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THE HYPOELLIPTIC LAPLACIAN ON THE COTANGENT BUNDLE

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Contents

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Introduction

In the present paper, we construct a 1-parameter deformation of classical Hodge theory. For nonzero values of the parameter b, the corresponding Laplacian is a second order hypoelliptic operator on the cotangent bundle, which is in general non-self-adjoint. As $b \to 0$, we recover classical Hodge theory. As $b \to +\infty$, the deformed

Laplacian converges to the generator of the geodesic flow. The constructions are as canonical as Hodge theory itself.

We will now explain the motivation which underlies these constructions. We will use freely the language of path integrals, since they will provide us with the proper geometric understanding of what is being done, if not always with adequate tools to make these ideas effective.

Let X be a compact connected Riemannian manifold of dimension n. In [W82], Witten explained how to give an analytic proof of the Morse inequalities using a deformation of the de Rham complex of X. Indeed if f is a Morse function and if $T \in \mathbf{R}$, Witten replaced the de Rham operator d^X by its twist $d_T^X = e^{-Tf} d^X e^{Tf}$ and showed that as $T \to +\infty$, the harmonic forms of the corresponding Laplacian localize near the critical points of f. He argued that as $T \to +\infty$, the subcomplex associated to asymptotically small eigenvalues could be identified with a finite dimensional complex associated to the instanton gradient lines connecting the critical points, which was later identified to be the Thom-Smale complex of the gradient field ∇f for a generic metric g^{TX}. The identification of the small eigenvalues complex to the Thom-Smale complex was established rigorously in [HSj85]. Moreover, in [BZ92, BZ94], these ideas were used to give a new proof of the Cheeger-Müller theorem [C79, Mü78] which gives the equality of the Reidemeister torsion [Re35] with the Ray-Singer analytic torsion [RS71].

A first motivation for this paper is to try to extend this line of arguments to LX, the space of smooth loops in X parametrized by $S_1 \simeq \mathbf{R}/\mathbf{Z}$. Indeed let g^{TX} be the given Riemannian metric on X, and let $V : X \to \mathbf{R}$ be a smooth function. For $a > 0$, consider the Lagrangian

$$L_a(x, \dot{x}) = \frac{a}{2} |\dot{x}|^2 - V(x) \tag{0.1}$$

and the corresponding functional on LX

$$I_a(x) = \int_0^1 L_a(x, \dot{x})\, ds. \tag{0.2}$$

When $a = 1$, we will use the notation $L = L_a, I = I_a$. When $a = 0, V = -f$ and f is Morse, the critical points of I_0 are exactly the trivial loops which are the critical points of f. When $V = 0$, I is the energy, and the set FLX of closed geodesics in X is the critical set of I. Morse theory on LX for the energy functional was the key tool in the proof by Bott [Bo59] of Bott periodicity. The obvious obstacle to the naive extension of Witten's deformation principle to LX is that LX is infinite dimensional and that the construction of the corresponding Hodge Laplacian is notoriously difficult, not to mention the proof of a Hodge theorem itself. On LX, it seems that there is no possibility to even start a Witten deformation.

Critical input in the positive direction came from the results we obtained jointly with Goette in [BG04]. In that paper, we showed that the difference between two natural versions of the equivariant Ray-Singer torsion can be expressed in terms of a new invariant, the V-invariant of a compact manifold equipped with a group action. Indeed assume that X is equipped with an isometric action of a compact Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G. Then if $K \in \mathfrak{g}$, $V_K(X) \in \mathbf{R}$ is an invariant of (X, K) which is computable locally. It vanishes when X is even dimensional. If f is a G-invariant Morse-Bott function on X with critical manifold $B \subset X$, then a simple formula relates $V_K(X)$ and $V_K(B)$. The main result of [BG04] shows that the difference of the analytic torsions evaluated at $K \in \mathfrak{g}$ can be expressed in terms

of $V_K(X)$. Let us observe at this stage that the compatibility of the main formula of [BG04] to the Cheeger-Müller theorem forces $V_K(X)$ to behave the way it does with respect to invariant Morse-Bott functions.

