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INFINITELY MANY HYPERBOLIC 3-MANIFOLDS WHICH CONTAIN NO REEBLESS FOLIATION

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1. Introduction

It has long been realized that the presence of a Reebless foliation in a compact 3-manifold M reveals useful topological information about M. By Novikov [No65], M is irreducible with infinite fundamental group. By Palmeira [Pa78], M has universal cover $\mathbb{R}^3$. Building on work of Thurston and Gabai and Kazez [Ga98, GK98], Calegari [Ca] has shown that if M is also atoroidal, then $\pi_1(M)$ is Gromov negatively curved. Furthermore, Thurston has proposed an approach to demonstrating geometrization for such M. Many 3-manifolds contain Reebless foliations, and it has often been conjectured that all closed hyperbolic 3-manifolds do. (It is our impression that for many years Hatcher provided the sole voice of dissent.) In this paper, we give the first examples of closed hyperbolic 3-manifolds which contain no Reebless foliation.

Theorem A. *There exist infinitely many closed orientable hyperbolic* 3*-manifolds which do not contain a Reebless foliation.*

In particular, therefore, there exist infinitely many closed orientable hyperbolic 3-manifolds which do not contain an Anosov flow.

In 1989, Gabai and Oertel [GO89] introduced the notion of essential lamination. Empirically, these objects seemed easier to find than Reebless foliations, but it was not known whether or not essential laminations were in fact more prevalent. See [Br93, Cl91], [RS01] and [BNR] for related results. In this paper we give the first examples (again, an infinite family) of closed hyperbolic 3-manifolds which do not contain a Reebless foliation but which do contain essential laminations.

Theorem C. *There exist infinitely many closed orientable hyperbolic* 3*-manifolds which contain neither a Reebless foliation nor a transversely oriented essential lamination but which do contain essential laminations.*

In contrast, as discussed in Section 2, we expect that a subset of the set of examples of Theorem A will lead also to examples of closed orientable hyperbolic

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3-manifolds which contain no essential lamination. Indeed, Fenley has recently announced a proof of this claim.

We establish nonexistence of Reebless foliations and transversely oriented essential laminations by proving the nonexistence of nontrivial fundamental group actions on leaf spaces. Let Λ be any Reebless foliation in M. Denote its lift to the universal cover $\tilde{M}$ of M by $\tilde{\Lambda}$. Then the *leaf space* (see Section 6) of $\tilde{\Lambda}$, $T_{\tilde{\Lambda}}$, is a second countable but not necessarily Hausdorff simply-connected 1-manifold, and the action of $\pi_1(M)$ on $\tilde{M}$ induces a nontrivial action of $\pi_1(M)$ on $T_{\tilde{\Lambda}}$ by homeomorphisms ([HR57, Pa78]; see also [Ba98, CC]). (An action of a group G on a topological space X is called *trivial* if there is an $x \in X$ such that for all $g \in G$, x and xg are not separated in X.) We obtain Theorem A by describing an infinite family of closed hyperbolic 3-manifolds whose fundamental groups do not act nontrivially on simply-connected (second countable but not necessarily Hausdorff) 1-manifolds. More generally, we investigate group actions on $\mathbb{R}$-order trees [GK97]. Let Λ be any essential lamination in M, and denote by $\tilde{\Lambda}$ its lift to $\tilde{M}$. Then the *leaf space* (as defined in Section 6) of $\tilde{\Lambda}$, $T_{\tilde{\Lambda}}$, is an $\mathbb{R}$-order tree, and the action of $\pi_1(M)$ on $\tilde{M}$ induces a nontrivial action of $\pi_1(M)$ on $T_{\tilde{\Lambda}}$ by homeomorphisms [GO89, GK97]. Using the same set of examples, but instead ruling out nontrivial orientation preserving actions by the fundamental groups on $\mathbb{R}$-order trees, we obtain Theorem C.

Here is a brief outline of the structure of the paper. In Section 2, we describe the family of examples of Theorems A and C. They form a subset of a family of examples proposed by Hatcher [Ha92]. In Section 3, we begin by examining the case that the simply-connected 1-manifold is $\mathbb{R}$. In Section 4, we build on a paper of Barbot [Ba98] and also the well-known work on isometric actions on real trees to investigate actions on non-Hausdorff 1-manifolds. In Section 5, we pass from simply-connected 1-manifolds to the more general world of $\mathbb{R}$-order trees. In particular, we obtain the following results.

Corollary 5.7. *G acts nontrivially on an $\mathbb{R}$-order tree, then G acts nontrivially on a Hausdorff $\mathbb{R}$-order tree.*

Proposition 5.10. *If G acts nontrivially on an oriented $\mathbb{R}$-order tree by orientation preserving order tree automorphisms, then G acts nontrivially on a simply-connected 1-manifold (by orientation preserving homeomorphisms).*

In Section 6, we recall the definition of leaf space and relate the existence of Reebless foliations (and essential laminations) to the existence of actions on simply-connected 1-manifolds ($\mathbb{R}$-order trees). In Sections 7 and 8, we prove the nonexistence of group actions for the examples. Finally, in the appendix, we make precise the notion of Denjoy blow-up for simply-connected 1-manifolds and order trees.

We note that recently Calegari and Dunfield [CD03] announced that they too can generate examples of closed hyperbolic 3-manifolds containing no taut foliation. Their approach is also via group actions but from a different viewpoint. They obtain their examples by using their result that any atoroidal 3-manifold with a taut foliation has a finite abelian cover whose fundamental group is left-orderable.

2. The examples

Once-punctured torus bundles over S^1,

$$M_\phi = (F \times I)/\phi$$

(where F is a compact orientable surface of genus one with single boundary component and $\phi \in Homeo(F)$), and manifolds $\hat{M}_\phi(\rho)$ obtained by ρ-surgery on M_ϕ have been much studied. See, for example, [Ba89, Ba90, BH96, BPZ, BMR, CJR, FH82, Ha92, HMW, Jo77, Ma00, Mc87, NR92, Pr83, Pr84, Re95, RW99, Th88, We85]. In particular, it is well known that M_ϕ is uniquely determined (up to homeomorphism) by $\phi_\sharp : H_1(F) \to H_1(F)$ [Ni17], where by fixing a basis for F, we obtain a matrix $\phi_\sharp \in GL_2(\mathbb{Z})$. By Thurston [Th88], M_ϕ is hyperbolic if and only if ϕ is pseudo-Anosov, and an elementary eigenvalue argument reveals that ϕ is pseudo-Anosov if and only if $|trace(\phi_\sharp)| > 2$. By Thurston's Orbifold Theorem, $\hat{M}_\phi(\rho)$ admits a geometric decomposition (cf. [CHK]). And when M_ϕ is hyperbolic (namely, when $|trace(\phi_\sharp)| > 2$), Thurston's Hyperbolic Dehn Surgery Theorem [Th79] reveals that $\hat{M}_\phi(\rho)$ is hyperbolic for all but at most finitely many choices of ρ. In fact, since $\hat{M}_\phi(\rho)$ admits a geometric decomposition, $\hat{M}_\phi(\rho)$ is hyperbolic for all but at most 12 values of ρ [Ag00, La00]. (For improvements on this bound for these manifolds, see also the comments following Theorem 13 of [BH96].)

We next summarize what is known about the existence of Reebless foliations and, more generally, essential laminations in the closed manifolds $\hat{M}_\phi(\rho)$. For simplicity of exposition, we restrict attention to the case that M_ϕ is hyperbolic and orientable. Recall that a *slope* is an isotopy class of unoriented simple closed curves in ∂M. Let $|\langle \zeta, \eta \rangle|$ denote the absolute value of the homological intersection number of representatives of slopes ζ and η. Note that this number is well defined even though the homology classes of ζ and η are defined only up to ± 1.

Essential surfaces in M_ϕ are classified in [FH82] and [CJR]. In particular, it follows from these classifications that there are essential surfaces in $\hat{M}_\phi(\rho)$ for at most finitely many ρ.

Fixing a Riemannian metric on F and choosing the corresponding pseudo-Anosov representative for ϕ [Th88], let f^s and f^u denote, respectively, the stable and unstable laminations fixed by ϕ. Let γ denote the isotopy class of a closed orbit of the pseudo-Anosov flow of ϕ restricted to ∂M. Choose transverse orientations for f^s and f^u. Notice that

- $\phi : f^s \to f^s$ preserves the transverse orientation iff $trace(\phi_\sharp) > 2$ iff $|\gamma \cap \partial F| = 1$.
- $\phi : f^s \to f^s$ reverses the transverse orientation iff $trace(\phi_\sharp) < -2$ iff $|\gamma \cap \partial F| = 2$.

Now let $\Lambda^s = (f^s \times I)/\phi$ and $\Lambda^u = (f^u \times I)/\phi$ denote the suspension laminations. Notice that Λ^s and Λ^u are transversely oriented if and only if $trace(\phi_\sharp) > 2$. By Theorem 5.3 of [GO89], Λ^s and Λ^u are essential in $\hat{M}_\phi(\rho)$ for all ρ not isotopic to γ when $|\gamma \cap \partial F| = 1$, and for all ρ satisfying $|\langle \rho, \gamma \rangle| \geq 2$ when $|\gamma \cap \partial F| = 2$. Furthermore, as noted by Thurston, Fried and Ghys, these suspension laminations extend to taut foliations exactly when they are transversely orientable (cf. [Ga97]). Namely, when $|\gamma \cap \partial F| = 1$ and ρ is not isotopic to γ, and when $|\gamma \cap \partial F| = 2$ and $\langle \rho, \gamma \rangle$ is a nonzero even integer, the suspension laminations extend to taut foliations in $\hat{M}_\phi(\rho)$. Otherwise, they do not.

There exists a family of taut foliations discovered by Hatcher [Ha92]. If $|\gamma \cap \partial F| = 1$, then $\hat{M}_\phi(\rho)$ contains taut foliations transverse to the pseudo-Anosov flow inherited from M_ϕ for all slopes ρ not isotopic to γ. If $|\gamma \cap \partial F| = 2$, the situation is again a little more complicated to describe. When $|\gamma \cap \partial F| = 2$, there exist

exactly two slopes, μ_1, μ_2 say, determined by the intersection number conditions $|\langle \mu_i, \partial F\rangle| = 1$ and $|\langle \mu_i, \gamma\rangle| = 2$. Fixing a basis on ∂M yields a canonical identification of the set of slopes with $\mathbb{Q} \cup \{\infty\}$, which in turn embeds as a dense subset of S^1 [Ro77]. We can therefore think of the boundary slopes $\{\mu_1, \mu_2\}$ as disconnecting S^1 into two open subintervals; denote by (μ_1, μ_2) the interval which does not contain γ. Then $\hat{M}_\phi(\rho)$ contains taut foliations transverse to the pseudo-Anosov flow inherited from M_ϕ for all slopes ρ in the interval (μ_1, μ_2).

Up to minor modifications, we have just listed all essential laminations known to exist in manifolds $\hat{M}_\phi(\rho)$.

Question (Hatcher [Ha92]). Is this list complete?

We are almost ready to describe the examples considered in this paper. First however we must fix a coordinate system on ∂M. As is standard, we describe a coordinate system on ∂M by specifying two oriented simple closed curves, called the longitude, λ, and the meridian, μ, respectively, and satisfying $\langle \lambda, \mu\rangle = 1$. Given any essential simple closed curve γ in T, we define

$$slope \;\; \gamma = \frac{\langle \gamma, \lambda\rangle}{\langle \mu, \gamma\rangle}.$$

(See, for example, [Ro77], p. 259.) Note that the slope of λ is therefore $\frac{0}{1}$; the slope of μ, $\frac{1}{0}$. We follow convention and set $\lambda = \partial F$, with the orientation inherited from F. When $|\langle \gamma, \partial F\rangle| = 1$, we choose $\mu = \gamma$. Otherwise, $|\langle \gamma, \partial F\rangle| = 2$ and we choose μ so that γ has slope $\frac{2}{1}$. Let ρ have slope $\frac{p}{q}$. Note that if $trace(\phi_\sharp) > 2$, then $|\langle \rho, \gamma\rangle| = 2|q|$, and if $trace(\phi_\sharp) < -2$, then $|\langle \rho, \gamma\rangle| = |p - 2q|$.

We can now summarize the existence results described above as follows.

(1) $\hat{M}_\phi(\frac{p}{q})$ contains an essential surface for at most finitely many $\frac{p}{q}$.
(2) $\hat{M}_\phi(\frac{p}{q})$ contains a taut foliation if one of the following is true:
   - $trace(\phi_\sharp) > 2$ and $\frac{p}{q} \neq \frac{1}{0}$.
   - $trace(\phi_\sharp) < -2$ and $\frac{p}{q} \in (-\infty, 1)$.
   - $trace(\phi_\sharp) < -2$ and p is even.
(3) $\hat{M}_\phi(\frac{p}{q})$ contains a transversely oriented essential lamination if one of the following is true:
   - $trace(\phi_\sharp) > 2$ and $\frac{p}{q} \neq \frac{1}{0}$.
   - $trace(\phi_\sharp) < -2$ and $\frac{p}{q} \in (-\infty, 1)$.
(4) $\hat{M}_\phi(\frac{p}{q})$ contains an essential lamination if one of the following is true:
   - $trace(\phi_\sharp) > 2$ and $\frac{p}{q} \neq \frac{1}{0}$.
   - $trace(\phi_\sharp) < -2$ and $|p - 2q| \geq 2$.

Next we fix a standard group presentation for $\pi_1(\hat{M}_\phi(\frac{p}{q}))$. Isotope μ as necessary so that $|\lambda \cap \mu| = 1$ and set $\{x_0\} = \lambda \cap \mu$. Let $t = [\mu] \in \pi_1(\hat{M}_\phi(\frac{p}{q}), x_0)$, and choose a basis a, b for $\pi_1(F, x_0)$. Let $\phi_* : \pi_1(F, x_0) \to \pi_1(F, x_0)$ be the map induced by $\phi : F \to F$. Then $\pi_1(\hat{M}_\phi(\frac{p}{q}))$ has group presentation

$$\langle a, b, t | a^t = a\phi_*, b^t = b\phi_*, t^p\,[a, b]^q = 1\rangle,$$

where we use the notation $g^h := h^{-1}gh$ and $[g,h] := ghg^{-1}h^{-1}$. In this paper, we pass to the subset of these examples satisfying

$$\phi_\sharp = \begin{bmatrix} m & 1 \\ -1 & 0 \end{bmatrix}.$$

(To view these manifolds in an alternate context, namely, as surgeries on the Whitehead link, see [HMW].) Now for integers p, q, m with $\gcd(p,q) = 1$, define $G(p,q,m)$ to be the group generated by t, a and b subject to the relations

(R1) $a^t = aba^{m-1}$,
(R2) $b^t = a^{-1}$, and
(R3) $t^p\,[a,b]^q = 1$.

Note that relations (R1) and (R2) imply that

$$t\,[a,b] = [a,b]\,t.$$

Since by Nielsen [Ni17],

$$Aut^+(F) \cong \{f \in Aut(\pi_1(F)) | f \text{ fixes } [a,b]\} \cong SL_2(\mathbb{Z}),$$

we conclude that

$$\pi_1(\hat{M}_\phi(\frac{p}{q})) \cong G(p,q,m).$$

Now let T be a simply-connected (second countable but not necessarily Hausdorff) 1-manifold. Since T is a simply-connected 1-manifold, it possesses exactly two orientations. Orient T and let $Homeo^+(T)$ be the subgroup of $Homeo(T)$ consisting of the orientation preserving homeomorphisms of T.

Convention. *Throughout this paper, we assume that all group actions on all sets are from the right. This includes the action of Homeo(X) on X for any space X.*

So we are interested in continuous (right) actions of $G(p,q,m)$ on T, that is, homomorphisms Φ from $G(p,q,m)$ to the group $Homeo(T)$ of homeomorphisms of T. As noted in the introduction, we say that a subgroup H of $Homeo(T)$ *has a global fixed point*, or that H acts *trivially* on T, if there is some $x \in T$ such that xh, x are nonseparated in T for all $h \in H$. We will prove the following result.

Theorem 2.1. *Suppose m, p, q are integers satisfying $m < -2$, $p \geq q \geq 1$, and $(p,q) = 1$. Suppose further that both m and p are odd. Then the image of any homomorphism*

$$\Phi : G(p,q,m) \to Homeo^+(T)$$

has a global fixed point.

Proof. This is proved in Sections 3, 7 and 8. □

Since the commutator quotient $H_1(G(p,q,m))$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_{|m-2|}$, the restriction to orientation preserving homeomorphisms is no restriction at all when p and m are both odd, as in this case $\mathbb{Z}_2$ is not a quotient of G. In fact, slightly more is true.

Lemma 2.2. *Let X be any oriented manifold and let*

$$\Psi : G(p,q,m) \to Homeo(X)$$

be any homomorphism.

(1) *If m is odd, then $\Psi(a), \Psi(b) \in Homeo^+(X)$.*

(2) *If p is odd, then $\Psi(t) \in Homeo^+(X)$.*

Proof. Note we first that $\big[Homeo(X) : Homeo^+(X)\big] \leq 2$. So in particular, $[Homeo(X), Homeo(X)] \subseteq Homeo^+(X)$, and any even power of any homeomorphism is in $Homeo^+(X)$. Now these two facts together with relation (R1) imply that when m is odd, we have $\Psi(b) \in Homeo^+(X)$. But relation (R2) guarantees that $\Psi(b) \in Homeo^+(X)$ if and only if $\Psi(a) \in Homeo^+(X)$, establishing the first claim of the lemma. Now if p is odd, we have $\Psi(t) \in Homeo^+(X)$ if and only if $\Psi(t^p) \in Homeo^+(X)$. So relation (R3) and the fact that $[a, b] \in Homeo^+(X)$ imply the second claim. □

Thus Theorem 2.1 gives the following result.

Corollary 2.3. *Suppose m, p, q are integers satisfying $m < -2$, $p \geq q \geq 1$, and $(p, q) = 1$. Suppose further that both m and p are odd. Then the image of any homomorphism $\Phi : G(p, q, m) \to Homeo(T)$ has a global fixed point.*

Theorem A. *There exist infinitely many closed hyperbolic 3-manifolds which do not contain a Reebless foliation.*

Proof. As noted in the introduction, if M contains a Reebless foliation, then $\pi_1(M)$ acts nontrivially on a simply-connected 1-manifold [HR57, Pa78]. A simple proof is as follows. Since M is hyperbolic, it is taut [Go75]. In particular, there is a homotopically nontrivial simple closed curve transverse to every leaf. This implies that the orbit of every leaf in the universal cover contains a pair of leaves which are joined in T by an embedded interval; in particular, the orbit contains separated leaves. So the action is nontrivial.

