text{for } x' \in Q \ \text{ and } \delta < y < 2\delta, \end{cases}$$

and then solve (using Theorem 2)

$$\begin{aligned} \operatorname{div} \hat{Z} &= \hat{g} && \text{in } \hat{Q}, \\ \hat{Z} &= Q && \text{on } \partial\hat{Q}, \end{aligned}$$

with

$$\|\hat{Z}\|_{W^{1,p}(\hat{Q})} \le C(d)\|\tilde{g}\|_{L^p(Q)}.$$

The restriction $\tilde{Z}$ of $\hat{Z}$ to $Q' \times (0,\delta)$ satisfies the desired properties.

Also, it is clear by scaling that the constant in (7.21) is independent of δ.

Returning to $(x',y) \in U$, set

$$Z(x',y) = \tilde{Z}\big(x', y - \psi(x')\big);$$

it is easy to see, using (7.18) and (7.21), that

$$\|\operatorname{div} Z - g\|_{L^p(U)} \le C(d)\varepsilon_0\|g\|_{L^p(U)}$$

and

$$\|Z\|_{W^{1,p}(U)} \le C(d)(1+\varepsilon_0)\|g\|_{L^p(U)}.$$

Choosing ε_0 such that $C(d)\varepsilon_0 < 1$ and iterating this construction yields the lemma. □

The variant necessary for Theorem 3′ is

Lemma 6′. *Assume (7.18). Then given* $g \in L^d(U)$ *there is some* $Z \in C^0(\bar{U}) \cap W^{1,p}(U)$ *satisfying (7.19), (7.20) and*

$$\|Z\|_{L^\infty(U)} + \|Z\|_{W^{1,d}(U)} \le C(d)\|g\|_{L^d(U)}.$$

Next, we remove the smallness condition (7.18) on the Lipschitz constant of ψ.

Lemma 7. *With the same notation as in Lemma 6, assume only that* $\psi \in \operatorname{Lip}(Q')$.

Then, given any $g \in L^p(U)$, *there is some* $Z \in W^{1,p}(U)$ *satisfying (7.19), (7.20) and*

$$\|Z\|_{W^{1,p}(U)} \le C(p, d, \|\nabla\psi\|_{L^\infty(Q')})\|g\|_{L^p(U)}.$$

Moreover Z *can be chosen to depend linearly on* g.

Proof. Consider the dilation $x' \mapsto \tilde{x}' = Nx'$ (only in x', not in the full x-variable). Set $\tilde{Q}' = NQ'$ and define on $\tilde{Q}'$ the function

$$\tilde{\psi}(\tilde{x}') = \psi(\tilde{x}'/N).$$

Fix an integer N sufficiently large so that

$$\|\nabla\tilde{\psi}\|_{L^\infty(\tilde{Q}')} = \frac{1}{N}\|\nabla\psi\|_{L^\infty(Q')} \le \varepsilon_0(d)$$

where $\varepsilon_0(d)$ comes from (7.18).

Set

$$\tilde{g}(\tilde{x}', y) = g\left(\frac{\tilde{x}'}{N}, y\right).$$

Divide the cube $\tilde{Q}'$ (of side $N\delta$) into N^{d-1} cubes of side δ and apply, in each of them, Lemma 6 to $\tilde{\psi}$ and $\tilde{g}$. By gluing the corresponding solutions (this is possible because all these solutions vanish on the lateral boundaries of their domains), we obtain some $\tilde{Z}(\tilde{x}', y) \in W^{1,p}(\tilde{U})$ satisfying

$$\begin{cases} \operatorname{div}_{\tilde{x}',y}\tilde{Z} = \tilde{g} & \text{in } \tilde{U} = \{(\tilde{x}', y) \in \tilde{Q}' \times \mathbb{R};\ \tilde{\psi}(\tilde{x}') < y < \tilde{\psi}(\tilde{x}') + \delta\}, \\ \tilde{Z} = 0 & \text{on } \{y = \tilde{\psi}(\tilde{x}'); \tilde{x}' \in \tilde{Q}'\}, \end{cases}$$

and the corresponding $W^{1,p}$-estimate for $\tilde{Z}$.

We now return to the variables $(x', y) \in U$. Write the components of $\tilde{Z}$ as

$$\tilde{Z} = (\tilde{Z}', \tilde{Z}_d)$$

and set

$$Z(x', y) = \left(\frac{1}{N}\tilde{Z}'(Nx', y), \tilde{Z}_d(Nx', y)\right).$$

It is easy to check that Z satisfies all the required properties. □

The variant necessary for Theorem 3′ is

Lemma 7′. *With the same notation as in Lemma 6, assume only that* $\psi \in \operatorname{Lip}(Q')$.