However, when working with Goette on [BG04], the author discovered more, at least at a formal level. Namely let $K = \dot{x}$ be the canonical vector field on LX which generates the obvious action of S_1. The natural L_2 metric on TLX is S_1-invariant. The author observed that at least formally, the Ray-Singer torsion $T(X)$ of the trivial vector bundle on X is given by

$$T(X) = V_K(LX). \tag{0.3}$$

Let us derive a simple formal consequence of (0.3). In fact if X is even dimensional and oriented, there are formal reasons to say that in this case LX is even dimensional, for instance by inspection of the spectrum of the covariant derivative $\frac{D}{Dt}$ along a loop LX acting on smooth periodic vector fields. Since this operator is antisymmetric, its nonzero spectrum comes by distinct pairs, and its kernel is even dimensional because TX is oriented. Previously described properties of the V-invariant should imply that T vanishes identically, which is indeed correct by Poincaré duality. Note that the above considerations also extend to general flat vector bundles. This statement has to be qualified in the sense that anything related to cohomology, or equivalently to high temperature and large loops, tends to be difficult to see in this formal analogy.

More generally, the author also found out that the analytic torsion forms he constructed with Lott [BLo95] are also versions of the V-invariant (which in general produces even cohomology classes on the base of a proper fibration).

But more is true. In his hidden agenda, the author also found that the proofs he gave with Zhang [BZ92, BZ94] of the Cheeger-Müller theorem are exactly the infinite dimensional analogues of his proof with Goette [BG04] on the behaviour of the V-invariant with respect to invariant Morse-Bott functions, when considering instead the invariant Morse function I_0 in (0.2), with $V = -f$, so that

$$I_0(x) = \int_0^1 f(x_s)\, ds. \tag{0.4}$$

This statement has to be qualified again. Instanton effects between critical points of f are invisible in any formal statement about V-invariants. The above analogy also extends to the case of fibrations. In this case the corresponding results for analytic torsion forms were established in [BG01].

It is then natural to try extending the above approach to the more general functionals I_a, and more specifically to the functional I with $V = 0$, which is the energy, whose critical set FLX consists of closed geodesics. Indeed assume that I is a Morse-Bott function, which happens typically on spheres, on tori, on manifolds with strictly negative curvature (in which case the closed geodesics are isolated) and on symmetric spaces. If, admittedly, the Ray-Singer torsion is the V-invariant of LX, it should localize on the critical manifold FLX of I. More precisely, one should express the Ray-Singer torsion as the V-invariant of the manifold FLX. The predictions one can infer from such a formal approach are stunning, at least at a formal level. Indeed easy computations reveal that such a statement is precisely contained in assertions made by Fried [Fr86, Fr88], which Fried himself proved in a number of cases, relating the Reidemeister torsion to the value at 0 of the dynamical zeta function associated to the geodesic flow. Fried's assertions were

proved for general symmetric spaces by Moscovici and Stanton [MoSt91] using the Selberg trace formula. Again the fact that the predictions one gets from a formal approach to the V-invariant on LX has to be qualified. Indeed it is a nontrivial fact that the dynamical zeta function extends holomorphically at $s = 0$. The above formal approach sidesteps this difficult question, at least to first approximation.

It follows in particular from the above considerations that one can think of the Cheeger-Müller theorem on one hand and of the Fried-Moscovici-Stanton results on the other hand as two versions of the same 'result' of localization of the V-invariant. The question which we will address in the present paper is to try building the proper objects which will eventually lead to a proof of both results with a similar proof.

Now we will explain how one can approach this question, first from the point of view of the path integral, which leads to geometrically easily understandable considerations, and from the analytic point of view.