Theorem A therefore follows immediately from Corollary 2.3 as soon as we show that the set

$$\mathcal{M} = \left\{ \hat{M}_\phi(\frac{p}{q}) | m < -2 \text{ is odd, } p \text{ is odd}, (p, q) = 1, \frac{p}{q} \geq 1 \right\}$$

contains infinitely many distinct hyperbolic 3-manifolds. As we noted above, Thurston's Hyperbolic Dehn Surgery Theorem [Th82] guarantees that for any given m, $\hat{M}_\phi(\rho)$ is hyperbolic for all except possibly finitely many $\frac{p}{q}$. Furthermore, since

$$H_1(\hat{M}_\phi(\rho)) = \mathbb{Z}/p \oplus \mathbb{Z}/(m-2),$$

it follows that the set $\mathcal{M}$ is infinite. □

Theorem 5.10. *Any nontrivial orientation preserving action on an oriented $\mathbb{R}$-order tree T_0 canonically induces a nontrivial orientation preserving action on a related oriented simply-connected 1-manifold X.*

Proof. This is proved in Section 5. □

From Corollary 2.3, we therefore also obtain

Theorem B. *There exist infinitely many closed hyperbolic 3-manifolds which do not contain a transversely oriented essential lamination.*

Furthermore, infinitely many of the manifolds in $\mathcal{M}$ do contain essential laminations: the essential laminations Λ^s and Λ^u.

Theorem C. *There exist infinitely many closed hyperbolic 3-manifolds which contain neither a Reebless foliation nor a transversely oriented essential lamination, but which do contain essential laminations.*

On the other hand, when $|p-2q|=1$ and $m<-2$, there are no known essential laminations in $\hat{M}_\phi(\frac{p}{q})$. In fact, restricting to these cases and adding the condition that m be odd, we conjecture that there exist no nontrivial actions of such $\pi_1(\hat{M}_\phi(\frac{p}{q}))$ on $\mathbb{R}$-order trees, and hence that there are infinitely many hyperbolic 3-manifolds which contain no essential lamination. As noted in the introduction, Fenley has announced a proof of this conjecture, without the condition that m be odd, for $m<-3$.

We also turn our attention to $\mathbb{R}$-covered foliations. When $T=\mathbb{R}$, it is possible to make slightly stronger statements.

Proposition 3.1. *If $m<0$ and $p\geq q\geq 1$, $(p,q)=1$, then the image of any homomorphism $\Phi: G(p,q,m)\to Homeo^+(\mathbb{R})$ is trivial.*

Corollary 2.4. *If $m<0$ and $p\geq q\geq 1$, $(p,q)=1$, then $\hat{M}_\phi(\frac{p}{q})$ contains no transversely oriented $\mathbb{R}$-covered foliations.*

Corollary 3.2. *If $m<0$ and $p\geq q\geq 1$, $(p,q)=1$, and m,p are both odd, then the image of any homomorphism $\Phi: G(p,q,m)\to Homeo(\mathbb{R})$ is trivial.*

Corollary 2.5. *If $m<0$ and $p\geq q\geq 1$, $(p,q)=1$, and both p and m are odd, then $\hat{M}_\phi(\frac{p}{q})$ contains no $\mathbb{R}$-covered foliations.*

Finally, we note that in [RSS], we examine all $\phi_\sharp\in SL_2(\mathbb{Z})$ with odd negative trace. By working with a standard normal form for $\phi_\sharp$, we obtain the conclusions of Proposition 3.1, Corollary 3.2, Corollary 2.4, and Corollary 2.5 for this larger family. We suspect that the conclusion of Theorem 2.1 is also true. So our restriction to ϕ satisfying

$$\phi_\sharp=\begin{bmatrix} m & 1\\ -1 & 0\end{bmatrix}$$

seems likely to be merely a convenience. On the other hand, we have yet really to understand the condition that m be odd. Notice that in contrast with the condition that p be odd, which is topologically necessary (since when p is even, M contains a Reebless foliation), the role of m odd is still unclear. The condition that m be odd does seem to be important in our proof of Theorem 2.1.

Question. Is the condition that m be odd necessary to rule out nontrivial group actions? If yes, is the condition that m be odd necessary to rule out existence of Reebless foliations?

3. The case $T=\mathbb{R}$

As a warm-up and for those readers primarily interested in actions on $\mathbb{R}$, we first prove

Proposition 3.1. *If $m<0$ and $p\geq q\geq 1$, $(p,q)=1$, then the image of any homomorphism $\Phi: G(p,q,m)\to Homeo^+(\mathbb{R})$ is trivial.*

As noted in Section 2, the restriction to orientation preserving homeomorphisms is no restriction at all when p and m are both odd. Thus Proposition 3.1 gives the following result.

Corollary 3.2. *If $m < 0$ and $p \geq q \geq 1$, $(p,q) = 1$, and m, p are both odd, then the image of any homomorphism $\Phi : G(p,q,m) \to Homeo(\mathbb{R})$ is trivial.*

Our proof of Proposition 3.1 is by contradiction; so we assume that there is some nontrivial homomorphism $\Phi : G(p,q,m) \to Homeo^+(\mathbb{R})$. Note that if every representation describes an action with global fixed point, then necessarily every representation is trivial, by the following argument. Fix a nontrivial homomorphism $\phi : G \to Homeo^+(\mathbb{R})$ and observe that $F := \{x \mid x\phi(\gamma) = x \text{ for each } \gamma \in G\}$ is a closed, proper subset of $\mathbb{R}$. Each component of the nonempty set $\mathbb{R} \setminus F$ is homeomorphic to $\mathbb{R}$ and is invariant under the given action. Furthermore, by construction the action on each component has no global fixed point and is orientation preserving.

So we may equivalently assume that there is some representation describing an action with no global fixed point.

Set

- $\tau := \Phi(t)$,
- $\alpha := \Phi(a)$,
- $\beta := \Phi(b)$, and
- $\gamma := \Phi([a,b])$.

Note that $\gamma = \alpha\beta\alpha^{-1}\beta^{-1}$. Using the relations which define $G(p,q,m)$, we see that we have

(A) $\tau^{-1}\alpha\tau = \alpha\beta\alpha^{m-1}$,
(B) $\tau^{-1}\beta\tau = \alpha^{-1}$,
(C) $\tau^p = \gamma^{-q}$, and
(D) $\tau\gamma = \gamma\tau$.

Condition (B) guarantees that the image of Φ is generated by both $\{\tau, \alpha\}$ and $\{\tau, \beta\}$, and the next lemma follows.

Lemma 3.3. *There is no $x \in \mathbb{R}$ which is fixed by τ and at least one of α, β.*

Lemma 3.4. *Let g, h be elements of a group G such that $gh = hg$ and such that there exist relatively prime integers p, q with $g^p = h^{-q}$. Then there is some $k \in G$ such that $g = k^q$ and $h = k^{-p}$.*

Proof. Take integers r, s with $rp + sq = 1$ and verify that $k = g^s h^{-r}$ has the desired properties. □

We conclude that there is some $\kappa \in \text{Image}(\Phi)$ such that:

(E1) $\tau = \kappa^q$,
(E2) $\gamma = \kappa^{-p}$.

Another relation which will be used repeatedly and without reference is:

(F) $\tau^{-1}\alpha\tau = \alpha\beta\alpha^{m-1} = \gamma\beta\alpha^m$.

And finally, we highlight the following elementary but key fact:

If g is orientation preserving and $x < y$, then $xg < yg$.

3.1. **A general lemma for posets.** Our proof of Proposition 3.1 involves examining the fixed point sets of κ and α. The following general lemma about actions of $G(p,q,m)$ on *partially ordered sets (posets)* will be of use not only for certain cases in this section, but also when proving Theorem 2.1 for general T. We say that a group G acts on a poset P if we have a homomorphism from G to the group

of order preserving bijections on P. Note that in this lemma, the existence of k is guaranteed by Lemma 3.4.

Lemma 3.5. *Let $G = G(p, q, m)$ act on a partially ordered set P. Assume $m < 0$ and $p \geq q \geq 1$, $(p, q) = 1$. Let $k \in G$ satisfy $k^q = t$ and $k^{-p} = [a, b]$. If some $x \in P$ satisfies either of the conditions*

(1) *$xk = x$ and x, xa are related in P, or*
(2) *$xa = x$ and x, xk are related in P,*

then x is fixed by every $g \in G$.

Proof. Say condition (1) holds, so $xk = x$. Then $xt = x\,[a, b] = x$. If $xa = x$, then x is fixed by every element of G, so assume (for contradiction) that $xa \neq x$. Replacing P with the poset P^{op} (so $y \leq_P z$ if and only if $z \leq_{P^{op}} y$) if necessary, we may (and do) assume that $xa > x$. Then

$$xba^m = x\,[a, b]\,ba^m = xt^{-1}at = xat > xt = x,$$

and since $m < 0$, we have

$$xb > xa^{-m} > x. \tag{3.1}$$

However, we have

$$xbt = xt^{-1}bt = xa^{-1} < x,$$

so

$$xb < xt^{-1} = x, \tag{3.2}$$

and equations (3.1) and (3.2) give the desired contradiction.

Now say condition (2) holds, so $xa = x$. As above, we may assume (for contradiction) that $xk > x$. Note that

$$xt^{-1}b = xa^{-1}t^{-1} = xt^{-1}.$$

Since

$$x\,[a, b] = xk^{-p} \leq xk^{-q} = xt^{-1},$$

we have

$$xt^{-1}at = x\,[a, b]\,ba^m \leq xt^{-1}ba^m = xt^{-1}a^m < xa^m = x,$$

so

$$xt^{-1}a < xt^{-1}. \tag{3.3}$$

On the other hand, we have

$$xba^{-1} = xaba^{-1} = x\,[a, b]\,b \leq xt^{-1}b = xt^{-1},$$

so

$$xt^{-1}a \geq xb > xt^{-1}b = xt^{-1}. \tag{3.4}$$

Now equations (3.3) and (3.4) give the desired contradiction. □

3.2. **Completing the proof of Proposition 3.1.** As noted above, a homomorphism from $G(p,q,m)$ to $Homeo^+(\mathbb{R})$ determines an action of $G(p,q,m)$ on the poset $\mathbb{R}$ (with the usual linear order). Since all pairs x, y of elements of $\mathbb{R}$ are related in this order, the next result follows immediately from Lemma 3.5.

Corollary 3.6. *Assume $p \geq q \geq 1$, $(p,q) = 1$, and $m < 0$. If the image of $\Phi : G(p,q,m) \to Homeo^+(\mathbb{R})$ has no global fixed point, then $Fix(\kappa) = Fix(\alpha) = \emptyset$.*

We complete the proof of Proposition 3.1 by showing that necessarily $Fix(\alpha) \neq \emptyset$.

Suppose, by way of contradiction, that $Fix(\alpha) = \emptyset$. By the intermediate value theorem, either $x\alpha < x$ for all $x \in \mathbb{R}$ or $x\alpha > x$ for all $x \in \mathbb{R}$. We may orient $\mathbb{R}$ so that $x\alpha > x$ for all $x \in \mathbb{R}$. So

$$(x\tau^{-1})\alpha\tau > (x\tau^{-1})\tau = x$$

for all $x \in \mathbb{R}$. On the other hand, $x\beta = x\tau\alpha^{-1}\tau^{-1} < x$, which implies $x\alpha\beta\alpha^{-1} < x$ for all $x \in \mathbb{R}$, and hence

$$(x\tau^{-1})\alpha\tau = x\alpha\beta\alpha^{m-1} = x(\alpha\beta\alpha^{-1})\alpha^m < x$$

for all $x \in \mathbb{R}$. Hence, $Fix(\alpha) = \emptyset$ is impossible and, necessarily, the image of Φ has a global fixed point.

4. Non-Hausdorff 1-manifolds

Let T be a (path-connected and) simply-connected 1-manifold. We will assume that T is second countable but not necessarily Hausdorff. Since T is path-connected, there is a path between any two points. In general, however, unique minimal paths do not exist. Given $x, y \in T$, we often consider instead the *geodesic spine*

$$[[x,y]] = \{z \in T | x, y \text{ lie in distinct components of } T \setminus \{z\}\} \cup \{x, y\}$$

from x to y (see p. 563, [Ba98]). Note that $[[x,y]]$ is the intersection of all paths from x to y (Theorem 3.6, [RS01]). Moreover, $[[x,y]]$ is the union of a finite number $n \geq 1$ of disjoint (possibly degenerate) closed intervals (Proposition 2.3, [Ba98])

$$[[x,y]] = [x, y_1] \cup [x_2, y_2] \cup \cdots \cup [x_n, y],$$

where y_i is not separated from x_{i+1}. (To obtain finiteness, it suffices to note that if $\rho : [0,1] \to T$ is any path from x to y, then $\rho([0,1])$ is compact and hence has a finite open cover by sets homeomorphic to $\mathbb{R}$.) Let $((x,y)), [[x,y)), ((x,y]]$ denote $[[x,y]] \setminus \{x,y\}, [[x,y]] \setminus \{y\}$ and $[[x,y]] \setminus \{x\}$, respectively. As in [Ba98], set

$$d(x,y) = n - 1.$$

Only in exceptional cases is d a metric. In general, it certainly might be true that $x \neq y$ but $d(x,y) = 0$. For example, if $T = \mathbb{R}$, then $d \equiv 0$. And, in general, only a modified version of the triangle inequality holds. See, for example, Figure 1. (On the other hand, replacing d by $d+1$ on $T \times T \setminus \{(x,x) : x \in T\}$ does yield a metric on T.)

Lemma 4.1. *Let $x, y, z \in T$. Then*

- [*Proposition 2.5*,[Ba98]]. *If $y \in [[x,z]]$, then $d(x,z) = d(x,y) + d(y,z)$.*
- $d(x,z) \leq d(x,y) + d(y,z) + 1$.

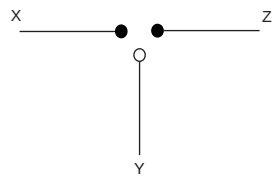

FIGURE 1. $d(x,z) = 1 > 0 = d(x,y) + d(y,z)$.

Proof. If x, y and z lie on a common geodesic spine, then $d(x,z) \leq d(x,y) + d(y,z)$. So we may assume that x, y and z satisfy one of the configurations of Figure 2, as described in Theorem 3.10, [RS01]. It is easy to verify that in the first three cases,

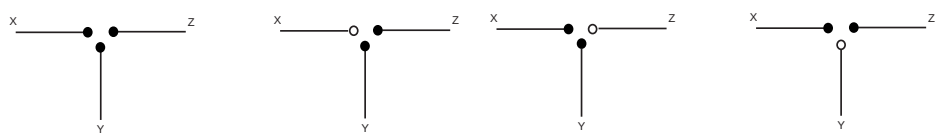

FIGURE 2.

$d(x,z) \leq d(x,y) + d(y,z)$, whereas in the fourth case, $d(x,z) \leq d(x,y) + d(y,z) + 1$ is the best possible. □

We shall call a subset X of T *spine-connected* if for all $x, y \in X$, $[[x,y]] \subset X$.

Lemma 4.2. *Suppose $X \subset T$ is spine-connected and $X \subset Y \subset \overline{X}$. Then Y is spine-connected.*

Proof. Consider distinct points $y_1, y_2 \in Y$ and let $z \in ((y_1, y_2))$. Let U_1, U_2 be neighbourhoods of y_1,y_2, respectively, which are homeomorphic to $\mathbb{R}$ and which lie in $T \setminus \{z\}$. Since U_1 and U_2 are separated by z with $U_1 \cap X \neq \emptyset$ and $U_2 \cap X \neq \emptyset$, necessarily $z \in X$. Hence, $[[y_1, y_2]] \subset Y$. □

Since T is simply-connected, it is orientable. Orient T by choosing either one of the two possible orientations.

Definition 4.3. (See Section 2.1, [Ba98].) For $x \in T$, let I_x be an open set in T containing x which is homeomorphic (as an oriented manifold) to $\mathbb{R}$ (such an I_x exists since T is an oriented 1-manifold). Let I_x^+ be the set of elements of $I_x \setminus \{x\}$ which can be reached from x by walking in the positive direction according to the orientation on T; let $I_x^- = (I_x \setminus I_x^+) \setminus \{x\}$. Since T is simply-connected, $T \setminus \{x\}$ has at least two connected components, and since T is a 1-manifold, $T \setminus \{x\}$ has exactly two connected components. Let x^+ be the component containing I_x^+ and let x^- be the component containing I_x^-.

We now define a partial order $\leq$ on T.

Definition 4.4. For $x, y \in T$, we say that

$$x \leq y \Longleftrightarrow x^+ \supseteq y^+.$$

Note that for distinct elements $x, y \in T$, $y^+ \subseteq x^+$ if and only if both $y \in x^+$ and $x \in y^-$. A straightforward induction on $d(x,y)$ therefore yields the following:

Lemma 4.5. *For all $x, y \in T$, we have:*

- $x^+ \supseteq y^+$ *if and only if* $x^- \subseteq y^-$.
- x, y *are comparable with respect to* $\leq$ *if and only if* $d(x, y)$ *is even.*

Define a relation $\sim$ on T by

- $x \sim y$ if and only if x and y are not separated in T.

Set

$$[x] = \{y \in T | y \sim x\}.$$

If $x \sim y$, let $T_{\{x,y\}}$ denote the submanifold defined as follows:

- if $x \in y^+$ (equivalently, if $y \in x^+$), set $T_{\{x,y\}} = \bigcap_{z \sim x \text{ and } z \sim y} z^+$, and
- if $x \in y^-$ (equivalently, if $y \in x^-$), set $T_{\{x,y\}} = \bigcap_{z \sim x \text{ and } z \sim y} z^-$.

The relation $\sim$ is reflexive and symmetric, but not necessarily transitive. However, since T has countable basis, there are at most countably many points b satisfying $a \sim b$ and $b \sim c$ but $a \nsim c$. (For an example of such a point b, see Figure 3. For a precise description of this phenomenon, see the appendix or [Ba98].) Hence, by

FIGURE 3. Denjoy blow-up at b.

blowing up these countably many points to closed nondegenerate intervals in the spirit of Denjoy [De32, Sc74] (see the appendix for details), we obtain a related simply-connected 1-manifold T' on which the relation $\sim$ is transitive, and hence an equivalence relation. Note that all trees T' obtained in this way are homeomorphic.

Definition 4.6. Let T be a simply-connected 1-manifold. Form the quotient space

$$T_H = T'/\sim,$$

where T' is a 1-manifold obtained from T as above. Since T' is uniquely determined up to homeomorphism, so is T_H. Call T_H the *Hausdorff tree associated to* T. Let $p : T \to T_H$ denote the corresponding quotient map.