Then, given any $g \in L^d(U)$, *there is some* $Z \in C^0(\bar{U}) \cap W^{1,p}(U)$ *satisfying (7.19), (7.20) and*

$$\|Z\|_{L^\infty(U)} + \|Z\|_{W^{1,p}(U)} \le C(d, \|\nabla\psi\|_{L^\infty(Q')})\|g\|_{L^p(U)}.$$

We now return to the

Proof of Lemma 5. Consider a finite covering of $\partial\Omega$ by a collection of cubes Q_i, $i = 1, \dots, k$, of side δ such that in each Q_i, $\partial\Omega \cap Q_i$ admits a Lipschitz parametrization ψ_i. To this covering we associate functions $\theta_0, \theta_1, \dots, \theta_k$ such that

$$\theta_0 + \sum_{i=1}^{k} \theta_i = 1 \qquad \text{on } \Omega,$$
$$\theta_0 \in C_0^\infty(\Omega) \text{ and } \theta_i \in C_0^\infty(Q_i) \text{ for } i = 1, \dots, k.$$

Given $g \in L^p(\Omega)$ solve, using Lemma 7, for $i = 1, 2, \dots, k$,

$$\begin{cases} \operatorname{div} Z_i = g & \text{in } U_i, \\ Z_i = 0 & \text{on } \partial\Omega \cap Q_i. \end{cases}$$

Next solve

$$\operatorname{div} Z_0 = g \qquad \text{in } \ \Omega,$$

for example $Z_0 = \operatorname{grad}(\Delta)^{-1}$ where Δ^{-1} is used with zero Dirichlet condition on $\partial\Omega$.

Note that

$$Z = \sum_{i=0}^{k} \theta_i Z_i \in W_0^{1,p}$$

and

$$\operatorname{div} Z = g + \sum_{i=0}^{k} \nabla\theta_i \cdot Z_i.$$

All the conclusions of Lemma 5 hold with

$$Sg = Z.$$

□

Proof of Lemma $5'$. We make the same construction as above, using Lemma $7'$ in place of Lemma 7 and Theorem 2 to solve $\operatorname{div} Z_0 = g$ in any large cube containing Ω. □

Theorem $2'$ is an immediate consequence of Lemma 5 and the following general functional analysis argument applied with $E = W_0^{1,p}, F = L^p_\#$ and $T = \operatorname{div}$. (Note that $T^* = \operatorname{grad}$ is injective on $F^* = L^q_\#$, since Ω is connected.)

Lemma 8. *Let E, F be two Banach spaces and let T be a bounded operator from E into F. Assume*

$$N(T^*) = \{0\}. \tag{7.22}$$

$$\begin{cases} \textit{There is a bounded operator } S \textit{ from } F \textit{ into } E \textit{ and} \\ \textit{a compact operator } K \textit{ from } F \textit{ into itself such that} \\ \qquad\qquad T \circ S = I + K. \end{cases} \tag{7.23}$$

Then T admits a right inverse.

Proof. First we note that T is onto. Indeed, in view of (7.22) it suffices to show that T (or equivalently T^*) has closed range. This is an obvious consequence of the inequality

$$\|f\| \le C\|T^*f\| + \|K^*f\| \qquad \forall f \in F^*$$

(which follows from (7.23)).

Next, let X be a complementing subspace for $N(I+K)$ in F and set $Y = R(I+K)$. Since $u = (I+K)_{|X}$ is an isomorphism onto Y, its inverse $u^{-1} : Y \to X \subset F$ satisfies

$$(I+K)\circ u^{-1} = I \text{ on } Y. \tag{7.24}$$

Let Q be a projector from F onto Y; since $R(I-Q)$ is finite dimensional, we may choose a base (e_α) of $R(I-Q)$ and write

$$f = Qf + \sum_\alpha \langle e_\alpha^*, f\rangle e_\alpha \qquad \forall f \in F, \tag{7.25}$$

for some e_α^*'s in F^*.