Indeed assume temporarily that Y is a smooth manifold and that $\left(E, g^E, \nabla^E\right)$ is a real Euclidean vector bundle equipped with a Euclidean connection. In [MaQ86], Mathai and Quillen have produced Gaussian representatives Φ^E of the Thom class using the Berezin integral formalism. If s is any smooth section of E, then $s^*\Phi\left(E, g^E, \nabla^E\right)$ is a closed differential form on Y which represents the Euler class of E. This form is difficult to write explicitly. We just give its leading term in the form

$$s^*\Phi\left(E, g^E, \nabla^E\right) = \exp\left(-\frac{1}{2}\left|s\right|^2 + \dots\right). \tag{0.5}$$

Scaling s by a parameter $T \in \mathbf{R}$, we thus get a family of closed differential forms a_T on E. When s is generic, a_T interpolates between the Chern-Gauss-Bonnet form of E for $T = 0$ and the current of integration on the zero set $Z \subset Y$ of s for $T = +\infty$.

Now if $V = -f$, the gradient ∇I_0 is given by

$$\nabla I_0 = \int_0^1 \nabla f\left(x_s\right) ds, \tag{0.6}$$

so that

$$\left|\nabla I_0\right|^2 = \int_0^1 \left|\nabla f\left(x_s\right)\right|^2 ds. \tag{0.7}$$

Let g^{TLX} be the obvious lift of g^{TX} to an S_1-invariant metric on TLX, and let ∇^{TLX} be the corresponding Levi-Civita connection on TLX. Inspection of the proof in [BZ92] shows that introducing the Witten twist is equivalent to producing the Mathai-Quillen form $\left(T\nabla I_0\right)^*\Phi\left(TLX, g^{TLX}, \nabla^{TLX}\right)$ on LX. Scaling f by T then forces localization on the critical points of f. Indeed the leading (bosonic) term in the path integral is given by

$$\exp\left(-\frac{1}{2}\left|K\right|^2 - \frac{T^2}{2}\left|\nabla I_0\right|^2 + \dots\right). \tag{0.8}$$

The energy $\left|K\right|^2$ in (0.8) reflects the presence of the classical Laplacian Δ^X in Witten's Laplacian.

At this stage, the obvious thing to do is to replace I_0 by I in the path integral over LX and to construct the form $a_T = \left(T\nabla I\right)^*\Phi\left(TLX, g^{TLX}, \nabla^{TLX}\right)$ on LX. Now observe that if $V = -f$,

$$\nabla I = -\ddot{x}_s + \nabla f\left(x_s\right). \tag{0.9}$$

If we replace ∇I_0 by ∇I in (0.8), we find that if $f = 0$, the leading term is

$$\exp\left(-\frac{1}{2}\int_0^1 |\dot{x}|^2 - \frac{T^2}{2}\int_0^1 |\ddot{x}|^2\, ds \ldots\right). \tag{0.10}$$

By (0.10), we can infer that for $T > 0$, the path integral will be supported by paths of finite energy, which never happens for $T = 0$. Moreover the Markov property of the corresponding process will be preserved only if one considers the process $(x_s, p_s) = (x_s, \dot{x}_s)$, which should be thought of as living in T^*X. The corresponding process looks like a physical Brownian motion, whose speed $\dot{x}$ is an Ornstein-Uhlenbeck process and whose generator is no longer self-adjoint. More precisely, if w is a n-dimensional Brownian motion, the dynamics of x is described by the stochastic differential equation

$$\ddot{x} = \frac{1}{T}\left(-\dot{x} + \dot{w}\right). \tag{0.11}$$

Observe that when $T \to 0$, equation (0.11) becomes $\dot{x} = \dot{w}$, i.e. x is a classical Brownian motion, and when $T \to +\infty$, it degenerates into $\ddot{x} = 0$, which is the equation of geodesics in X. The dynamics associated to (0.11) interpolates between classical Brownian motion and the geodesic flow.