When proving Theorem 2.1, we will often examine subsets of T whose images in T_H are homeomorphic to subintervals of $\mathbb{R}$. Working in T_H rather than T whenever possible allows us to avoid tedious case analyses when examining such subsets of T. For example, notice that if [[x,y]] is a geodesic spine in T, then $p([[x, y]]) \subset T_H$ is homeomorphic to a closed interval in $\mathbb{R}$. Other examples of such subsets are *bridges*, which we now define. If X, Y are disjoint, nonempty, spine-connected subsets of T, the bridge from X to Y is simply the intersection of all paths in T with one endpoint in X and the other in Y. Similarly, if X, Y are disjoint, nonempty connected subsets of T_H, then the bridge from X to Y in T_H is the intersection of all paths in T_H with one endpoint in X and the other in Y. Such a bridge in T_H is always of the form $[x, y]$ for some $x \in \overline{X}$ and $y \in \overline{Y}$. Some possible structures at $[x]$ of a bridge in T whose image in T_H is $[x, y]$ are illustrated in Figure 4. (The bridge near X is represented schematically by the connected vertical segment and X is represented schematically by the horizontal segments.)

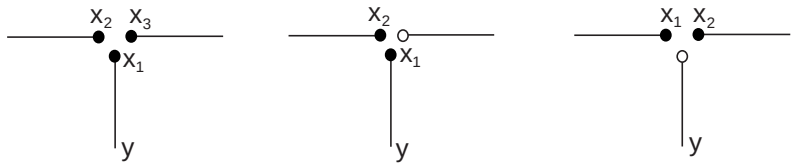

FIGURE 4.

Now suppose that G is any group acting on T. Let $g \in G$. Note that if g is orientation preserving (reversing), it preserves (reverses, respectively) the partial order $\leq$ on T. As usual, write

$$Fix(g) = \{x \in T | xg = x\}.$$

Write

$$Nonsep(g) = \{x \in T | xg \sim x\},$$

for the set of points not separated by g. We shall say that $x \in T$ is a *global fixed point* for the action of G on T if $xg \sim x$ for all $g \in G$. If there exists a global fixed point, we call the action *trivial*. By extending linearly over the blown-up intervals, any action of G on T induces an action of G on T'. Moreover, the action of G on T is trivial if and only if the induced action of G on T' is trivial.

Without loss of generality therefore, and with gain an increased simplicity of exposition, we make the following assumption throughout the rest of the paper: $\sim$ *is transitive on* T.

Define the *characteristic set* associated to g by

$$C_g = \{x \in T | d(x, xg) \text{ is even}\}.$$

Note that in [Ba98], Barbot calls this set the *fundamental axis*. We will reserve the term axis for the case $Nonsep(g) = \emptyset$.

Lemma 4.7 (See also [Ba98, Proposition 2.7(2)]). *Let $x \in T$. Then $x \in C_g$ if and only if x and xg are comparable with respect to the partial order $\leq$.*

Proof. This follows immediately from Lemma 4.5. □

Proposition 4.8 ([Ba98, Proposition 2.10]). *Suppose $Nonsep(g) = \emptyset$. Then $C_g \neq \emptyset$ and for any $x \in C_g$,*

$$C_g = \bigcup_{n \in \mathbb{Z}} [[xg^n, xg^{n+1}]].$$

For an alternate approach to the proof of Proposition 4.8, beginning with the characterization of C_g given in Corollary 4.11, see also Theorem 5.6 [RS01] or Theorem 3.4 [Fe].

Corollary 4.9. *If $Nonsep(g) = \emptyset$, then $Nonsep(g^n) = \emptyset$, for all nonzero integers n.*

Hence, when $Nonsep(g) = \emptyset$, $A_g := C_g$ is an *axis* for g in the spirit of Tits-Serre (Proposition 24, Section 6.4, [Se77]). We note in passing that any fact from the theory of group actions on $\mathbb{R}$-trees which depends only on the combinatorial properties of existence of such axes still holds true in this setting. In fact, existence merely of the characteristic set C_g for arbitrary g is sometimes (although certainly not always) sufficient for the generalization of well-known arguments. (Good surveys

on isometric actions on real trees can be found in [Ch01, Mo92, Pa95, Sh87, Sh91]. See also [CM87, CV96].)

From Proposition 4.8, it follows that in T_H, $p(A_g) \approx \mathbb{R}$, and in T, either $A_g \approx \mathbb{R}$ or $A_g = \bigcup_{-\infty}^{\infty}[x_i, y_i]$, where $[x_i, y_i]$ is homeomorphic to a closed interval in $\mathbb{R}$, $[x_i, y_i] \cap [x_j, y_j] = \emptyset$ when $i \neq j$, $x_i \neq y_i$, and $y_i \sim x_{i+1}$ for all i, j. In each case, the action of g on A_g is conjugate to an action by translations, and there is a natural linear order $\preceq_g$ on A_g satisfying $x \prec_g xg$ for all $x \in A_g$. (In general, $\preceq_g$ agrees with neither $\leq$ nor the opposite partial order $\geq$.)

Suppose Y is a g-invariant embedded copy of $\mathbb{R}$ in T on which g acts freely. Then we call Y a *local axis* for g. Note that if x lies in a local axis for g, then $d(x, xg) = 0$.

Now suppose $Nonsep(g) \neq \emptyset$ and let $T_i, i \in \mathcal{I}$, denote the path components of $T \setminus Nonsep(g)$. Notice that each T_i is path-connected and open. So $\overline{T}_i \cap T_j = \emptyset$ for all $i \neq j$ and hence $\overline{T}_i \setminus T_i \subset Nonsep(g) \subset \overline{C}_g$ for all i. In particular, $\overline{T}_i \setminus T_i = \emptyset$ only if $Nonsep(g) = \emptyset$. Notice also that $T_i g = T_j$ for some $j \in \mathcal{I}$. When $T_i g = T_j \neq T_i$, then since T is simply-connected, $\overline{T}_i \setminus T_i \neq \emptyset$ can consist of at most one, and hence exactly one, point $x \in Nonsep(g)$, and $\overline{T}_j \setminus T_j$ is the point xg, where $xg \sim x$ and $xg \neq x$ (namely, the situation pictured in Figure 5 must hold). In this case, we call

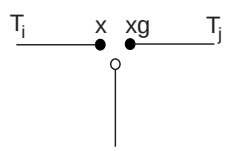

FIGURE 5.

x the *root* of T_i in T (and xg the root of T_j in T). On the other hand, whenever $T_i g = T_i$, g acts freely on T_i, and hence this local action has an axis $A_g^i \subset T_i$. Using distance to an element of $\overline{T}_i \cap Nonsep(g)$, one can check that such an A_g^i is homeomorphic to $\mathbb{R}$ and hence is an example of a local axis for g; in fact, all local axes for g arise in this way. We summarize some of these observations in the following lemma.

Lemma 4.10. *Suppose $Nonsep(g) \neq \emptyset$. Then:*

- *$Nonsep(g) \subset Nonsep(g^n)$ and hence $C_g \subset C_{g^n}$, for any n.*
- *$C_g = \{x \in T | d(x, xg) = 0\}$.*
- *$C_g = Fix(g) \cup \{x \in T | x$ lies on a local axis for $g\}$.*

Corollary 4.11. *For any $g \in G$, we have*

$$C_g = \{x \in T | x \in [[xg^{-1}, xg]]\}.$$

Proof. Suppose $x \in T$ satisfies $x \in [[xg^{-1}, xg]]$. If $d(xg^{-1}, xg) = 0$, then $d(x, xg) = 0$ and so Lemma 4.10 guarantees that $x \in C_g$. So suppose that $d(xg^{-1}, xg) > 0$. By Lemma 4.1,

$$d(xg^{-1}, xg) = d(xg^{-1}, x) + d(x, xg) = 2d(x, xg)$$

and so $x \in C_{g^2}$. By Lemma 4.10, $Nonsep(g^2) = \emptyset$, and so $C_{g^2} = A_{g^2}$. But $Nonsep(g) \subset Nonsep(g^2) = \emptyset$ and in particular, $x \in C_{g^2} = A_{g^2} = A_g = C_g$.

The reverse inclusion follows immediately from Proposition 4.8 and Lemma 4.10. $\square$

Corollary 4.12. *If there is some $x \in T$ such that $d(x, xg) \neq 0$ is even, then $Nonsep(g) = \emptyset$.*

Corollary 4.13. *Let $g \in G$. Then both C_g and $C_g \cup Nonsep(g)$ are spine-connected.*

Proof. If $Nonsep(g) = \emptyset$, then $C_g \cup Nonsep(g) = A_g$ is spine-connected. So we may assume $Nonsep(g) \neq \emptyset$. By Lemma 4.2, it suffices to prove that C_g is spine-connected.

Consider first the special case that $x \in Fix(g)$ and $y \in C_g$. Since $d(x, y) = d(x, yg)$ and $d(y, yg) = 0$, necessarily x, y and yg are collinear in T (i.e., lie on a common geodesic spine). If g is orientation reversing, then $[[x, y]] \subset [[y, yg]] = [y, yg]$. And if g is orientation preserving, then either $yg \in [[x, y]]$ or $y \in [[x, y]]g$. So either $[[x, y]]g \subset [[x, y]]$ or $[[x, y]]g^{-1} \subset [[x, y]]$, and hence, for every $z \in [[x, y]]$, $d(z, zg) = 0$. In either case, $[[x, y]] \subset C_g$.

Consider next the special case that $x, y \in C_g \setminus Fix(g)$. If x and y lie in a common component of $T \setminus Nonsep(g)$, then necessarily x and y lie on a common local axis, and hence $[[x, y]] = [x, y]$ is also contained in this common local axis (and hence in C_g). Otherwise, x and y are separated by some $z \in Nonsep(g)$. Now $z \in Fix(g)$ since otherwise $d(x, xg)$ and $d(y, yg)$ are odd. So $[[x, y]] = [[x, z]] \cup [[z, y]]$ is contained in C_g by the first special case. □

It will sometimes be useful to consider an object obtained by adding one point $\hat{x}$, called an *ideal point* of T, to T for each $\sim$-equivalence class $[x]$ in T which contains more than one point. This object, denoted by $\hat{T}$, is called the *completion* of T. (Compare with Section 5 of [RS01].) We say that an ideal point $\hat{x}$ is a *source* if whenever y, z are distinct elements of $[x]$ we have $y \in z^-$, and we say that $\hat{x}$ is a *sink* if whenever y, z are distinct elements of $[x]$ we have $y \in z^+$. Note that every ideal point $\hat{x}$ is either a source or a sink. The action of any subgroup of $Homeo(T)$ extends to an action on $\hat{T}$ in the obvious way, that is, we set $\hat{x}g = \hat{y}$ if $[x]g = [y]$. We want to extend our partial order on T to $\hat{T}$ so that group actions on $\hat{T}$ obtained from orientation preserving actions on T preserve this extended partial order. For an ideal point $\hat{x}$, we define

$$\hat{x}^+ = \begin{cases} \bigcup_{y \in [x]} (\{y\} \cup y^+), & \hat{x} \text{ a source,} \\ \bigcap_{y \in [x]} y^+, & \hat{x} \text{ a sink,} \end{cases}$$

and set

$$\hat{x}^- = T \setminus \hat{x}^+.$$

Note that $\hat{x}^+, \hat{x}^- \subseteq T$. It is straightforward to show that if H is any group of orientation preserving homeomorphisms of T, then for $x, y \in \hat{T}$ and $h \in H$, we have $x^+ \subseteq y^+$ if and only if $(xh)^+ \subseteq (yh)^+$. So, we have the following result, which will allow us to invoke Lemma 3.5 more often in the proof of Theorem 2.1 than would be possible without the introduction of $\hat{T}$.

Proposition 4.14. *Define the relation $\leq$ on $\hat{T}$ by $x \leq y$ if $y^+ \subseteq x^+$. This relation is a partial order which extends the order $\leq$ on T defined in Definition 4.4. In addition, if $H \leq Homeo^+(T)$, then the induced action of H on $\hat{T}$ is order preserving.*

5. $\mathbb{R}$-order trees

Both simply-connected 1-manifolds and their associated Hausdorff trees are special cases of a more general tree-like object, the $\mathbb{R}$-order tree. An *order tree* T [GO89] is a set T together with a collection $\mathcal{S}$ of linearly ordered subsets called *segments*. If σ is a segment, then $-\sigma$ denotes the same subset with reverse order. The segments satisfy :

(1) Each segment σ has distinct least and greatest elements, which we will denote by $i(\sigma)$ and $f(\sigma)$, respectively. (We also write $\sigma = [i(\sigma), f(\sigma)]$.)
(2) If σ is a segment, so is $-\sigma$.
(3) A closed nondegenerate (i.e., containing more than one element) subinterval of a segment is a segment.
(4) Given $x, y \in T$, there exists a path from x to y; namely, a sequence $\sigma_1, \ldots, \sigma_k$ of segments such that $i(\sigma_1) = x$, $f(\sigma_k) = y$, and $f(\sigma_j) = i(\sigma_{j+1})$ for all j.
(5) Given a cyclic path $\sigma_0\sigma_1 \cdots \sigma_{k-1}$ (where cyclic means $f(\sigma_{k-1}) = i(\sigma_0)$), there is a subdivision of the path $\sigma_0\sigma_1 \cdots \sigma_{k-1}$ to a path $\rho_0 \cdots \rho_{n-1}$ so that after cancelling all adjacent pairs of the form $(\rho)(-\rho)$, we have the empty sequence.
(6) If $f(\sigma_1) = i(\sigma_2) = \sigma_1 \cap \sigma_2$, then $\sigma_1 \cup \sigma_2$ is a segment.

An $\mathbb{R}$-*order tree* [GK97] is an order tree satisfying also:

(7) Each segment is order isomorphic to a closed interval in $\mathbb{R}$.
(8) T is a countable union of segments.

T is topologized by giving segments the order topology and then declaring a set $U \subset T$ to be open in T if and only if $U \cap \sigma$ is open in σ for every segment σ. Note that defining axiom (4) guarantees that T is path-connected and that defining axiom (5) guarantees that T is simply-connected.

An *orientation* of an order tree is a choice of subset $\mathcal{S}_+ \subset \mathcal{S}$ such that

- $\mathcal{S}_+ \cap (-\mathcal{S}_+) = \emptyset$, where $-\mathcal{S}_+ = \{-\sigma | \sigma \in \mathcal{S}_+\}$.
- A closed nondegenerate subinterval of a segment in $\mathcal{S}_+$ is in $\mathcal{S}_+$.
- Any two elements of T can be joined by a sequence $\sigma_1, \ldots, \sigma_k$ of segments in $\mathcal{S}_+ \cup (-\mathcal{S}_+)$ such that $f(\sigma_j) = i(\sigma_{j+1})$ for all j.
- If $\sigma_1, \sigma_2 \in \mathcal{S}_+$, and $f(\sigma_1) = i(\sigma_2) = \sigma_1 \cap \sigma_2$, then $\sigma_1 \cup \sigma_2 \in \mathcal{S}_+$.

Since there are no nontrivial cyclic words, orientations always exist. In contrast to the situation when T is a simply-connected 1-manifold and therefore has exactly two orientations, there are generally many possible choices of orientation for an $\mathbb{R}$-order tree. Note that if $\mathcal{S}_0$ is a collection of linearly ordered subsets of T such that $\mathcal{S}_0 \cup -\mathcal{S}_0$ satisfies conditions (1), (4) and (5) of the definition of order tree, then there is a unique smallest set $\mathcal{S}$ containing $\mathcal{S}_0$ and also satisfying all six defining conditions.

As we will discuss further in Section 6, if Λ is an essential lamination in M with no isolated leaves, then its lift to the universal cover of M has leaf space an $\mathbb{R}$-order tree [GO89]. If Λ is a transversely oriented essential lamination with no isolated leaves in M, then its lift to the universal cover of M has leaf space an oriented $\mathbb{R}$-order tree.

Now let T be any simply-connected 1-manifold for which $\sim$ is transitive. Let T_H denote the associated Hausdorff tree. It is easy to see that the (oriented) 1-manifold structure on T induces canonical (oriented) $\mathbb{R}$-order tree structures on T and T_H,

respectively. Choose either of the two orientations of T. Let $\mathcal{S}_+ = \{\sigma \subset T | \sigma$ is homeomorphic to a nondegenerate closed interval in $\mathbb{R}$, with linear order inherited from the orientation of $T\}$, and let $\mathcal{S}$ be the smallest set containing $\mathcal{S}_+$ and also satisfying the axioms defining an order tree. Then T together with $\mathcal{S}$ is an $\mathbb{R}$-order tree, and $\mathcal{S}_+$ is an order tree orientation for T. Next, let $(\mathcal{S}_H)_+ = \{p(\sigma) | \sigma \in \mathcal{S}_+\}$, and let $\mathcal{S}_H$ be the smallest set containing $(\mathcal{S}_H)_+$ and also satisfying the axioms defining an order tree. Then T_H together with $\mathcal{S}_H$ is an $\mathbb{R}$-order tree, and $(\mathcal{S}_H)_+$ is an order tree orientation for T_H. (The first two defining conditions of order tree orientation are clearly satisfied. The third condition follows from the finiteness of $d(x, y)$ for every pair $x, y \in T$; namely, from Proposition 2.3 of [Ba98]. The fourth condition follows from the fact that since T is an oriented 1-manifold, it is not possible to find distinct $x, y \in T$ and $\sigma_1, \sigma_2 \in \mathcal{S}_+$, such that $f(\sigma_1) = x \sim y = i(\sigma_2)$.) So any orientation of T projects to an orientation of T_H as an $\mathbb{R}$-order tree. In contrast, not all order tree orientations of T_H lift to orientations of T.

Now let T be any order tree. A function $\phi : T \to T$ is called an *(order tree) automorphism* if ϕ is a set bijection satisfying $[x, y]\phi = [x\phi, y\phi] \in \mathcal{S} \iff [x, y] \in \mathcal{S}$. We say that ϕ is orientation preserving if $\sigma\phi \in \mathcal{S}_+ \iff \sigma \in \mathcal{S}_+$. Set $Aut(T) = \{\phi : T \to T | \phi$ is an order tree automorphism$\}$, and set $Aut^+(T) = \{\phi \in Aut(T) | \phi$ is orientation preserving$\}$. Let G be any group. A right action of G on T *as an order tree* is a mapping

$$T \times G \to T : (x, g) \mapsto xg = x\Phi(g),$$

for some homomorphism

$$\Phi : G \to Aut(T).$$

An orientation preserving action is an action satisfying $\Phi(G) \subset Aut^+(T)$. Now consider the special case that T is a simply-connected 1-manifold with canonically induced $\mathbb{R}$-order tree structure. Then $\phi \in Aut(T) \iff \phi \in Homeo(T)$. So an action of G on T as a 1-manifold is also an action of G on T as an $\mathbb{R}$-order tree, and an action of G on T as an $\mathbb{R}$-order tree (still with canonically induced $\mathbb{R}$-order tree structure) is also an action of G on T as a 1-manifold.