Since we showed that T is onto, one has, for each α, some $\bar{e}_\alpha \in E$ satisfying

$$T\bar{e}_\alpha = e_\alpha \qquad \forall\alpha. \tag{7.26}$$

Consider the operator $S_1 : F \to E$ defined for every $f \in F$, by

$$S_1 f = S \circ u^{-1} \circ Qf + \sum_\alpha \langle e_\alpha^*, f\rangle \bar{e}_\alpha.$$

Using (7.24), (7.25) and (7.26) we see that

$$\begin{aligned} T\circ S_1 f &= (I+K)\circ u^{-1}\circ Qf + \sum_\alpha \langle e_\alpha^*, f\rangle e_\alpha \\ &= Qf + \sum_\alpha \langle e_\alpha^*, f\rangle e_\alpha = f \end{aligned}$$

for every $f \in F$. Thus S_1 is a right inverse for T. □

Proof of Theorem 3′. Given $f \in L^d$ write, using Lemma 5′,

$$f = \operatorname{div} Y_1 + R$$

with $Y_1 \in C^0(\bar\Omega) \cap W_0^{1,d}(\Omega)$ and $R \in W_0^{1,d}(\Omega)$ (and the corresponding estimates).

If $\int f = 0$, then $\int R = 0$ and we may apply Theorem 2′ in any L^p (since $W^{1,d} \subset L^p$, $\forall p < \infty$). In particular, if we choose $p > d$, we obtain $Y_2 \in W_0^{1,p}(\Omega)$ such that

$$R = \operatorname{div} Y_2.$$

By the Sobolev imbedding, $Y_2 \in C^0(\bar\Omega)$ and $Y = Y_1 + Y_2$ satisfies all the required properties. □

8. ESTIMATION OF THE PHASE IN $H^{1/2} + W^{1,1}$. PROOF OF THEOREM 4

We return in this last section to the question discussed in the Introduction concerning the control of the phase φ in terms of $\|e^{i\varphi}\|_{H^{1/2}}$.

Let φ be a smooth real-valued function on $\mathbb{T}^d$ and set $g = e^{i\varphi}$. The main result is the estimate

$$\|\varphi\|_{H^{1/2}+W^{1,1}} \le C(d)(1 + \|g\|_{H^{1/2}})\|g\|_{H^{1/2}}. \tag{8.1}$$

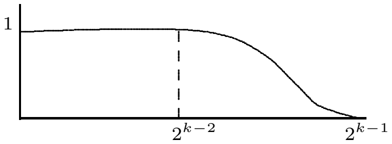
Write g as a Fourier series

$$g = \sum_{\xi\in\mathbb{Z}^d} \hat{g}(\xi) e^{ix\xi}.$$

The $H^{1/2}$-component in the decomposition of φ will be obtained as a paraproduct of g and $\bar{g}$,

$$P = \sum_k \Big[\sum_{\xi_2} \lambda_k(|\xi_2|)\overline{\hat{g}(\xi_2)}e^{-ix\xi_2}\Big]\Big[\sum_{2^k\le|\xi_1|<2^{k+1}} \hat{g}(\xi_1)e^{ix\xi_1}\Big], \tag{8.2}$$

where for each k we let $0 \le \lambda_k \le 1$ be a smooth function on $\mathbb{R}_+$:



We claim that

$$\|P\|_{H^{1/2}} \le C\|g\|_\infty\|g\|_{H^{1/2}} \tag{8.3}$$

and

$$\|\varphi - \frac{1}{i}P\|_{W^{1,1}} \le C\|g\|^2_{H^{1/2}}. \tag{8.4}$$

Proof of (8.3). This is totally obvious from the construction

$$\begin{aligned}\|P\|^2_{H^{1/2}} &\sim \sum_k 2^k \Big\|\Big[\sum_{\xi_2} \lambda_k(|\xi_2|)\overline{\hat{g}(\xi_2)}e^{-ix\xi_2}\Big]\Big[\sum_{2^k\le|\xi_1|<2^{k+1}} \hat{g}(\xi_1)e^{ix\xi_1}\Big]\Big\|_2^2\\ &\le \sum_k 2^k \Big\|\sum \lambda_k(|\xi|)\overline{\hat{g}(\xi)}e^{-ix\xi}\Big\|^2_\infty \Big[\sum_{|\xi|\sim 2^k} |\hat{g}(\xi)|^2\Big]\\ &\le C\|g\|^2_\infty\|g\|^2_{H^{1/2}}. \qquad (8.5)\end{aligned}$$