The question then arises of how to produce effectively the analogue of the Witten twist of the de Rham operator d^X on X associated to the functional I, so that ultimately we produce the right path integral on LX. This is a much more modest objective than to produce the genuine Witten deformation of d^{LX} on LX. Producing this twist is precisely what we will do in the present paper. It should be clear from the above that one can hope to produce such a deformation only by working directly on the cotangent bundle T^*X and by considering instead the de Rham operator d^{T^*X}. Introducing a classical Witten twist by $e^{|p|^2/2}$ is natural in this context, but it will certainly not produce a path integral like (0.10). So we have to work with other objects than functions on T^*X. However the other natural structure which is at our disposal is its symplectic form ω.

Indeed if E is a Euclidean vector space, for $b \in \mathbf{R}$, consider the bilinear form ϕ_b on $E \oplus E^*$ given by

$$\phi_b = \begin{pmatrix} 1 & -b \\ b & 0 \end{pmatrix}. \tag{0.12}$$

Note that for $b \neq 0$, it is nondegenerate. The parameter b gives a deformation of the symplectic form of $E \oplus E^*$. The idea is then to take the 'adjoint' of d^{T^*X} with respect to ϕ_b acting on $TX \oplus T^*X$. The obvious Laplacian associated to ϕ_b is the vertical Laplacian along the fibre Δ^V, the contribution of b being invisible, cross derivatives in x, p disappearing by antisymmetry. However interaction with the twist by $e^{c|p|^2/2}$ (with $c = \pm 1/b^2$) produces ultimately a Laplacian $\mathfrak{L}_b$ on T^*X which is such that $\frac{\partial}{\partial u} - \mathfrak{L}_b$ is hypoelliptic. Inspection of the corresponding path integral shows that it produces exactly the path integral suggested in (0.10), with $T = b^2$. The construction of the Laplacian $\mathfrak{L}_b$ is as canonical as the construction of the standard Laplacian of X and can even look deceptively simpler. As the path integral suggests, it interpolates in the appropriate sense between the scaled standard Laplacian $\Box^X/2$ of X for $b = 0$ and the geodesic flow of T^*X for $b = +\infty$. The Laplacian $\mathfrak{L}_b$ is no longer classically self-adjoint, but it is still self-adjoint with respect to a nonpositive Hermitian form. In the case of S_1, up to scaling, when

acting on functions,

$$\mathfrak{L}_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial p^2} + 2p\frac{\partial}{\partial p}\right) - \frac{1}{b}p\frac{\partial}{\partial x}. \tag{0.13}$$

In the case $X = S_1$, the operator $\frac{\partial}{\partial u} - \mathfrak{L}_b$ is essentially the operator considered by Kolmogorov in [Ko34] in his study of certain random processes. This operator is the typical example to which the Hörmander theorem on second order differential operators [Hö67] can be applied.

The purpose of this paper is to present in detail the construction of the hypoelliptic Laplacian, to exhibit its relation to standard Hodge theory, and to demonstrate its potential as a deformation object which connects standard Hodge theory to the geodesic flow. Its effective applications in the geometric context are deferred to joint work with Lebeau [BL05].

The construction is obtained using the superconnection formalism of Quillen [Q85a]. We also treat families of manifolds, since experience has repeatedly shown that family considerations are useful, even when ultimately considering the case of one single manifold. In that context, earlier work by Lott and the author [BLo95] is used as a model to construct our deformed theory.

We have tried to make connections with path integrals as transparent as possible, even while not providing the supporting analytic details. Finally let us point out that S_1 indeed plays a special role in the story. Indeed, since our ultimate purpose is to deform a manifold into closed geodesics, one should expect the circle to be a fixed point of this deformation. That this is indeed the case is shown in the proper sense in subsection 3.10.

This paper is organized as follows. In Section 1, we develop the finite dimensional Hodge theory for finite dimensional complexes equipped with Hermitian forms which are not necessarily positive. Few results of classical Hodge theory survive, but these are enough to develop the corresponding machinery to construct the associated generalized Ray-Singer torsion and Quillen metrics. The reason for inclusion of this section is that our generalized Laplacians will precisely be associated to nonpositive Hermitian forms on the de Rham complex.