Many of the properties of simply-connected 1-manifolds hold true for or generalize to $\mathbb{R}$-order trees. (In particular, the notation used in this section for $\mathbb{R}$-order trees T is consistent with the notation used in Section 4 in the special case that T is a simply-connected 1-manifold.)

Given $x, y \in T$, we again consider the *geodesic spine*

$$[[x, y]] = \{z \in T | x, y \text{ lie in distinct components of } T \setminus \{z\}\} \cup \{x, y\}$$

from x to y. Again, $[[x, y]]$ is the intersection of all paths from x to y (Theorem 3.6, [RS01]). Moreover, $[[x, y]]$ is the union of a finite number $n \geq 1$ of disjoint (possibly degenerate) closed intervals (Axiom (4))

$$[[x, y]] = [x, y_1] \cup [x_2, y_2] \cup \cdots \cup [x_n, y],$$

where y_i is not separated from x_{i+1}. Let $((x, y)), [[x, y)), ((x, y]]$ denote $[[x, y]] \setminus \{x, y\}, [[x, y]] \setminus \{y\}$ and $[[x, y]] \setminus \{x\}$, respectively. Set

$$d(x, y) = n - 1,$$

and note that only a modified version of the triangle inequality holds.

Lemma 5.1. *Let $x, y, z \in T$. Then*

- *If $y \in [[x, z]]$, then $d(x, z) = d(x, y) + d(y, z)$.*

- $d(x,z) \leq d(x,y) + d(y,z) + 1$.

Again, we call a subset X of T *spine-connected* if for all $x, y \in X$, $[[x,y]] \subset X$. Define a relation $\sim$ on T by

- $x \sim y$ if and only if x and y are not separated in T.

Set

$$[x] = \{y \in T | y \sim x\}.$$

If $x \sim y$, let $T_{\{x,y\}}$ denote the component of $T \setminus [x]$ which has both x and y as limit points. Note that alternatively, $\sim$ can be defined in terms of segments as follows:

$$x \sim y$$

if and only if $\sigma_1 \cap \sigma_2 \neq \emptyset$ for every pair $\sigma_1, \sigma_2 \in \mathcal{S}_+$ such that $x \in \sigma_1 \setminus \{i(\sigma_1), f(\sigma_1)\}$ and $y \in \sigma_2 \setminus \{i(\sigma_2), f(\sigma_2)\}$. The relation $\sim$ is reflexive and symmetric, but not necessarily transitive. However, as described in the appendix, there is a naturally associated $\mathbb{R}$-order tree T' on which $\sim$ is transitive and hence an equivalence relation.

The notions of characteristic set, axis and local axis also generalize to the case of order trees. Define the characteristic set of g to be

$$C_g = \{x \in T | x \in [[xg^{-1}, xg]]\}.$$

Again we note that any fact from the theory of group actions on $\mathbb{R}$-trees which depends only on the combinatorial properties of existence of such characteristic sets still holds true in this setting.

Proposition 5.2 ([RS01, Theorem 5.6])**.** *Suppose $Nonsep(g) = \emptyset$. Then $C_g \neq \emptyset$, with*

$$C_g = \bigcup_i [[xg^{i-1}, xg^i]]$$

for any $x \in C_g$.

Hence, when $Nonsep(g) = \emptyset$, it again makes sense (in the sense of Tits-Serre) to call $A_g := C_g$ an axis for g.

And again, by local axis for g we mean either an axis for g or, when $Nonsep(g) \neq \emptyset$, any subset of T order isomorphic to $\mathbb{R}$ on which g acts freely.

Lemma 5.3. *Suppose $Nonsep(g) \neq \emptyset$. Then*

- *$Nonsep(g) \subset Nonsep(g^n)$ and hence $C_g \subset C_{g^n}$, for any n.*
- *$C_g = Fix(g) \cup \{x \in T | x$ lies on a local axis for $g\}$.*

Corollary 5.4. *Let $g \in G$. Then both C_g and $C_g \cup Nonsep(g)$ are spine-connected.*

The proof of Theorem 5.6 of [RS01] also yields the following.

Proposition 5.5. *Let T be any $\mathbb{R}$-order tree. Let $x \in T$, $g \in Homeo(T)$ with $xg \neq x$. Suppose further that $[[xg^{-1}, x]] \cap [[x, xg]] = \{x\}$. For $n \in \mathbb{Z}$ set $I_n = [[xg^n, xg^{n+1}]]$. Then*

(1) *If $j, k \in \mathbb{Z}$ with $j < k$, then*

$$I_j \cap I_k = \begin{cases} \{xg^k\}, & k = j+1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

(2) *$\bigcup_{n \in \mathbb{Z}} I_n$ is a local axis for g.*

Now let T be any oriented $\mathbb{R}$-order tree and let T_H denote its associated (oriented) Hausdorff tree (Definition 9.3).

Lemma 5.6. *Any nontrivial action of G on T canonically induces a nontrivial action of G on T_H.*

Proof. If $x \sim y$, then $xg \sim yg$ for any $g \in G$. So $[x]g = [xg]$ defines a homeomorphism of T_H. This induced action is trivial if and only if the action of G on T is trivial. □

Note that with respect to this induced action, $Fix_{T_H}(g) = p(Nonsep(g))$.

Corollary 5.7. *If G acts nontrivially on an $\mathbb{R}$-order tree, then G acts nontrivially on a Hausdorff $\mathbb{R}$-order tree.*

Lemma 5.8. *Suppose T_H has orientation inherited from T. Then any orientation preserving action on T_H canonically induces an orientation preserving action on T. This induced action on T is nontrivial if and only if the given action of G on T_H is nontrivial.*

Proof. Let $\mathcal{C} = \{x \in T | |[x]| > 1\}$. Since $p : T \to T_H$ restricted to $T \setminus \mathcal{C}$ is injective (and $\mathcal{C}$ is G-invariant), we may set

$$xg = p^{-1}([x]g)$$

for all $x \in T \setminus \mathcal{C}$. Now consider any $z \in \mathcal{C}$. Since $\mathcal{C}$ is countable, $z \in \sigma$ for some segment $\sigma = [x, y]$ with $x, y \in T \setminus \mathcal{C}$, and so we may set

$$zg = p^{-1}([z]g) \cap [xg, yg].$$

□

Next we introduce a notion of *incidence* for order trees. Fix an orientation on T and let $x \in T$. Define an equivalence relation $\approx_f$ on the set $S(x, f) = \{\sigma \in \mathcal{S}_+ | f(\sigma) = x\}$ by $\sigma_1 \approx_f \sigma_2$ if and only if both $f(\sigma_1) = f(\sigma_2) = x$ and $\{x\} \subsetneq \sigma_1 \cap \sigma_2$. For each $\sigma \in S(x, f)$, let $r_\sigma = \{\tau \in S(x, f) | \tau \approx_f \sigma\}$ and call r_σ an *incoming ray* at x. Let $R(x, f) = \{r_\sigma | \sigma \in S(x, f)\}$. Call $n_f(x) = |R(x, f)|$ the *in-degree* at x. Similarly, define an equivalence relation $\approx_o$ on the set $S(x, o) = \{\sigma \in \mathcal{S}_+ | i(\sigma) = x\}$ by $\sigma_1 \approx_o \sigma_2$ if and only if both $i(\sigma_1) = i(\sigma_2) = x$ and $\{x\} \subsetneq \sigma_1 \cap \sigma_2$. For each $\sigma \in S(x, o)$, let $r_\sigma = \{\tau \in S(x, o) | \tau \approx_o \sigma\}$ and call r_σ an *outgoing ray* at x. Let $R(x, o) = \{r_\sigma | \sigma \in S(x, o)\}$. Call $n_o(x) = |R(x, o)|$ the *out-degree* at x. We say that a segment σ is incident to x if $\sigma \in S(x, o) \cup S(x, f)$, and we say that a ray r_σ is incident to x if $r_\sigma \in R(x, o) \cup R(x, f)$. Call $x \in T$ *regular* if $n_o(x) = n_f(x) = 1$. Call $x \in T$ a *branch point* if it is not regular, and let $\mathcal{B}$ denote the set of branch points of T. Note that if $\mathcal{B} = \emptyset$, then T can also be given the structure of a simply-connected 1-manifold.

Now consider any $x \in \mathcal{B}$. If the out-degree $n_o(x) = 0$ (in-degree $n_f(x) = 0$), call x a *sink* (respectively, *source*). If $n_o(x) = 1$ and $n_f(x) > 1$, call the single element $r_\sigma \in R(x, o)$ the *distinguished ray* at x. Symmetrically, if $n_f(x) = 1$ and $n_o(x) > 1$, call the single element $r_\sigma \in R(x, f)$ the *distinguished ray* at x.

Lemma 5.9. *Let T_0 be an oriented $\mathbb{R}$-order tree such that at every $x \in \mathcal{B}$, there is a distinguished ray. Then any nontrivial orientation preserving action on T_0 canonically induces a nontrivial orientation preserving action on a related oriented simply-connected 1-manifold X.*

Proof. It suffices to describe X as an $\mathbb{R}$-order tree with no branch points.

By assumption, every branch point $x \in \mathcal{B}$ has a distinguished ray $\hat{r}_x$. Construct X from $T_0 \setminus \mathcal{B}$ by adding in place of each $x \in \mathcal{B}$ a family of points $\{x_{r_\sigma}\}$, where r_σ ranges over all rays in $R(x, f)$ if $\hat{r}_x$ is outgoing and ranges over all rays in $R(x, o)$ if $\hat{r}_x$ is ingoing. We describe an $\mathbb{R}$-order tree structure on X as follows. Let σ be any segment in the orientation $(\mathcal{S}_0)_+$ of T_0. Let $x \in \sigma \cap \mathcal{B}$. Either $x \in \{i(\sigma), f(\sigma)\}$ or $x \in \sigma \setminus \{i(\sigma), f(\sigma)\}$. If $x \in \{i(\sigma), f(\sigma)\}$, and σ does not represent the distinguished ray at x, then set $x' = x_{r_\sigma}$. If $x \in \sigma \setminus \{i(\sigma), f(\sigma)\}$, let τ be a nondegenerate subinterval of σ representing a ray incident to x such that $r_\tau \neq \hat{r}_x$, and set $x' = x_{r_\tau}$. Now let X'_σ denote the set of points x' thus defined. Set

$$\sigma' = (\sigma \setminus \mathcal{B}) \cup X'_\sigma,$$

with linear order induced in the natural way from the linear order on σ. Let

$$\mathcal{S}_+ = \{\sigma' | \sigma \in (\mathcal{S}_0)_+\}$$

and let S be the smallest set containing $\mathcal{S}_+$ and also satisfying the axioms defining an order tree. Then $(X, \mathcal{S})$ is an $\mathbb{R}$-order tree, and $\mathcal{S}_+$ is an orientation of X.

Extend the action of G on $T \setminus \mathcal{B}$ to an action on X by setting $x_{r_\sigma} g = x_{r_{\sigma g}}$. □

Proposition 5.10. *Any nontrivial orientation preserving action on an oriented $\mathbb{R}$-order tree T_0 canonically induces a nontrivial orientation preserving action on a related oriented simply-connected 1-manifold X.*

Proof. We show that any nontrivial orientation preserving action on an oriented $\mathbb{R}$-order tree T_0 canonically induces a nontrivial orientation preserving action on an oriented $\mathbb{R}$-order tree T such that at every $x \in \mathcal{B}$, there is a distinguished ray. Lemma 5.9 then applies.

First, let $\mathcal{D}$ denote the set of branch points $x \in T_0$ with both in-degree and out-degree greater than one. Let T denote the linear Denjoy blow-up of T_0 along $\mathcal{D}$ with respect to the orientation on T_0 and extend the action of G to T as described in Section 9.3. Let $\mathcal{B}$ now denote the branch points of T. Note that if $x \in \mathcal{B}$, then either x has a distinguished ray or else it is either a sink or a source.

Finally, we introduce distinguished rays at all sinks and sources in $\mathcal{B}$. At every sink $x \in \mathcal{B}$, attach a set σ_x order isomorphic to $[0, \infty)$ (so that precisely the endpoint of σ_x is identified with x). Symmetrically, at every source $x \in \mathcal{B}$, attach a set σ_x order isomorphic to $(-\infty, 0]$ (so that precisely the endpoint of σ_x is identified with x). Let the set of segments associated to this new tree be the smallest set satisfying the defining axioms and containing $\mathcal{S}$, the segments of T, together with all nondegenerate subintervals of the σ_x. Extend the action of G linearly over the sets σ_x.

□

6. Spaces of leaves

Let M be any closed 3-manifold containing an essential lamination Λ [GO89]. By Theorem 6.1 of [GO89], the universal cover $\tilde{M}$ of M is homeomorphic to $\mathbb{R}^3$. Lift Λ to a lamination $\tilde{\Lambda}$ of $\tilde{M}$. Now define an equivalence relation $\equiv$ on $\tilde{M}$ by

$$x \equiv y$$

if and only if either x, y lie on a common leaf of $\tilde{\Lambda}$ or x, y both lie in the union of some complementary region with its boundary leaves.

$$T_{\tilde{\Lambda}} = \tilde{M}/\equiv$$

is called the *leaf space* of $\tilde{\Lambda}$. Remark that when Λ is not a foliation and therefore has complementary regions, $T_{\tilde{\Lambda}}$ is not really the "space of leaves" but rather a natural quotient of this space.

When Λ is a Reebless foliation, $T_{\tilde{\Lambda}}$ is a second countable but not necessarily Hausdorff simply-connected 1-manifold, and the action of $\pi_1(M)$ on $\tilde{M}$ induces a nontrivial action of $\pi_1(M)$ on $T_{\tilde{\Lambda}}$ by homeomorphisms ([HR57, Pa78]; see also [Ba98, CC]).

Proposition 6.1 ([HR57, Pa78])**.** *If M contains a Reebless foliation, then $\pi_1(M)$ acts nontrivially on a simply-connected 1-manifold.*

Corollary 6.2. *If M contains a Reebless foliation Λ and $\pi_1(M)$ contains no index two subgroup, then Λ is necessarily transversely orientable, and $\pi_1(M)$ acts nontrivially on a simply-connected 1-manifold by orientation preserving homeomorphisms.*

More generally, when Λ is an essential lamination with no isolated leaves, $T_{\tilde{\Lambda}}$ is an $\mathbb{R}$-order tree [GO89]. Roughly speaking, segments in the $\mathbb{R}$-order tree arise from a family of well-chosen transversals τ to $\tilde{\Lambda}$: if $\tilde{\Lambda}$ (equivalently, Λ) has no isolated leaves, then each $\tilde{\Lambda} \cap \tau$ is a closed perfect set, and hence, by a devil's staircase-like argument (cf. [Be99]), $\tau/\equiv$ is order isomorphic to $\mathbb{R}$. As remarked in [GK97], if M contains an essential lamination, then M contains an essential lamination with no isolated leaves (isolated leaves can simply be replaced by products as described in [Ga92]). On the leaf space level, this replacement of isolated leaves by products results in the Denjoy blow-up operation as defined in the appendix.

Now consider the action of $\pi_1(M)$ on $T_{\tilde{\Lambda}}$ induced by the action of $\pi_1(M)$ on $\tilde{M}$ by deck transformations. This action has no global fixed point. (See, for example, Proposition 8.1 of [RS01].) Furthermore, if Λ is transversely oriented, then the transverse orientation on Λ lifts to a transverse orientation on $\tilde{\Lambda}$, and hence induces an orientation on $T_{\tilde{\Lambda}}$ which is preserved by the action of $\pi_1(M)$. (Note that if Λ is an essential surface, then by passing to a double cover of Λ as necessary, we may assume that Λ is transversely oriented.)

Proposition 6.3. *If M contains an oriented essential lamination Λ, then $\pi_1(M)$ acts nontrivially by orientation preserving order tree automorphisms on an oriented $\mathbb{R}$-order tree.*

7. Case I: $Nonsep(\kappa) = \emptyset$

This section is devoted to the proof of the following special case of Theorem 2.1.

Lemma 7.1. *Let $m < -2$ and $p \geq q \geq 1$, $(p, q) = 1$, with m and p both odd. Let $\Phi : G(p, q, m) \to Homeo^+(T)$ be a homomorphism with the property that $Nonsep(\kappa) = \emptyset$. Then the image of Φ has a global fixed point.*

Proof. As noted in Section 4, we may assume that $\sim$ is an equivalence relation on T. In much of the following argument, we work in T_H. When doing so, we often abuse notation and write x for $[x]$. However, we are careful to remind the reader of this whenever we think confusion might otherwise arise.

Suppose $Nonsep(\kappa) = \emptyset$. Consider the action of $G(p, q, m)$ on the Hausdorff tree T_H. In T_H, $A_\kappa \approx \mathbb{R}$ and there are exactly three possibilities for $A_\kappa \cap A_\kappa\alpha$:

(a) $A_\kappa \cap A_\kappa\alpha = A_\kappa$.
(b) $A_\kappa \cap A_\kappa\alpha$ is a nonempty proper closed connected subset I of A_κ.
(c) $A_\kappa \cap A_\kappa\alpha = \emptyset$.

Recall that $A_\kappa\alpha = A_{\alpha^{-1}\kappa\alpha}$ and $A_\kappa\alpha^{-1} = A_{\alpha\kappa\alpha^{-1}}$.

7.1. **Case (a).** If $A_\kappa \cap A_\kappa\alpha = A_\kappa$, then $A_\kappa \approx \mathbb{R}$ is invariant under $\operatorname{Im}\Phi$ and hence there is a fixed point in T_H by Corollary 3.2. Thus, there is a global fixed point in T by Lemma 5.6.

Although unnecessary for this proof, we observe here that an element which is orientation preserving as a homeomorphism of T can induce an orientation reversing homeomorphism on a copy of $\mathbb{R}$ properly embedded in T_H.