□

Proof of (8.4). We estimate for instance

$$\|\partial_1\varphi - \frac{1}{i}\partial_1 P\|_{L^1}. \tag{8.6}$$

Thus, letting $\xi = (\xi^1, \dots, \xi^d) \in \mathbb{Z}^d$,

$$\partial_1\varphi = \frac{1}{i}\bar{g}\partial_1 g = \sum_{\xi_1,\xi_2\in\mathbb{Z}^d} \xi_1^1\hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}\, e^{ix\cdot(\xi_1-\xi_2)} \tag{8.7}$$

and by (8.2)

$$\frac{1}{i}\partial_1 P = \sum_k \sum_{2^k\le|\xi_1|<2^{k+1},\xi_2} (\xi_1^1 - \xi_2^1)\lambda_k(|\xi_2|)\hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}e^{ix\cdot(\xi_1-\xi_2)}, \tag{8.8}$$

$$\partial_1\varphi - \frac{1}{i}\partial_1 P = \sum_k \sum_{2^k\le|\xi_1|<2^{k+1},\xi_2} m_k(\xi_1,\xi_2)\hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}e^{ix\cdot(\xi_1-\xi_2)}, \tag{8.9}$$

where by definition of λ_k

$$m_k(\xi_1,\xi_2) = \xi_1^1 - \lambda_k(|\xi_2|)(\xi_1^1 - \xi_2^1) = \begin{cases} \xi_2^1 & \text{if } |\xi_2| \le 2^{k-2},\\ \xi_1^1 & \text{if } |\xi_2| \ge 2^{k-1}.\end{cases} \tag{8.10}$$

Estimate

$$\|\partial_1\varphi - \frac{1}{i}\partial_1 P\|_1 \le \sum_{k_1,k_2} \Bigg\| \sum_{|\xi_1|\sim 2^{k_1}, |\xi_2|\sim 2^{k_2}} m_{k_1}(\xi_1,\xi_2)\hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}e^{ix\cdot(\xi_1-\xi_2)} \Bigg\|_1 . \tag{8.11}$$

Distinguish the contributions of

$$\sum_{k_1\sim k_2} + \sum_{k_1<k_2-4} + \sum_{k_1>k_2+4} = (8.12) + (8.13) + (8.14).$$

Clearly $2^{-k}m_k(\xi_1,\xi_2)$ restricted to $[|\xi_1|\sim 2^k]\times[|\xi_2|\sim 2^k]$ is a smooth multiplier satisfying the usual derivative bounds. Therefore

$$(8.12) \le C\sum_k 2^k \Bigg\| \sum_{|\xi_1|\sim 2^k} \hat{g}(\xi_1)e^{ix\xi_1}\Bigg\|_2 \Bigg\| \sum_{|\xi_2|\sim 2^k} \hat{g}(\xi_2)e^{ix\xi_2}\Bigg\|_2 \sim \|g\|^2_{H^{1/2}}. \tag{8.15}$$

If $k_1 < k_2 - 4$, then $|\xi_2| > 2^{k_1}$ and $m_{k_1}(\xi_1,\xi_2) = \xi_1^1$ by (8.10). Therefore

$$\begin{aligned}
(8.13) &= \sum_{k_1<k_2-4} \Bigg\| \sum_{|\xi_1|\sim 2^{k_1}, |\xi_2|\sim 2^{k_2}} \xi_1^1\, \hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}e^{ix\cdot(\xi_1-\xi_2)} \Bigg\|_1 \\
&\le \sum_{k_1<k_2-4} 2^{k_1} \Bigg\| \sum_{|\xi_1|\sim 2^{k_1}} \hat{g}(\xi_1)e^{ix\xi_1}\Bigg\|_2 \cdot \Bigg\| \sum_{|\xi_2|\sim 2^{k_2}} \hat{g}(\xi_2)e^{ix\xi_2}\Bigg\|_2 \\
&\le \sum_{k_1<k_2} 2^{k_1}\Bigg(\sum_{|\xi_1|<2^{k_1}} |\hat{g}(\xi_1)|^2\Bigg)^{1/2} \Bigg(\sum_{|\xi_2|\sim 2^{k_2}} |\hat{g}(\xi_2)|^2\Bigg)^{1/2} \le C\|g\|^2_{H^{1/2}}. \qquad (8.16)
\end{aligned}$$

If $k_1 > k_2 + 4$, then $|\xi_2| < 2^{k_1-2}$ and $m_{k_1}(\xi_1,\xi_2) = \xi_2^1$ and the bound on (8.14) is similar. □

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INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540
E-mail address: `bourgain@math.ias.edu`

ANALYSE NUMÉRIQUE, UNIVERSITÉ P. ET M. CURIE, B.C. 187, 4 PL. JUSSIEU, 75252 PARIS CEDEX 05, FRANCE
E-mail address: `brezis@ccr.jussieu.fr`
Current address: Department of Mathematics, Rutgers University, Hill Center, Busch Campus, 110 Frelinghuysen Rd., Piscataway, New Jersey 08854
E-mail address: `brezis@math.rutgers.edu`