In Section 2, we construct a natural deformation of the first order elliptic operator $\frac{1}{2}\left(d^X + d^{X*}\right)$, as a first order nonelliptic operator over T^*X. In Section 3, we give a Weitzenböck formula for our deformed Laplacian, and we show that Hörmander's theorem can be applied to this operator. We relate this new Laplacian to Mathai and Quillen's formulas for Thom forms, and we show that it interpolates between the standard Laplacian and the geodesic flow. Also the case of S_1 receives special attention.

Finally in Section 4, we apply the above constructions in the context of families, the model construction being our earlier work with Lott [BLo95]. Special attention is given to the proof of Weitzenböck's formulas in various forms. Indeed the construction is still a mixture of Riemannian and symplectic geometry. The motivation underlying the manipulation of these formulas is connected to future applications to local index theory.

The application of this machine to Hodge theory and generalized Ray-Singer metrics is now the object of work in progress with Lebeau [BL05].

In the entire paper, if $\mathcal{A}$ is a $\mathbf{Z}_2$-graded algebra, if $A, A' \in \mathcal{A}$, then $[A, A']$ denotes the supercommutator of A and A'.

The results which are contained in this paper have been announced in [B04b, B04c, B04d], and a survey of the motivation and results has been given in [B04e].

1. Generalized metrics and determinants

In this section, we develop elementary aspects of Hodge theory associated to finite dimensional complexes equipped with Hermitian forms of arbitrary signature. One of the main results of this theory is that, in general, the harmonic forms produce a subcomplex out of which one can extract the cohomology. It may even happen that the corresponding Laplacian is identically 0, which makes the general theory differ dramatically from standard Hodge theory, even in finite dimensions.

The consequences of this theory are quite important when applied to de Rham complexes, as will be established in later work with Lebeau [BL05]. We have included this section in the present paper so as to give some idea of what difficulties are involved when dealing with arbitrary Hermitian forms. In Sections 2 and 3, we will prove that our deformed Hodge theory fits in the algebraic framework developed in this section.

This section is organized as follows. In subsection 1.1, we recall briefly some properties of determinants. In subsection 1.2, we consider a finite dimensional complex (E, ∂), equipped with a Hermitian form g^E, which we will also call a generalized metric. In subsection 1.3, we develop the Hodge theory associated to g^E. Among these, we distinguish the generalized metrics of Hodge type, which are the ones for which the Hodge theorem holds. In subsection 1.4, we construct a corresponding generalized metric on $\det E$. In subsection 1.5, we compute explicitly this generalized metric in terms of the corresponding analytic torsion in the sense of Ray and Singer [RS71]. In subsections 1.6 and 1.7, we develop a simple theory of the determinant bundle by imitating Quillen [Q85b], i.e. by truncating the spectrum of the Laplacian. In subsections 1.8–1.10, we apply this construction to flat complexes. In subsection 1.11, we prove that if a family of generalized metrics depends analytically on $s \in \mathbf{R}$, if the set where the Hodge condition holds is nonempty, its complement is discrete. Finally in subsection 1.12, we consider equivariant determinants in the sense of [B95].

1.1. **Determinants.** If λ is a complex line bundle, let λ^{-1} be the corresponding dual line. If V is a complex vector space, put

$$\det V = \Lambda^{\max}(V). \tag{1.1}$$

Let V^* be the dual of V. Then

$$\det V^* = (\det V)^{-1}. \tag{1.2}$$

Let

$$(E^{\cdot}, \partial) : 0 \to E^0 \xrightarrow{\partial} \ldots \xrightarrow{\partial} E^n \to 0 \tag{1.3}$$

be a complex of finite dimensional complex vector spaces. Here $E^{\cdot} = \bigoplus_{0 \le i \le n} E^i$. Let $H^{\cdot}(E^{\cdot}) = \bigoplus_{0 \le i \le n} H^i(E^{\cdot})$ be the cohomology of $(E^{\cdot}, \partial)$. Put

$$\det E^{\cdot} = \bigotimes_{0 \le i \le n} \left(\det E^i\right)^{(-1)^i}, \quad \det H^{\cdot}(E^{\cdot}) = \bigotimes_{0 \le i \le n} \left(\det H^i(E^{\cdot})\right)^{(-1)^i}. \tag{1.4}$$

Then by [KM76], there is a canonical isomorphism

$$\det E^{\cdot} \simeq \det H^{\cdot}(E^{\cdot}). \tag{1.5}$$

It should be pointed out that in [KM76], the considered objects are pairs (λ, ϵ), with $\epsilon = \pm 1$, and that the canonical isomorphisms involve the ϵ explicitly. Here we will mostly disregard these questions of signs.