7.2. **Case (b).** Let $\preceq$ denote the total order on A_κ specified by $x \preceq x\kappa$ for all $x \in A_\kappa$. (When $d(x, x\kappa) \neq 0$, this total order bears no resemblance to the partial order $\leq$ on T.) With respect to this total order, let r (respectively, s) denote the lower bound (respectively, upper bound), if it exists, of $A_\kappa \cap A_\kappa\alpha$. Otherwise, set

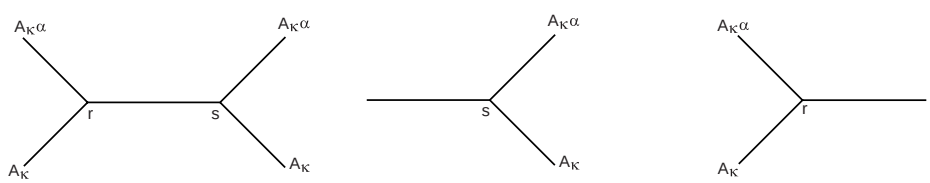

FIGURE 6.

$r = -\infty$ (respectively, $s = \infty$). Note that at least one of r and s is finite since we are in Case (b). For ease of exposition (namely, to avoid breaking into the three cases shown in Figure 6), we set $\pm\infty g = \pm\infty$ or $\pm\infty g = \mp\infty$, as necessary, for elements $g \in G(p, q, m)$.

Let $\preceq_\alpha$ denote a total order on $A_\kappa\alpha$ such that $\preceq$ and $\preceq_\alpha$ agree on $A_\kappa \cap A_\kappa\alpha$, and let $\preceq_{\alpha^{-1}}$ denote a total order on $A_\kappa\alpha^{-1}$ such that $\preceq$ and $\preceq_{\alpha^{-1}}$ agree on $A_\kappa \cap A_\kappa\alpha^{-1}$. When $r = s$, choose $\preceq_\alpha$ so that $\alpha^{-1}\kappa\alpha$ is increasing with respect to $\preceq_\alpha$ on $A_\kappa\alpha$, and choose $\preceq_{\alpha^{-1}}$ so that $\alpha\kappa\alpha^{-1}$ is increasing with respect to $\preceq_{\alpha^{-1}}$ on $A_\kappa\alpha^{-1}$. Note that $\preceq_\alpha$ and $\preceq_{\alpha^{-1}}$ are uniquely determined.

Lemma 7.2. *The following are equivalent:*

- $r\alpha^{-1} \preceq s\alpha^{-1}$ *on* A_κ.
- $\alpha^{-1}\kappa\alpha$ *is increasing with respect to* $\preceq_\alpha$ *on* $A_\kappa\alpha$.
- $\alpha\kappa\alpha^{-1}$ *is increasing with respect to* $\preceq_{\alpha^{-1}}$ *on* $A_\kappa\alpha^{-1}$.

Proof. Assume $r \neq s$. The map

$$\alpha : (A_\kappa, \prec) \to (A_\kappa\alpha, \prec_\alpha)$$

must be either order preserving or order reversing. Since $(r\alpha^{-1})\alpha = r \prec_\alpha s = (s\alpha^{-1})\alpha$, we see that α is order preserving if $r\alpha^{-1} \prec s\alpha^{-1}$ and order reversing if $s\alpha^{-1} \prec r\alpha^{-1}$. Since $r\alpha^{-1}, s\alpha^{-1} \in A_\kappa \cap A_\kappa\alpha^{-1}$, we have

$$r\alpha^{-1} \prec_{\alpha^{-1}} s\alpha^{-1} \iff r\alpha^{-1} \prec s\alpha^{-1}$$

(by definition of $\prec_{\alpha^{-1}}$). Therefore

$$\alpha^{-1} : (A_\kappa, \prec) \to (A_\kappa\alpha^{-1}, \prec_{\alpha^{-1}})$$

is order preserving if $r\alpha^{-1} \prec s\alpha^{-1}$ and order reversing if $s\alpha^{-1} \prec r\alpha^{-1}$.

Suppose $r\alpha^{-1} \prec s\alpha^{-1}$. Since $x \prec x\kappa$ for all $x \in A_\kappa$, we have $x\alpha^{-1} \prec_{\alpha^{-1}} x\kappa\alpha^{-1} = x\alpha^{-1}(\alpha\kappa\alpha^{-1})$ and $x\alpha \prec_\alpha x\kappa\alpha = x\alpha(\alpha^{-1}\kappa\alpha)$ for all $x \in A_\kappa$.

Symmetrically, if $s\alpha^{-1} \succ r\alpha^{-1}$, we have $x\alpha^{-1} \succ_{\alpha^{-1}} x\kappa\alpha^{-1} = x\alpha^{-1}(\alpha\kappa\alpha^{-1})$ and $x\alpha \succ_\alpha x\kappa\alpha = x\alpha(\alpha^{-1}\kappa\alpha)$.

□

Now note that by substituting $\beta = \tau\alpha^{-1}\tau^{-1}$ into $\alpha\beta\alpha^{-1}\beta^{-1} = \kappa^{-p}$, we obtain

$$(\alpha\tau\alpha^{-1})\tau^{-1} = \kappa^{-p+q}(\alpha^{-1}\tau^{-1}\alpha). \tag{7.1}$$

Let ω denote the element represented by the two words in (7.1). Using the axes A_κ and $A_{\alpha^{-1}\kappa\alpha}$, we will derive information about the translate

$$A_\kappa\omega = A_\kappa\kappa^{-p+q}(\alpha^{-1}\tau^{-1}\alpha) = A_\kappa(\alpha^{-1}\tau^{-1}\alpha).$$

Then, using instead the axes A_κ and $A_{\alpha\kappa\alpha^{-1}}$, we will derive information about the translate

$$A_\kappa\omega = A_\kappa(\alpha\tau\alpha^{-1})\tau^{-1}.$$

Happily, contradictions are plentiful.

We order $A_\kappa\omega$ by setting, for $x, y \in A_\kappa$, $x\omega \preceq_\omega y\omega$ if and only if $x \preceq y$.

Suppose first that $r\alpha^{-1} \preceq s\alpha^{-1}$. By Lemma 7.2, $r\alpha^{-1}\tau^{-1}\alpha \prec_\alpha r$ along $A_\kappa\alpha$. We compare $s\alpha^{-1}\tau^{-1}\alpha$ and r with respect to $\prec_\alpha$. It is straightforward to show

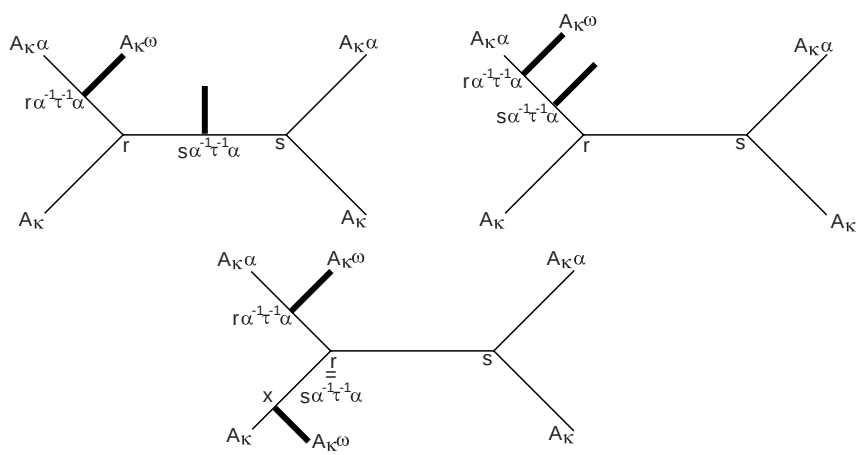

FIGURE 7.

that

- if $r \prec_\alpha s\alpha^{-1}\tau^{-1}\alpha$, then $A_\kappa \cap A_\kappa\omega = [r, s\alpha^{-1}\tau^{-1}\alpha]$ and the orders $\preceq$ and $\preceq_\omega$ agree on $A_\kappa \cap A_\kappa\omega$,
- if $s\alpha^{-1}\tau^{-1}\alpha \prec_\alpha r$, then $A_\kappa \cap A_\kappa\omega = \emptyset$, with $[r, s\alpha^{-1}\tau^{-1}\alpha]$ the bridge connecting A_κ and $A_\kappa\omega$, and
- if $r = s\alpha^{-1}\tau^{-1}\alpha$, then both r, s are finite, $A_\kappa \cap A_\kappa\omega = [x, r] = [x, s\alpha^{-1}\tau^{-1}\alpha]$ for some x, and either $x = r$ or the orders $\preceq$ and $\preceq_\omega$ disagree on $A_\kappa \cap A_\kappa\omega$.

These three possibilities are illustrated in Figure 7. Also by Lemma 7.2, we have $s\alpha^{-1} \prec_{\alpha^{-1}} s\tau\alpha^{-1} = s\alpha^{-1}(\alpha\tau\alpha^{-1})$ along $A_\kappa\alpha^{-1}$, and we compare $r\tau\alpha^{-1}$ and $s\alpha^{-1}$ with respect to $\prec_{\alpha^{-1}}$. After noting that $A_\kappa \cap A_\kappa\omega = (A_\kappa \cap A_\kappa\alpha\tau\alpha^{-1})\tau^{-1}$, it is

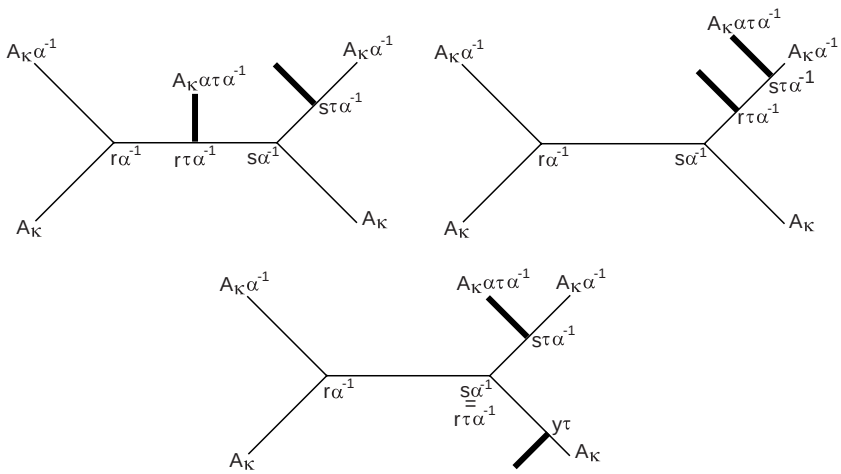

FIGURE 8.

straightforward to show that

- if $r\tau\alpha^{-1} \prec_{\alpha^{-1}} s\alpha^{-1}$, then $A_\kappa \cap A_\kappa\omega = [r\tau\alpha^{-1}\tau^{-1}, s\alpha^{-1}\tau^{-1}]$ and the orders $\preceq$ and $\preceq_\omega$ agree on $A_\kappa \cap A_\kappa\omega$,
- if $s\alpha^{-1} \prec_{\alpha^{-1}} r\tau\alpha^{-1}$, then $A_\kappa \cap A_\kappa\omega = \emptyset$, with $[s\alpha^{-1}\tau^{-1}, r\tau\alpha^{-1}\tau^{-1}]$ the bridge connecting A_κ and $A_\kappa\omega$, and
- if $r\tau\alpha^{-1} = s\alpha^{-1}$, then both r, s are finite, $A_\kappa \cap A_\kappa\omega = [s\alpha^{-1}\tau^{-1}, y] = [r\tau\alpha^{-1}\tau^{-1}, y]$ for some y, and either $y = s\alpha^{-1}\tau^{-1}$ or the orders $\preceq$ and $\preceq_\omega$ disagree on $A_\kappa \cap A_\kappa\omega$.

These three possibilities are illustrated in Figure 8. Hence, one of the following cases holds.

(1) $[r, s\alpha^{-1}\tau^{-1}\alpha] = [r\tau\alpha^{-1}\tau^{-1}, s\alpha^{-1}\tau^{-1}]$.
(2) $[r, s\alpha^{-1}\tau^{-1}\alpha] = [s\alpha^{-1}\tau^{-1}, r\tau\alpha^{-1}\tau^{-1}]$.
(3) $[x, r] = [s\alpha^{-1}\tau^{-1}, y]$, $r\alpha^{-1}\tau\alpha = s = r\tau$, and r, s are both finite.

In case (1), at least one of $r\tau$ and $s\alpha^{-1}\tau^{-1}$ (is finite and) lies in $Nonsep(\alpha) \cap A_\kappa$, and hence the following lemma yields the desired contradiction. (Considering $Nonsep(\alpha)$ in T rather than $Fix(\alpha)$ in T_H allows us to take advantage of the partial order $\leq$ defined on T.)

Lemma 7.3. *Suppose that in T we have $Nonsep(\alpha) \cap A_\kappa \neq \emptyset$.*

- *If $\alpha : (A_\kappa, \prec) \to (A_\kappa\alpha, \prec_\alpha)$ is orientation preserving, then necessarily the action is trivial.*
- *If $\alpha : (A_\kappa, \prec) \to (A_\kappa\alpha, \prec_\alpha)$ is orientation reversing and $p \neq 4q$, then necessarily the action is trivial.*

Proof. If $x \in Fix(\alpha) \cap A_\kappa \neq \emptyset$, then $d(x, x\kappa)$ is necessarily even, and hence x and $x\kappa$ are comparable with respect to the partial order $\leq$ on T. Lemma 3.5 therefore applies.

So we may assume that $Fix(\alpha) \cap A_\kappa = \emptyset$, and choose $x \in (Nonsep(\alpha) \setminus Fix(\alpha)) \cap A_\kappa$. Consider first the possibility that $x\alpha \in A_\kappa$ or $x\alpha^{-1} \in A_\kappa$ (and therefore $\alpha : (A_\kappa, \prec) \to (A_\kappa\alpha, \prec_\alpha)$ is orientation reversing). By replacing x with $x\alpha^{-1}$ as necessary, we may assume that $x, x\alpha \in A_\kappa$. We consider separately the cases

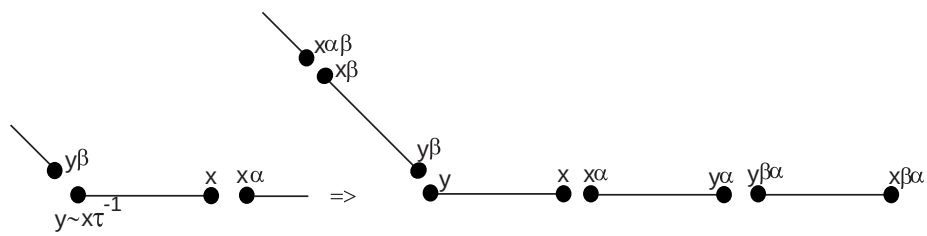

FIGURE 9.

$x \prec x\alpha$ and $x\alpha \prec x$. Note that since $x\alpha \sim x$, we have $x\tau^{-1}\beta \sim x\tau^{-1}$. Therefore, as illustrated in Figures 9 and 10, respectively, straightforward computations reveal that

$$d(x, x\alpha\beta\alpha^{-1}\beta^{-1}) = d(x\beta\alpha, x\alpha\beta) = 4(2nq),$$

where $d(y, y\kappa) = 2n$ for all $y \in A_\kappa$. So $2np = d(x, x\gamma) = 4(2nq)$, and hence $p = 4q$, which is impossible.

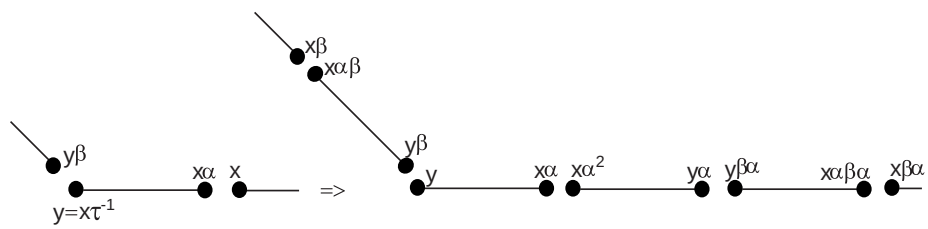

FIGURE 10.

Now assume that if $z \in Nonsep(\alpha) \cap A_\kappa$, then $z\alpha^{-1}, z\alpha \notin A_\kappa$. Consider $[x] \cap A_\kappa$. Either $[x] \cap A_\kappa = \{x\}$ or $[x] \cap A_\kappa = \{x, y\}$ for some $y \neq x$. In the first case, Lemma 3.5 applied to the ideal point determined by $[x]$ shows that the action of G on T is trivial. In the second case, note that $y\alpha \sim x\alpha \sim x \sim y$, but $y\alpha \notin \{x, y\}$ by assumption. We therefore have the situation modelled in Figure 11. For the details, proceed as follows, working now in T_H. We have

$$A_\kappa \cap A_\kappa\alpha = \{x\} = A_\kappa \cap A_\kappa\alpha^{-1}.$$

Hence,

$$A_\kappa \cap A_\kappa\beta = A_\kappa \cap A_\kappa\tau\alpha^{-1}\tau^{-1} = A_\kappa \cap A_\kappa\alpha^{-1}\tau^{-1} = (A_\kappa \cap A_\kappa\alpha^{-1})\tau^{-1} = \{x\tau^{-1}\}.$$

Furthermore,

$$A_\kappa \cap A_\kappa\alpha = \{x\} \Longrightarrow A_\kappa\beta \cap A_\kappa\alpha\beta = \{x\beta\}.$$

Note that $x\beta \neq x\tau^{-1}$, since otherwise $x\alpha = x\tau \in A_\kappa$. Therefore, by simple connectivity,

$$A_\kappa\alpha\beta \cap A_\kappa\alpha = \emptyset,$$

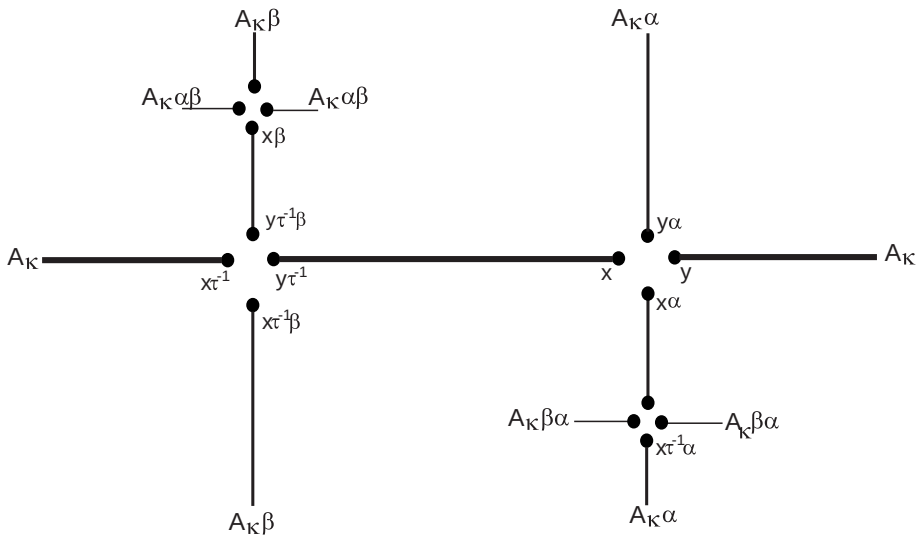

FIGURE 11.

whereas

$$A_\kappa \cap A_\kappa\beta = \{x\tau^{-1}\} \Longrightarrow A_\kappa\alpha \cap A_\kappa\beta\alpha = \{x\tau^{-1}\alpha\}.$$

Since

$$A_\kappa\gamma = A_\kappa \Longrightarrow A_\kappa\alpha\beta = A_\kappa\beta\alpha,$$

this is impossible and the lemma is proved. □

In case (3), we have $s\alpha^{-1}\tau^{-1} = r\alpha^{-1}$, so $A_\kappa \cap A_\kappa\omega = [r\alpha^{-1}, r]$. In particular, $r\alpha^{-1} \preceq r$. If $r\alpha^{-1} = r$ in T_H, then Lemma 7.3 applies. So, we may assume that $r\alpha^{-1} \prec r$. If $s \preceq s\alpha^{-1}$, then α determines a homeomorphism from the subinterval $[r, s]$ to $[r\alpha^{-1}, s\alpha^{-1}]$. Therefore, α fixes some element of $[r, s]$ and Lemma 7.3 applies again. So we may assume that $s\alpha^{-1} \prec s$. Recall that since $r\alpha^{-1} \prec s\alpha^{-1}$, the map $\alpha : (A_\kappa, \prec) \to (A_\kappa\alpha, \prec_\alpha)$ is order preserving. Thus $s \preceq_\alpha s\alpha$. Now, since $r\alpha^{-1}\tau\alpha = s = r\tau$, relation (B) gives

$$r\beta\alpha = r\tau\alpha^{-1}\tau^{-1}\alpha = r.$$

Therefore,

$$s = r\tau \prec r\kappa^p = r\gamma^{-1} = r\beta\alpha\beta^{-1}\alpha^{-1} = r\beta^{-1}\alpha^{-1} = r\tau\alpha\tau^{-1}\alpha^{-1} = s\alpha\tau^{-1}\alpha^{-1}.$$

So $s\alpha\tau^{-1} = r\kappa^p\alpha \in A_\kappa\alpha$ with $s \prec_\alpha s\alpha \prec_\alpha s\alpha\tau^{-1}$. But this in turn gives $s\alpha = (s\alpha\tau^{-1})\tau \notin A_\kappa\alpha$, a contradiction.