1.2. **A generalized metric on $E^{\cdot}$.** Let $h^{E^{\cdot}} = \bigoplus_{0 \leq i \leq n} h^{E^i}$ be a Hermitian form on $E^{\cdot} = \bigoplus_{0 \leq i \leq n} E^i$, which is assumed to be nondegenerate. Then the map

$$(a, b) \in E^{\cdot} \times E^{\cdot} \to \langle a, b \rangle_{h^{E^{\cdot}}} \tag{1.6}$$

is sesquilinear, that is, it is linear in a, antilinear in b, the E^i are mutually orthogonal in $E^{\cdot}$, and the pairing in (1.6) is nondegenerate. Moreover

$$\langle b, a \rangle_{h^{E^{\cdot}}} = \overline{\langle a, b \rangle}_{h^{E^{\cdot}}}. \tag{1.7}$$

Then h^E defines a $\mathbf{Z}$-graded isomorphism $E \to \overline{E}^{\cdot *}$, which we also denote by h^E. In the sequel, we will omit the subscript $h^{E^{\cdot}}$ in $\langle\ \rangle_{h^{E^{\cdot}}}$.

The above $h^{E^{\cdot}}$ will be called generalized metrics. Indeed classical Hermitian metrics are special cases of the $h^{E^{\cdot}}$.

If $A \in \mathrm{End}\,(E^{\cdot})$, A^* denotes its adjoint with respect to $h^{E^{\cdot}}$. Namely if $a, b \in E^{\cdot}$,

$$\langle Aa, b \rangle = \langle a, A^* b \rangle. \tag{1.8}$$

Then

$$(A^*)^* = A. \tag{1.9}$$

1.3. **The Hodge theory of $h^{E^{\cdot}}$.** Clearly $(E^{-\cdot}, \partial^*)$ is a complex.

Put

$$D = \partial + \partial^*. \tag{1.10}$$

Then D is an $h^{E^{\cdot}}$ self-adjoint odd endomorphism of $E^{\cdot}$. Moreover

$$D^2 = [\partial, \partial^*] \tag{1.11}$$

is the corresponding generalized Laplacian, which is also $h^{E^{\cdot}}$ self-adjoint. Of course, D^2 commutes with ∂ and with ∂^*.

In the sequel, Sp is our notation for the spectrum of an operator.

Proposition 1.1. *The spectra* $\mathrm{Sp} D$ *and* $\mathrm{Sp} D^2$ *are conjugation invariant.*

Proof. Let $\widetilde{D} \in \mathrm{End}\,(E^{\cdot *})$ be the transpose of D. Then since D is $h^{E^{\cdot}}$ self-adjoint,

$$D = \left(h^{E^{\cdot}}\right)^{-1} \overline{\widetilde{D}} h^{E^{\cdot}}. \tag{1.12}$$

Our proposition now follows from (1.12). □

Let

$$E^{\cdot} = \bigoplus_{\lambda \in \mathrm{Sp} D^2} E^{\cdot}_{\lambda} \tag{1.13}$$

be the decomposition of $E^{\cdot}$ into the Jordan blocks associated to D^2. Note that since ∂ commutes with D^2, the $E^{\cdot}_{\lambda}$ are indeed subcomplexes of $E^{\cdot}$. Moreover the Jordan filtration on the $E^{\cdot}_{\lambda}$ is a filtration of complexes.