In case (2), we obtain

$$r = s\alpha^{-1}\tau^{-1} \Longrightarrow s = r\tau\alpha$$

(so both r and s are finite) and

$$s\alpha^{-1}\tau^{-1}\alpha = r\tau\alpha^{-1}\tau^{-1} \Longrightarrow r\alpha = r\tau\alpha^{-1}\tau^{-1}.$$

Hence,

$$(r\alpha^{-1})(\alpha\tau\alpha^{-1})\tau^{-1} = r\tau\alpha^{-1}\tau^{-1} = r\alpha.$$

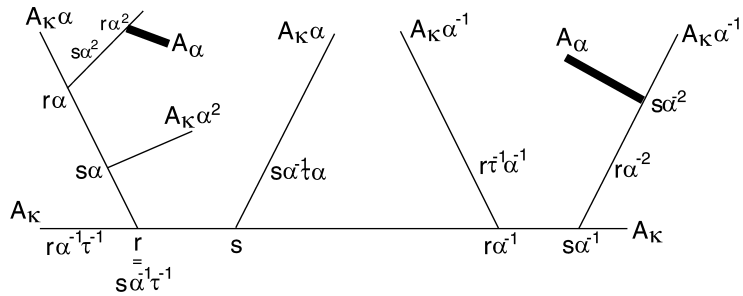

FIGURE 12.

Next apply (7.1) to the element $r\alpha^{-1}$:

$$\begin{array}{rrcl} & r\alpha & = & (r\alpha^{-1})(\kappa^{-p+q})(\alpha^{-1}\tau^{-1}\alpha) \\ \Longrightarrow & r\tau\alpha & = & (r\alpha^{-1})(\kappa^{-p+q}) \\ \Longrightarrow & s & = & (r\alpha^{-1})(\kappa^{-p+q}). \end{array}$$

Therefore,

$$s\kappa^{p-q} = r\alpha^{-1} \tag{7.2}$$

Since $p \geq q$, $s \prec r\alpha^{-1}$ on A_κ. So

$$r \preceq s \prec r\alpha^{-1} \preceq s\alpha^{-1}$$

on A_κ. Since $\alpha : (A_\kappa, \prec) \to (A_\kappa\alpha, \prec_\alpha)$ and $\alpha^{-1} : (A_\kappa, \prec) \to (A_\kappa\alpha^{-1}, \prec_{\alpha^{-1}})$ are order preserving, we see that

$$r\alpha \preceq_\alpha s\alpha \prec_\alpha r \preceq_\alpha s$$

on $A_\kappa\alpha$ and

$$r\alpha^{-1} \preceq_{\alpha^{-1}} s\alpha^{-1} \prec_{\alpha^{-1}} r\alpha^{-2} \preceq_{\alpha^{-1}} s\alpha^{-2}$$

on $A_\kappa\alpha^{-1}$. In particular, $[r, r\alpha^{-1}] = [r, s] \cup [s, r\alpha^{-1}]$, with $[r, r\alpha^{-1}]\alpha^n \subset A_\kappa\alpha^n$ and $[r, s]\alpha^n \cap [r, s]\alpha^{n+1} = \emptyset$, for all $n \in \mathbb{Z}$. This is illustrated in Figure 12. Therefore,

$$A_\alpha = \bigcup_{-\infty}^{\infty} [r, r\alpha^{-1}]\alpha^n$$

is a local axis or axis for α.

Next we investigate the orientation that A_κ inherits from T_H. For any vertices x, y both of which lie on one of the (simplicial) trees A_κ, $A_\kappa\alpha$, $A_\kappa\alpha^{-1}$ in T_H, let $f_{x,y}$ be the first edge in the simplicial path from x to y in the given tree.

After reversing the orientation of T_H if necessary, we may assume that $f_{r,r\tau}$ is positively oriented. Since $s\alpha^{-1} = r\tau$ and τ preserves orientation, we see that $f_{r,r\tau}$ and $f_{s\alpha^{-1},s\alpha^{-1}\tau}$ have the same orientation, as do $f_{r,r\tau^{-1}}$ and $f_{s\alpha^{-1},s\alpha^{-1}\tau^{-1}}$. Since α^{-1} preserves orientation, we see that $f_{r,r\alpha}$ and $f_{r\alpha^{-1},r} = f_{r\alpha^{-1},r\alpha^{-1}\tau^{-1}}$ have the same orientation, as do both edges from each pair $f_{r,r\alpha^{-1}} = f_{r,r\tau}$ and $f_{r\alpha^{-1},r\alpha^{-2}} = f_{r\alpha^{-1},r\alpha^{-1}\tau}$; $f_{r,r\tau^{-1}}$ and $f_{r\alpha^{-1},r\tau^{-1}\alpha^{-1}}$; $f_{s,s\alpha} = f_{s,s\tau^{-1}}$ and $f_{s\alpha^{-1},s} = f_{s\alpha^{-1},s\alpha^{-1}\tau^{-1}}$; $f_{s,s\alpha^{-1}} = f_{s,s\tau}$ and $f_{s\alpha^{-1},s\alpha^{-2}}$; and $f_{s,s\alpha^{-1}\tau\alpha}$ and $f_{s\alpha^{-1},s\alpha^{-1}\tau}$. Now, using (7.2), we see that $f_{s,s\tau}$ and $f_{r\alpha^{-1},r\alpha^{-1}\tau}$ have the same orientation, as do $f_{s,s\tau^{-1}}$ and $f_{r\alpha^{-1},r\alpha^{-1}\tau^{-1}}$. Finally, after applying $\tau\alpha$ to the interval on the right side of the equality

$$[r, r\alpha] = [s\alpha^{-1}\tau^{-1}, r\tau\alpha^{-1}\tau^{-1}],$$

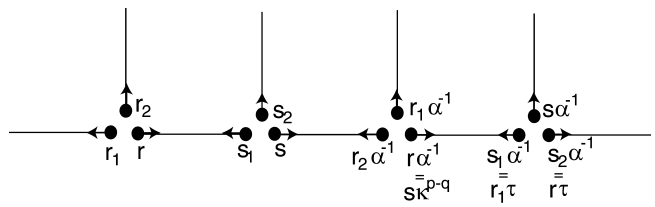

FIGURE 13.

we see that $f_{r,r\alpha}$ and $f_{s,r\tau} = f_{s,s\tau}$ have the same orientation. It follows that all of the edges under consideration have the same orientation, which we have assumed to be positive. We now see that each of the points $r, s, r\alpha^{-1}, s\alpha^{-1}$ in T_H corresponds to a pair of (distinct) nonseparated points along A_κ in T, and that the corresponding branching at $A_\kappa\alpha$ and $A_\kappa\alpha^{-1}$ is as shown in Figure 13.

So we change viewpoint and consider instead the non-Hausdorff 1-manifold T. Let $n \in \mathbb{N}$ such that $d(x, x\kappa) = 2n$ for all $x \in A_\kappa$. Then

$$\begin{aligned} 2nq &= d(r, r\tau) \\ &= d(r,s) + d(s, s\kappa^{p-q}) + d(r\alpha^{-1}, s\alpha^{-1}) \\ &= 2d(r,s) + 2n(p-q) \\ \implies d(r,s) &= n(2q-p). \end{aligned}$$

(In particular, $2q \geq p$.) Also,

$$d(r, r\alpha^{-1}) = d(r,s) + d(s, s\kappa^{p-q}) = n(2q-p) + 2n(p-q) = np.$$

(Therefore, np is necessarily even.) Finally, we use relation (A) from Section 3, in the form

$$\tau(\alpha\tau^{-1}\alpha^{-1})\tau^{-1}(\alpha\tau\alpha^{-1}) = \alpha^{m-2},$$

by applying each of the given words to the element r.

Let $v = r\tau(\alpha\tau^{-1}\alpha^{-1}) = (r\tau\alpha)\tau^{-1}\alpha^{-1} = s_2\tau^{-1}\alpha^{-1}$. Note that since $s_1 \prec s_1\alpha^{-1} = r_1\tau$, we have $s_1\tau^{-1} \prec r_1$ along A_κ, and hence $s_1\tau^{-1}\alpha^{-1} \prec_{\alpha^{-1}} r_1\alpha^{-1}$ along $A_\kappa\alpha^{-1}$. Therefore, since $v \sim s_1\tau^{-1}\alpha^{-1}$, we see that $[v, r_1\alpha^{-1}]$ is the bridge from v to A_κ. So $[v\tau^{-1}, r_1\alpha^{-1}\tau^{-1}]$ is the bridge from $v\tau^{-1}$ to A_κ and hence $[v\tau^{-1}, r_2\alpha^{-1}]$ is the bridge from $v\tau^{-1}$ to $A_\kappa\alpha^{-1}$. So

$$[r\alpha^{m-2}, r_2\alpha^{-1}(\alpha\tau\alpha^{-1})] = [v\tau^{-1}(\alpha\tau\alpha^{-1}), r_2\tau\alpha^{-1}]$$

is the bridge from $r\alpha^{m-2}$ to $A_\kappa\alpha^{-1}$. Since $r_2\tau\alpha^{-1} \sim s\alpha^{-2}$ and $r\alpha^{m-2} \in A_\alpha$, necessarily $r_2\tau\alpha^{-1} = s\alpha^{-2}$. (See Figure 14.)

By computing the length of the path $[v\tau^{-1}(\alpha\tau\alpha^{-1}), r\alpha^{-1}] = [r\alpha^{m-2}, r\alpha^{-1}]$, we obtain

$$\begin{aligned} d(r\alpha^{m-2}, r\alpha^{-1}) &= d(r\alpha^{m-2}, s\alpha^{-2}) + d(s\alpha^{-2}, r\alpha^{-1}) \\ &= d(v\tau^{-1}, r_2\alpha^{-1}) + d(s\alpha^{-1}, r) \\ &= [d(v\tau^{-1}, r_1\alpha^{-1}\tau^{-1}) + d(r_1\alpha^{-1}\tau^{-1}, s_1\alpha^{-1}\tau^{-1}) \\ &\quad + d(r_1, r_2\alpha^{-1})] + d(r\tau, r) \\ &= 4nq + np. \end{aligned}$$

On the other hand,

$$d(r\alpha^{m-2}, r\alpha^{-1}) = |m-1| d(r, r\alpha^{-1}) = |m-1|np = (|m|+1)np.$$

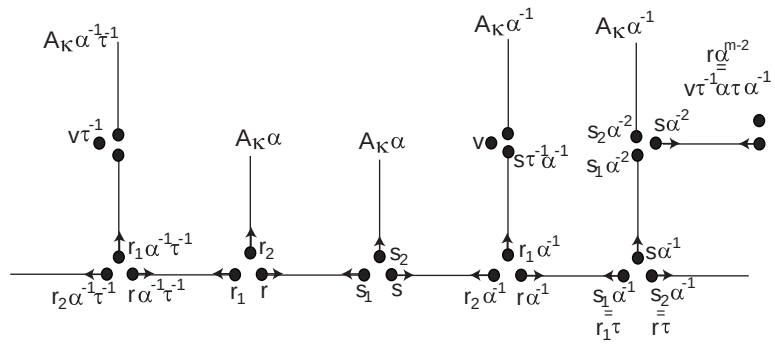

FIGURE 14.

So

$$4nq + np = (|m| + 1)np \Longrightarrow 4q = |m|p,$$

which is impossible since both p and m are odd, and hence we have reached our contradiction.

Since we have been working under the assumption that $r\alpha^{-1} \preceq s\alpha^{-1}$, we conclude therefore that $s\alpha^{-1} \prec r\alpha^{-1}$ (and hence $r \neq s$). Consider first the possibility that $[r, s] \cap [s\alpha^{-1}, r\alpha^{-1}] \neq \emptyset$. In this case, the Intermediate Value Theorem guarantees the existence of an element $x \in Fix(\alpha) \cap A_\kappa \subset T_H$. Since $p \neq 4q$, Lemma 7.3 therefore applies. So restrict attention to the case that $[r, s] \cap [s\alpha^{-1}, r\alpha^{-1}] = \emptyset$. By appealing to symmetry, we may assume that $r \prec s \prec s\alpha^{-1} \prec r\alpha^{-1}$. It follows that

$$[s, s\alpha^{-1}] \cap [s, s\alpha] = \{s\},$$

and hence that

$$A_\alpha = \bigcup_{n=-\infty}^{\infty} [s, s\alpha^{-1}]\alpha^n$$

is a local axis or axis for α. Since $A_\beta = A_\alpha\tau^{-1}$,

$$A_\beta \cap A_\kappa = [s\tau^{-1}, s\alpha^{-1}\tau^{-1}].$$

Suppose first that $s\alpha^{-1}\tau^{-1} \neq s$ and apply each of the words from the relation $\tau^{-1}\alpha\tau = \alpha\beta\alpha^{m-1}$ to s. Referring to the axes A_κ, A_α and A_β, a straightforward computation reveals that the bridge from $s\tau^{-1}\alpha\tau$ to A_κ has endpoint $s\tau$ at A_κ, whereas the bridge from $s\alpha\beta\alpha^{m-1}$ to A_κ has endpoint $s\alpha^{-1}$ at A_κ. We conclude that $s\alpha^{-1} = s\tau$.

Without loss of generality, we may assume that $f_{s,s\tau} = f_{s,s\alpha^{-1}}$ is positively oriented. Applying τ and α^{-1} to $f_{s,s\tau}$, we see that $f_{s\alpha^{-1},s\alpha^{-1}\tau}$ and $f_{s\alpha^{-1},s\alpha^{-2}}$ are also positively oriented. Now applying α to $f_{s\alpha^{-1},s\alpha^{-1}\tau} = f_{s\alpha^{-1},r\alpha^{-1}}$ shows that $f_{s,s\tau^{-1}} = f_{s,r}$ is positively oriented, and applying τ to $f_{s,s\tau^{-1}}$ shows that $f_{s\alpha^{-1},s\alpha^{-1}\tau^{-1}}$ is positively oriented. Finally, applying α to $f_{s\alpha^{-1},s\alpha^{-1}\tau^{-1}}$ shows that $f_{s,s\alpha}$ is positively oriented. Hence, in T we have the situation shown in Figure 15, with

$$d(s, s\alpha^{-1}) = d(s, s\tau) = 2nq \geq 2.$$

Let $u, v, w \in [s]$ be as given in Figure 15. Notice that

$$s\alpha\tau = (s\tau)(\tau^{-1}\alpha\tau) = (s\tau)\alpha\beta\alpha^{m-1}.$$

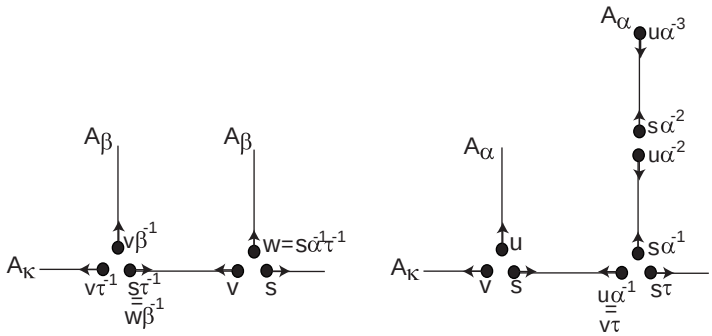

FIGURE 15.

Now $d(s\alpha\tau, s\tau) = d(s\alpha, s) = 2nq$, together with Lemma 4.1, guarantees that

$$s\alpha\tau \in (s\alpha^{-2})^{-}.$$

On the other hand, $u \neq s\tau\alpha \sim v$, and since $v \in A_\beta$, $d(v, v\beta) = 2nq$. And $d(v, v\beta) = 2nq$, together with Lemma 4.1, reveals that $v\beta \in (u\alpha)^{-}$. So $(s\tau\alpha)\beta \in (u\alpha)^{-} \cup \{u\alpha\}$, and hence

$$s\alpha\tau = (s\tau\alpha\beta)\alpha^{m-1} \in (u\alpha^m)^{-} \cup \{u\alpha^m\}.$$

Since $m \leq -3$, we have reached a contradiction.

7.3. **Case (c).** Suppose that $A_\kappa \cap A_\kappa\alpha = \emptyset$, and let $\rho = [r, s]$ be the bridge from A_κ to $A_\kappa\alpha$ in T_H. Once again we consider

$$A_\kappa\omega = A_\kappa \cdot \kappa^{-p+q}(\alpha^{-1}\tau^{-1}\alpha) = A_\kappa(\alpha^{-1}\tau^{-1}\alpha).$$

Since $[r, s]$ is the bridge from A_κ to $A_\kappa\alpha$, we see that $[r(\alpha^{-1}\tau^{-1}\alpha), s(\alpha^{-1}\tau^{-1}\alpha)]$ is the bridge from $A_\kappa\omega$ to $A_\kappa\alpha$. Since $s\alpha^{-1}\tau^{-1}\alpha \neq s$, we see that

$$[r(\alpha^{-1}\tau^{-1}\alpha), r] = [r(\alpha^{-1}\tau^{-1}\alpha), s(\alpha^{-1}\tau^{-1}\alpha)] \sqcup [s(\alpha^{-1}\tau^{-1}\alpha), r]$$

is the bridge from $A_\kappa\omega$ to A_κ. Next consider $A_\kappa\omega = A_\kappa(\alpha\tau\alpha^{-1})\tau^{-1}$. Since $[s\alpha^{-1}, r\alpha^{-1}]$ is the bridge from A_κ to $A_\kappa\alpha^{-1}$, we know that

$$[s\alpha^{-1}(\alpha\tau\alpha^{-1}), r\alpha^{-1}(\alpha\tau\alpha^{-1})]$$

is the bridge from $A_\kappa(\alpha\tau\alpha^{-1})$ to $A_\kappa\alpha^{-1}$. Hence, $[s\tau\alpha^{-1}, s\alpha^{-1}]$ is the bridge from $A_\kappa(\alpha\tau\alpha^{-1})$ to A_κ. So $[s\tau\alpha^{-1}\tau^{-1}, s\alpha^{-1}\tau^{-1}]$ is the bridge from $A_\kappa\omega$ to A_κ. Hence,

$$[r(\alpha^{-1}\tau^{-1}\alpha), r] = [s\tau\alpha^{-1}\tau^{-1}, s\alpha^{-1}\tau^{-1}].$$

(See Figure 16.) In particular, $r(\alpha^{-1}\tau^{-1}\alpha) = s\tau\alpha^{-1}\tau^{-1}$ and $s = r\tau\alpha$. Now apply

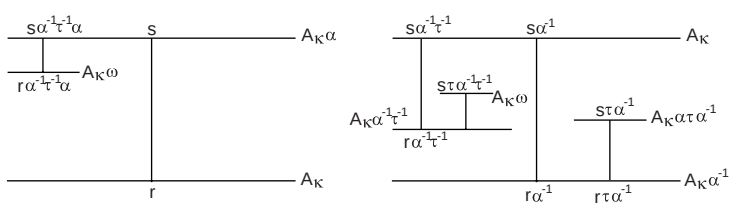

FIGURE 16.

both sides of (7.1) to the element $r\kappa^{p-q}$.

$$\begin{aligned} r\kappa^{p-q}\cdot(\alpha\tau\alpha^{-1})\tau^{-1} &= r\kappa^{p-q}\cdot\kappa^{-p+q}\cdot(\alpha^{-1}\tau^{-1}\alpha) \\ &= r(\alpha^{-1}\tau^{-1}\alpha) = s\tau\alpha^{-1}\tau^{-1} \\ &= (r\tau\alpha)(\tau\alpha^{-1}\tau^{-1}). \end{aligned}$$

So

$$r\kappa^{p-q} = r\tau \Longrightarrow r \in Fix(\kappa^{p-2q}).$$

However, $r \in A_\kappa$ means that no nontrivial power of κ fixes r. Since p is odd, we have $\kappa^{p-2q} \neq 1$, giving the final contradiction. □

We conclude this section by recording, for future reference, a lemma which follows easily from the above arguments restricted to the special case that $A_\kappa \approx \mathbb{R}$. Note that the only way in which we have used the fact that $Nonsep(\kappa) = \emptyset$ in this section is to guarantee the existence of the axis A_κ. The fact that A_κ is an axis is then used to control the structure of the intersections of A_κ with some of its translates and that of the bridges from A_κ to such translates in the case of empty intersection. The arguments in this section can be easily adjusted to obtain the following result, which will be used repeatedly in the next section, where we examine the case where $Nonsep(\kappa) \neq \emptyset$.

Lemma 7.4. *Suppose Y is a κ-invariant embedded copy of $\mathbb{R}$ in T on which κ acts freely. If*

- *$\emptyset \neq Y \cap Y\alpha \subset [r,s]$ for some $r,s \in Y$, or*
- *$Y \cap Y\alpha = \emptyset$, and the bridge from Y to $Y\alpha$ has the form $[[r,s]]$ for some $r \sim r' \in Y$, $s \sim s' \in Y\alpha$,*

then the action of G on T has a global fixed point.

8. Case II: $Nonsep(\kappa) \neq \emptyset$.

In this section we prove that if $Nonsep(\kappa) \neq \emptyset$, then necessarily the action of G on T is trivial. First we give some preliminary lemmas, whose primary import is the fact that in most cases, the argument reduces to the case that $T = \mathbb{R}$.

Lemma 8.1. *There is no $x \in T$ which is nonseparated by τ and at least one of α, β.*

Lemma 8.2. *If $u \sim v < w$, then $u \in w^-$, with $u < w \Leftrightarrow u, w$ are comparable $\Leftrightarrow w \in T_{\{u,v\}}$. Similarly, if $u \sim v > w$, then $u \in w^+$, with $u > w \Leftrightarrow u, w$ are comparable $\Leftrightarrow w \in T_{\{u,v\}}$.*

Lemma 8.3. *If $Fix(\tau) \cap Nonsep(\kappa) \cap C_\alpha \neq \emptyset$, then the action of G on T is trivial.*

Proof. Let $x \in Fix(\tau) \cap Nonsep(\kappa) \cap C_\alpha$. We modify slightly the arguments of Section 3.1. Once again, we may assume that $x < x\alpha$. Since $x \in Nonsep(\kappa)$, we have $x\gamma \sim x$. Therefore,

$$x\beta\alpha^m \sim x\gamma\beta\alpha^m = x\tau^{-1}\alpha\tau = (x\alpha)\tau > x\tau = x$$

and hence, by Lemma 8.2, $x\beta\alpha^m \in x^+$. Since $x\alpha^{-m} > x$, $x\beta \in x^+\alpha^{-m} \subset x^+$.

On the other hand, $x\beta\tau = x\tau^{-1}\beta\tau = x\alpha^{-1} < x \Rightarrow x\beta < x\tau^{-1} = x \Rightarrow x\beta \in x^-$. □

Lemma 8.4. *If $Nonsep(\kappa) \cap C_\alpha \neq \emptyset$, then the action of G on T is trivial.*

Proof. Let $x \in Nonsep(\kappa) \cap C_\alpha$. By Lemma 8.3, we may assume that $x \sim x\tau$ but $x \neq x\tau$. Set $T_0 = T_{\{x, x\tau\}}$. Again, we may assume that $x < x\alpha$.

If $x\alpha^{-1} \in T_0$ or $x\alpha \in T_0$, then the ideal point $\hat{x} \in \hat{T}$ is fixed by κ and related to $\hat{x}\alpha$, and Lemma 3.5 applies.

So we may assume that $x\alpha, x\alpha^{-1} \notin T_0$. Since $x < x\alpha$, either $T_0 \subset x^-$ and $x\alpha^{-1} \in y^+$ for some $y \sim x$, $y \neq x$, or $T_0 \subset x^+$ and $x\alpha \in y^-$ for some $y \sim x$, $y \neq x$. In each case, by Lemma 8.3, we may assume that $y\tau \neq y$. In fact, by reversing the orientation on T and exchanging the roles of x and y as necessary, we may assume that the first possibility holds; namely, $T_0 \subset x^-$ and $x\alpha^{-1} \in y^+$ for some $y \sim x$, $y \neq x$. These possibilities are illustrated in Figure 17. Notice that

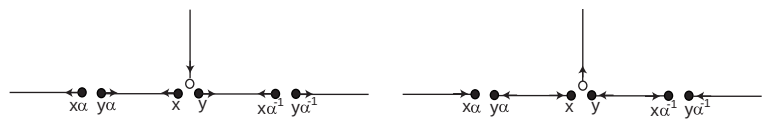

FIGURE 17.

$\{x, y\} \subset [[x\alpha^{-1}, x\alpha]]$ and so $d(x, x\alpha) = 2n > 0$ for some $n \in \mathbb{N}$. In particular, by Proposition 4.8 and Corollary 4.12, we have $Nonsep(\alpha) = \emptyset$ and $C_\alpha = A_\alpha$.

Consider first the case that $y \neq x\tau$. Since

$$x\tau^{-1}\alpha\tau \sim x\alpha\tau > x\tau,$$

we have

$$x\tau^{-1}\alpha\tau \in x\tau^+ \subseteq y^-. \tag{8.1}$$

Also, $x\gamma\tau\alpha^{-1}\tau^{-1} \sim x\alpha^{-1}\tau^{-1}$, and since $x\alpha^{-1} \in y^+$, we have

$$x\gamma\tau\alpha^{-1}\tau^{-1} \in (y\tau^{-1})^+ \subseteq x^-.$$

This gives

$$x\tau^{-1}\alpha\tau = x\gamma\beta\alpha^m = x\gamma\tau\alpha^{-1}\tau^{-1}\alpha^m \in (x\alpha^m)^- \subseteq (x\alpha^{-1})^-. \tag{8.2}$$

However, since $x \in (x\alpha^{-1})^+$, we have

$$y^- \cap (x\alpha^{-1})^- = \emptyset,$$

and (8.1) and (8.2) now give a contradiction.

So we may assume that $y = x\tau$. In this case,

$$d(x\tau\alpha\tau, x\tau) = d(x\tau\alpha, x) = 2n - 1 < d(x\tau\alpha^{-1}, x\tau) \Longrightarrow x\tau\alpha\tau \in (x\tau\alpha^{-1})^-.$$

Therefore, since $x\tau^{-1}\alpha\tau \sim x\tau\alpha\tau$, we have

$$x\tau^{-1}\alpha\tau \in (x\tau\alpha^{-1})^- \cup \{x\tau\alpha^{-1}\}. \tag{8.3}$$

On the other hand, since $d(x\alpha^{-1}\tau^{-1}, x) = 2n - 1$, we have $x\alpha^{-1}\tau^{-1} \in (x\alpha)^-$ and hence $(x\gamma\tau)\alpha^{-1}\tau^{-1} \in (x\alpha)^- \cup \{x\alpha\}$. So, since $m \leq -3$, we have

$$x\tau^{-1}\alpha\tau = x\gamma\tau\alpha^{-1}\tau^{-1}\alpha^m \in (x\alpha^{m+1})^- \cup \{x\alpha^{m+1}\} \subset (x\alpha^{-2})^- \cup \{x\alpha^{-2}\}. \tag{8.4}$$

Since $y \in (x\alpha^{-1})^+$, we have $x\tau\alpha^{-1} = y\alpha^{-1} \in (x\alpha^{-2})^+$, and it follows that

$$((x\tau\alpha^{-1})^- \cup \{x\tau\alpha^{-1}\}) \cap ((x\alpha^{-2})^- \cup \{x\alpha^{-2}\}) = \emptyset. \tag{8.5}$$

Now (8.3), (8.4) and (8.5) together give a contradiction. □

Lemma 8.5. *If $Nonsep(\kappa) \neq \emptyset$ and $Nonsep(\alpha) \cap C_\kappa \neq \emptyset$, then the action of G on T is trivial.*

Proof. Let $x \in Nonsep(\alpha) \cap C_\kappa$. By Lemma 4.10, either $x \in Fix(\kappa)$ or x lies on some local axis $A^i_\kappa \approx \mathbb{R}$ (in T) for κ. By Lemma 8.1, we may assume that x lies on some local axis A^i_κ (in T). Then either $x \in Fix(\alpha)$ or the ideal point $\hat{x} \in \hat{T}$ is fixed by α and related to $\hat{x}\kappa$. In either case, Lemma 3.5 applies. □

Let $\{T_i\}_{i\in\mathcal{I}}$ denote the path components of $T \setminus Nonsep(\kappa)$; so $T \setminus Nonsep(\kappa) = \bigsqcup_{i\in\mathcal{I}} T_i$. Notice that for each $i \in \mathcal{I}$ we have $T_i\kappa = T_j$ for some $j \in \mathcal{I}$. Moreover, whenever $T_i\kappa = T_i$, κ acts freely on T_i with local axis $A^i_\kappa \subset T_i$ (and since $Nonsep(\kappa) \neq \emptyset$, $A^i_\kappa \approx \mathbb{R}$).

Similarly, if $Nonsep(\alpha) \neq \emptyset$, let $\{X_j\}_{j\in\mathcal{J}}$ denote the set of path components of $T \setminus Nonsep(\alpha)$. Again, either $X_j\alpha = X_j$, and α acts freely on X_j with local axis A^j_α, or $X_j\alpha = X_k \neq X_j$. When $Nonsep(\alpha) = \emptyset$, we write $T = X_1$ and let A^1_α denote the axis for α.

Lemma 8.6. *If G acts nontrivially on T, then*

- *$C_\kappa \cup Nonsep(\kappa) \subset X_{j_0}$ for some $j_0 \in \mathcal{J}$, and*
- *$C_\alpha \cup Nonsep(\alpha) \subset T_{i_0}$ for some $i_0 \in \mathcal{I}$.*

Proof. By Lemma 8.5, $(C_\kappa \cup Nonsep(\kappa)) \cap Nonsep(\alpha) = \emptyset$. By Corollary 4.13 therefore, $C_\kappa \cup Nonsep(\kappa) \subset X_{j_0}$ for some $j_0 \in \mathcal{J}$. A symmetric argument proves the second statement. □

Proposition 8.7. *Suppose $Nonsep(\kappa) \neq \emptyset$. Then the action of G on T is trivial.*

Proof. Let i_0, j_0 be as guaranteed in Lemma 8.6.

[Case 1] Suppose first that $T_{i_0}\kappa = T_{i_0}$. As remarked above, $A^{i_0}_\kappa \approx \mathbb{R}$. By Lemma 8.6, $Nonsep(\kappa) \cup A^{i_0}_\kappa \subset X_{j_0}$.

Consider first the possibility that $X_{j_0}\alpha = X_{j_0}$, and hence $A^{j_0}_\alpha \subset T_{i_0}$. In fact, $T_{i_0} \cap X_{j_0}$ is a subtree of T containing both $A^{i_0}_\kappa$ and $A^{j_0}_\alpha$. Therefore, if $A^{i_0}_\kappa \cap A^{j_0}_\alpha = \emptyset$, the bridge from $A^{i_0}_\kappa$ to $A^{j_0}_\alpha$ lies in $T_{i_0} \cap X_{j_0}$. If either of the two potential endpoints of $A^{i_0}_\kappa$ (respectively, $A^{j_0}_\alpha$) exist in T, they are in $Nonsep(\kappa)$ (respectively, $Nonsep(\alpha)$) and hence are not elements of T_{i_0} (respectively, X_{j_0}), and therefore cannot be on the bridge. Hence this bridge has the form $[[u, v]]$ or $[[u, v))$, where u and v are not separated from points in $A^{i_0}_\kappa$ and $A^{j_0}_\alpha$, respectively. (See Figure 18.) Computing

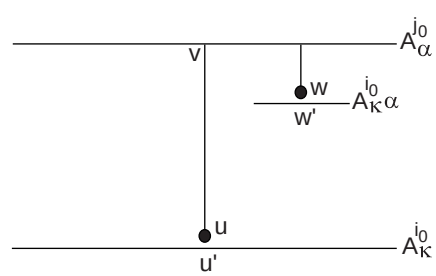

FIGURE 18.

$A^{i_0}_\kappa\alpha$ in this case, we see that $A^{i_0}_\kappa \cap A^{i_0}_\kappa\alpha = \emptyset$, with the bridge from $A^{i_0}_\kappa$ to $A^{i_0}_\kappa\alpha$ of the form $[[u, w]]$ for some $w \sim w' \in A^{i_0}_\kappa\alpha$. So Lemma 7.4 reveals that the action of

G on T is necessarily trivial. On the other hand, if $A_\kappa^{i_0} \cap A_\alpha^{j_0} \neq \emptyset$, then Lemma 8.1 guarantees that $A_\kappa^{i_0} \cap A_\alpha^{j_0} \subset [u, v]$ for some $u, v \in A_\kappa^{i_0}$. (See Figure 19.) Computing

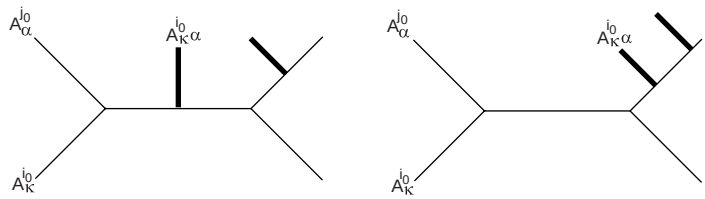

FIGURE 19.

$A_\kappa^{i_0}\alpha$ in this case, we see that one of the two conditions of Lemma 7.4 is satisfied, and so once again, the action of G on T must be trivial.

Next consider the possibility that $X_{j_0}\alpha = X_{j_1} \neq X_{j_0}$. Let y and $y\alpha$ denote the roots of X_{j_0} and $X_{j_0}\alpha$, respectively. Let $[[y, r]]$ denote the bridge from y to $A_\kappa^{i_0}$ in T. By Lemma 8.1, we may assume that $r \sim r'$ for some $r' \in A_\kappa^{i_0}$. So $A_\kappa^{i_0} \cap A_\kappa^{i_0}\alpha = \emptyset$ with bridge $[[r, r\alpha]]$. Again, by Lemma 7.4, the action of G on T has a global fixed point.

[Case 2] Finally, we assume that $T_{i_0}\kappa = T_{i_1} \neq T_{i_0}$. Let x and $x\kappa$ denote the roots of T_{i_0} and T_{i_1}, respectively. Set $T_0 = T_{\{x, x\kappa\}}$. Without loss of generality, we may assume that $T_0 \subset x^+$.

If $X_{j_0}\alpha = X_{j_0}$, then $A_\alpha^{j_0} \subset T_{i_0}$. Therefore, since $x \in Nonsep(\kappa) \subset X_{j_0}$, the bridge from x to $A_\alpha^{j_0}$ is of the form $[[x, r]]$ or $[[x, r))$. Deleting x from this bridge, we obtain $((x, r]]$ (or $((x, r))$, respectively), which lies in $T_{i_0} \cap X_{j_0}$. In particular, $r (\nsim x)$ is nonseparated from a point in $A_\alpha^{j_0}$. If $X_{j_0}\alpha = X_{j_1} \neq X_{j_0}$, let y and $y\alpha$ denote the roots of X_{j_0} and X_{j_1}, respectively. Note that since $x \in X_{j_0}$, $y \in [x, x\alpha]$. These two possibilities are illustrated in Figure 20. Note that in either case, $d(x, x\alpha)$

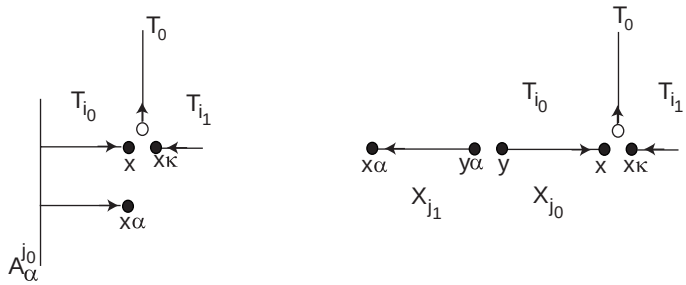

FIGURE 20.

is odd; so $d(x, x\alpha) = 2n - 1$ for some $n \in \mathbb{N}$.

Consider first the case that $x \neq x\tau$. This is illustrated in Figure 21. Note that since $d(x\tau\alpha, x\alpha\tau) = 4n$ and $(x\tau\alpha, x\alpha\tau] \subset (x\tau\alpha)^+$, we have

$$((x\tau\alpha, x\alpha\tau])\alpha^{-1}\tau^{-1} = (x, x\alpha\tau\alpha^{-1}\tau^{-1}] \subset x^+.$$

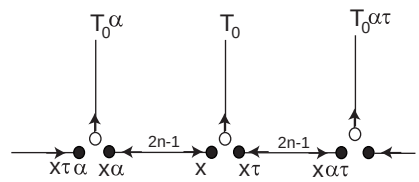

FIGURE 21.

Therefore, since $x\alpha \in x^-$, we have $x \in [x\alpha, x\alpha\tau\alpha^{-1}\tau^{-1}]$ and hence,

$$\begin{aligned} d(x\alpha, x\alpha\tau\alpha^{-1}\tau^{-1}) &= d(x\alpha, x) + d(x, x\alpha\tau\alpha^{-1}\tau^{-1}) \\ &= (2n-1) + d(x\tau\alpha, x\alpha\tau) \\ &= 6n-1. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(x\alpha, x\alpha\tau\alpha^{-1}\tau^{-1}) &= d(x\alpha, x\kappa^{-p+q}\alpha^{-1}\tau^{-1}\alpha) \\ &= d(x, x\kappa^{-p+q}\alpha^{-1}\tau^{-1}) \leq d(x, x\alpha^{-1}\tau^{-1}) + 1 \\ &= d(x\tau\alpha, x) + 1 \\ &= 2n+1. \end{aligned}$$

Since $n \geq 1$, this is impossible.

Therefore, we may assume that $x = x\tau$. So $x\alpha\tau \in x^-$. Notice that $d(x, x\alpha) = d(x, x\alpha\tau)$, and either

- $x\alpha \in (x\alpha\tau)^-$, or
- $x\alpha\tau$ separates $x\alpha$ and x.

Consider first the case that $x\alpha \in (x\alpha\tau)^-$. Note that since $x\tau = x, x\kappa \neq x$ and $(p,q) = 1$, we have $x\gamma \neq x$. (See Figure 22.) Now $x\alpha \in (x\alpha\tau)^-$ implies that

$$x = (x\alpha)\alpha^{-1}\tau^{-1} \in (x\alpha\tau\alpha^{-1}\tau^{-1})^- = (x\kappa^{-p+q}\alpha^{-1}\tau^{-1}\alpha)^-$$

which implies in turn that

$$x\alpha^{-1} \in (x\kappa^{-p+q}\alpha^{-1}\tau^{-1})^-.$$

This gives

$$x\alpha^{-1} \in (x\alpha^{-1}\tau^{-1})^+.$$

But $x\alpha \in x^-$ implies that

$$x = (x\alpha)(\alpha^{-1}\tau^{-1}) \in (x\alpha^{-1}\tau^{-1})^-.$$

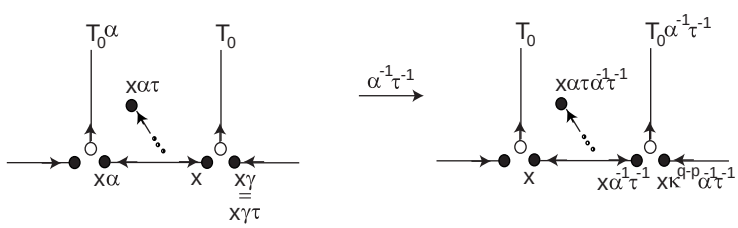

FIGURE 22.

Therefore,

$$\{x\alpha^{-1}\tau^{-1}, x\kappa^{-p+q}\alpha^{-1}\tau^{-1}\} \subset [x\alpha^{-1}, x],$$

and hence

$$d(x, x\alpha^{-1}) > d(x, x\alpha^{-1}\tau^{-1}) = d(x, x\alpha^{-1}),$$

which is impossible.

Hence, necessarily, $x\alpha\tau$ separates $x\alpha$ and x. In particular, $[x, x\alpha]\tau \subset [x, x\alpha]$ and $d(x, x\alpha) = d(x, x\alpha\tau)$ together imply that $d(x\alpha, x\alpha\tau) = 0$. If $X_{j_0}\alpha = X_{j_1} \neq X_{j_0}$, then $\{y, y\alpha\} \subset [x, x\alpha]$ gives $y\tau = y$. So $y \in Fix(\tau) \cap Nonsep(\alpha)$, and by Lemma 8.1, the action of G on T is trivial.

Therefore, we may assume that $X_{j_0}\alpha = X_{j_0}$. Since $d(x\alpha, x\alpha\tau) = 0$, we have

$$d(x\alpha^{-1}\tau, x\gamma\alpha^{-1}) = d((x\alpha)(\alpha^{-1}\tau^{-1}\alpha^{-1}\tau), (x\alpha\tau)(\alpha^{-1}\tau^{-1}\alpha^{-1}\tau)) = 0.$$

Therefore,

$$x\alpha^{-1}\tau \in (x\gamma\alpha^{-1})^{-} \cup T_0\alpha^{-1}.$$

But we also have $d(x\alpha^{-1}\tau, x) = d(x\alpha^{-1}, x)$, and hence $x\alpha^{-1}\tau \in T_0\alpha^{-1}$. But this means $x\alpha^{-1} \in [x, x\alpha^{-1}\tau]$, which gives $x\alpha^{-1}\tau^{-1} \in [x, x\alpha^{-1}]$. Therefore,

$$\begin{aligned} x\alpha\tau &= x\tau^{-1}\alpha\tau \\ &= x\gamma\beta\alpha^m \\ &= x\gamma\tau\alpha^{-1}\tau^{-1}\alpha^m \\ &\sim x\alpha^{-1}\tau^{-1}\alpha^m \in [x\alpha^m, x\alpha^{m-1}]. \end{aligned}$$

Since $x\alpha\tau \in [x, x\alpha]$, this is impossible.

□

9. Appendix: Denjoy blow-ups

9.1. **Denjoy blow-up of a 1-manifold.** We describe a well-known operation from [De32] in which countably many points in a closed subinterval of $\mathbb{R}$ are "blown up" into nondegenerate closed subintervals so as to obtain a new closed subinterval of $\mathbb{R}$. Topologically, it is straightforward to check that this operation is well defined and that it extends to arbitrary (not necessarily Hausdorff) 1-manifolds. For completeness, we do so here.

Let X be any oriented 1-manifold. Let $\mathcal{C} \subset X$ be countable. For every $c \in \mathcal{C}$, let $[c_1, c_2]$ denote an associated closed interval in $\mathbb{R}$, with standard orientation satisfying $c_1 < c_2$. We assume that the intervals $[c_1, c_2], c \in \mathcal{C}$, are pairwise disjoint and disjoint from X. Let Y be the set obtained from X by replacing each $c \in \mathcal{C}$ with the corresponding interval $[c_1, c_2]$, and define a topology on Y as follows.

Let $\{I_x | x \in X\}$ be any oriented basis for X satisfying $I_x \approx \mathbb{R}$ for every $x \in X$. For each $x \in X$, set

$$J_x = (I_x \cap (X \setminus \mathcal{C})) \cup \Big(\bigcup_{c \in \mathcal{C} \cap I_x} [c_1, c_2] \Big),$$

with linear ordering determined uniquely by the following conditions:

- If $y < z$ for some $y, z \in I_x \setminus \mathcal{C}$ or $y, z \in [c_1, c_2]$, for some $c \in \mathcal{C}$, then $y < z$.
- If $c < z$ for some $z \in I_x \setminus \mathcal{C}$ and for some $c \in \mathcal{C} \cap I_x$, then $c_2 < z$.
- If $c > z$ for some $z \in I_x \setminus \mathcal{C}$ and for some $c \in \mathcal{C} \cap I_x$, then $c_1 > z$.

Then, for every $c \in \mathcal{C}$, let $J_c^- = \{x \in J_c | x < c\}$ and let $J_c^+ = \{x \in J_c | c < x\}$, and for every $r \in \mathbb{Q} \cap [c_1, c_2]$, set

$$J_{r,-} = J_c^- \cup [c_1, r)$$

and set

$$J_{r,+} = (r, c_2] \cup J_c^+.$$

Let $\mathcal{B}$ be any countable basis for $\bigcup_{c \in \mathcal{C}} (c_1, c_2)$ consisting of sets homeomorphic to $\mathbb{R}$. Finally, let $\mathcal{T}$ be the topology on Y with basis

$$\mathcal{B} \cup \{J_x | x \in X\} \cup \{J_{r,-} | r \in \mathbb{Q} \cap [c_1, c_2], \exists c \in \mathcal{C}\} \cup \{J_{r,+} | r \in \mathbb{Q} \cap [c_1, c_2], \exists c \in \mathcal{C}\}.$$

Note that if X has countable basis consisting of sets homeomorphic to $\mathbb{R}$, then so does Y. So the space $(Y, \mathcal{T})$ is again an oriented 1-manifold. Moreover, if X is simply-connected, so is Y. Notice that if we remove the requirement that manifolds be second countable, then we may remove the condition that $\mathcal{C}$ be countable in this construction.

Definition 9.1. $(Y, \mathcal{T})$ is called the *Denjoy blow-up of X along $\mathcal{C}$*. If we begin with an action of G on X and extend this action linearly over the intervals $[c_1, c_2], c \in \mathcal{C}$, we call the resulting action of G on Y the *(canonically) induced action*.

Now let

$$\mathcal{C} = \{x \in X | \exists y, z \in [x] \text{ such that } y \not\sim z\}.$$

Since X has countable basis, $\mathcal{C}$ is necessarily countable. Note that $\mathcal{C}$ is the set of points at which $\sim$ fails to be transitive. Since for each point $c \in \mathcal{C}$, $[c]$ splits up into two subsets on which $\sim$ is transitive, we will blow the point c up into a segment $[c_1, c_2]$ which then splits the set $[c] \setminus \mathcal{C}$ into two sets, $[c_1]$ and $[c_2]$, and $\sim$ will be transitive on each of these sets. More precisely, choose an orientation for X, and let X' denote the Denjoy blow-up of X along $\mathcal{C}$.

Lemma 9.2. *The relation $\sim$ is transitive on X'.*

Proof. Let $\mathcal{C}' = \{x \in X' | \exists y, z \in [x] \text{ with } y \not\sim z\}$. We wish to show that $\mathcal{C}' = \emptyset$.

Let $c \in \mathcal{C}$. Consider $y, z \in [c]$ with $y \not\sim z$. Then $\exists y', z'$ such that $[[y', y)) = [[y', c))$ and $[[z', z)) = [[z', c))$ but $[[y', c)) \cap [[z', c)) = \emptyset$. Hence, since X is a 1-manifold, any other set of the form $[[r, c))$ must intersect either $[[y', c))$ or $[[z', c))$ (but of course not both) in some set $[[r', c))$. In addition, since X is an oriented 1-manifold, exactly one of y' and z' is in c^+. Hence if $w \in [c]$, w distinct from y, z, c, then necessarily $\exists w'$ such that $[[w', c)) = [[w', w))$, which implies that $w \sim y$ or $w \sim z$, but not both. Moreover, if $v \in [c]$, v distinct from y, z, c, w, and either $w \sim y$ and $v \sim y$, or $w \sim z$ and $v \sim z$, then $w \sim v$.

Hence $[c] \setminus \mathcal{C}$ splits into two subsets: $[c]_y = \{b \in [c] | b \sim y\}$ and $[c]_z = \{b \in [c] | b \sim z\}$. Moreover, notice that in X', either $[c_1] = [c]_y$ and $[c_2] = [c]_z$ if $y' \in c^+$, or vice versa if $y' \in c^-$. See Figure 23. In particular, $\mathcal{C}' \cap \{c_1, c_2 | c \in \mathcal{C}\} = \emptyset$. But since $|[x]| = 1$ for all $x \in (c_1, c_2)$, for all $c \in \mathcal{C}$, we know that $\mathcal{C}' \subset \{c_1, c_2 | c \in \mathcal{C}\}$. So $\mathcal{C}' = \emptyset$.

$\square$

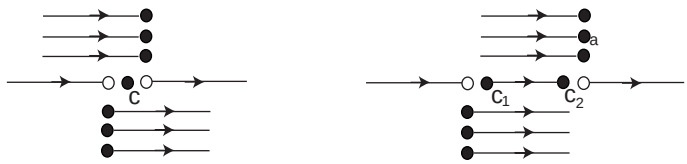

FIGURE 23.

9.2. **Star Denjoy blow-up of an order tree.** We now describe a similar blow-up construction for an $\mathbb{R}$-order tree T which will result in an order tree T' on which the relation $\sim$ is transitive. As in the 1-manifold case, we need to replace the set

$$\mathcal{C} = \{x \in T | \exists y, z \in [x] \text{ such that } y \not\sim z\}.$$

However, since T may not be a 1-manifold, it is no longer the case that for each $x \in \mathcal{C}$, $[x]$ splits up into just two sets on which $\sim$ is transitive. Instead $[x]$ splits into at most countably many such subsets, one for each $T_{\{x,y\}}$ where $y \sim x, y \neq x$. So we replace the point x by a union of segments, one for each such tree $T_{\{x,y\}}$, all identified at exactly one common endpoint into a star shape. Then if we denote the center of the star by x, and the segment $[x, x_y]$ corresponds to the tree $T_{\{x,y\}}$, we define a set of segments $\mathcal{S}'$ for T' in the obvious way so that both x_y and y are limit points of the distinguished ray of the tree $T_{\{x,y\}}$.

If T is oriented, the orientation extends naturally to T'. If we begin with an action of G on T, we may extend to an action on T' in the natural way. This blow-up insures that the relation $\sim$ is transitive on T'.

Definition 9.3. Hence, given an $\mathbb{R}$-order tree T we may define the *Hausdorff tree associated to* T as follows. Set

$$T_H = \{[x] | x \in T'\},$$

and set

$$\mathcal{S}_H = \{[[i(\sigma)], [f(\sigma)]] | \sigma \in \mathcal{S}'\}.$$

Then $(T_H, \mathcal{S}_H)$ is a Hausdorff $\mathbb{R}$-order tree, which we call the Hausdorff tree associated to the oriented $\mathbb{R}$-order tree T. Given an orientation $(S')_+$ for T, we say that the orientation

$$(\mathcal{S}_H)_+ = \{[[i(\sigma)], [f(\sigma)]] | \sigma \in (S')_+\}$$

is the orientation on T_H induced by, or inherited from, the orientation on T. Define

$$p : T \to T_H : x \mapsto [x].$$

9.3. **Linear Denjoy blow-up of an oriented order tree.** Now let T be any oriented $\mathbb{R}$-order tree, with set $\mathcal{S}$ of segments and orientation $\mathcal{S}_+$. Occasionally it is useful to allow the Denjoy blow-up of points to intervals in a way more closely following the construction of Section 9.1. In this construction, the orientation of T plays a crucial role. We proceed as follows. Let $\mathcal{C} \subset T$ be countable. Let $<$ denote the partial order on T induced by the orientation $\mathcal{S}_+$. Again, for every $c \in \mathcal{C}$, let $[c_1, c_2]$ denote an associated closed interval in $\mathbb{R}$, with standard orientation satisfying $c_1 < c_2$. We assume that the intervals $[c_1, c_2], c \in \mathcal{C}$, are pairwise disjoint and disjoint from T. Let Y be the set obtained from T by replacing each $c \in \mathcal{C}$ with the corresponding interval $[c_1, c_2]$, and put an $\mathbb{R}$-order tree structure on Y as follows.

For each $\sigma \in \mathcal{S}_+$, set

$$\sigma' = (\sigma \cap (T \setminus \mathcal{C})) \cup \Big(\bigcup_{c \in \mathcal{C} \cap \sigma} [c_1, c_2] \Big),$$

with linear ordering determined uniquely by the following conditions:

- If $x < y$ for some $x, y \in \sigma \setminus \mathcal{C}$ or $x, y \in [c_1, c_2]$, for some $c \in \mathcal{C}$, then $x < y$.
- If $c < y$ for some $y \in \sigma \setminus \mathcal{C}$ and for some $c \in \mathcal{C} \cap \sigma$, then $c_2 < y$.
- If $c > y$ for some $y \in \sigma \setminus \mathcal{C}$ and for some $c \in \mathcal{C} \cap \sigma$, then $c_1 > y$.

Let $(\mathcal{S}')_+$ be the smallest orientation on Y containing $\{\sigma' | \sigma \in \mathcal{S}_+\}$. Let $\mathcal{S}'$ be the smallest set satisfying the defining axioms of $\mathbb{R}$-order tree and also containing $(S')_+$.

Then $(T', \mathcal{S}')$ with orientation $(\mathcal{S}')_+$ is an oriented $\mathbb{R}$-order tree. Notice that if we remove the requirement that T be second countable, then we may remove the condition that $\mathcal{C}$ be countable in this construction.

Definition 9.4. (T', S') is called the *linear Denjoy blow-up of T along $\mathcal{C}$ with respect to the orientation $\mathcal{S}_+$*. If we begin with an action of G on T and extend this action linearly over the intervals $[c_1, c_2], c \in \mathcal{C}$, we call the resulting action of G on Y the *(canonically) induced action*.

Acknowledgments

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