We will consider in particular the subcomplex $(E^{\cdot}_0, \partial)$. Note that for $k \in \mathbf{N}$ large enough,

$$E^{\cdot}_0 = \ker \left(D^2\right)^k. \tag{1.14}$$

If $h^{E^{\cdot}}$ is a Hermitian metric, the complex (E_0, ∂) is trivial, i.e. $\partial = 0$.

Set

$$E_*^{\cdot} = \bigoplus_{\lambda \neq 0} E_\lambda^{\cdot}. \tag{1.15}$$

Then the complex $E^{\cdot}$ splits as

$$E^{\cdot} = E_0^{\cdot} \oplus E_*^{\cdot}. \tag{1.16}$$

Observe that for $k \in \mathbf{N}$ large enough,

$$E_*^{\cdot} = \operatorname{Im}\left(D^2\right)^k. \tag{1.17}$$

Therefore, for $k \in \mathbf{N}$ large enough, the splitting (1.16) of the complex $E^{\cdot}$ is just the splitting

$$E^{\cdot} = \ker\left(D^2\right)^k \oplus \operatorname{Im}\left(D^2\right)^k. \tag{1.18}$$

Theorem 1.2. *The complex $(E_*^{\cdot}, \partial)$ is acyclic. In particular*

$$H^{\cdot}(E) = H^{\cdot}(E_0^{\cdot}). \tag{1.19}$$

The vector spaces $E_0^{\cdot}$ and $E_*^{\cdot}$ are orthogonal with respect to $h^{E^{\cdot}}$, and the restrictions of $h^{E^{\cdot}}$ to $E_0^{\cdot}$ and $E_*^{\cdot}$ are nondegenerate. Moreover,

$$E_*^{\cdot} = \operatorname{Im}\partial|_{E_*^{\cdot}} \oplus \operatorname{Im}\partial^*|_{E_*^{\cdot}}, \tag{1.20}$$

and the decomposition (1.20) *is $h^{E^{\cdot}}$ orthogonal.*

Proof. Clearly D^2 acts as an invertible operator on $E_*^{\cdot}$, so that on $E_*^{\cdot}$,

$$1 = \left[\partial, \partial^*\left(D^2\right)^{-1}\right]. \tag{1.21}$$

From (1.21), we deduce that $(E_*^{\cdot}, \partial)$ is acyclic. Using (1.16), we get (1.19).

Take $a \in E_0^{\cdot}, b \in E^{\cdot}$. Using (1.14) and the fact that D^2 is $h^{E^{\cdot}}$ self-adjoint, we get

$$\left\langle a, \left(D^2\right)^k b\right\rangle = 0. \tag{1.22}$$

Since $\left(D^2\right)^k$ acts as an invertible map on $E_*^{\cdot}$, we deduce that $E_0^{\cdot}$ and $E_*^{\cdot}$ are mutually orthogonal in $E^{\cdot}$ with respect to $h^{E^{\cdot}}$. Since $h^{E^{\cdot}}$ is nondegenerate, its restrictions to $E_0^{\cdot}$ and $E_*^{\cdot}$ are also nondegenerate.

By (1.21), $\operatorname{Im}\partial|_{E_*^{\cdot}}$ and $\operatorname{Im}\partial^*|_{E_*^{\cdot}}$ span $E_*^{\cdot}$. Moreover these vector spaces are $h^{E^{\cdot}}$ orthogonal. Since the restriction of $h^{E^{\cdot}}$ to $E_*^{\cdot}$ is nondegenerate, we get (1.20). The proof of our theorem is completed. □

Remark 1.3. To prove that in (1.20), $\operatorname{Im}\partial|_{E_*^{\cdot}} \cap \operatorname{Im}\partial^*|_{E_*^{\cdot}} = 0$, note that this intersection lies in $\ker D^2|_{E_*^{\cdot}}$ and that this last vector space is reduced to 0.

Definition 1.4. We will say that the generalized metric $h^{E^{\cdot}}$ is of Hodge type if

$$E_0^{\cdot} = \ker\,\partial \cap \ker\,\partial^*. \tag{1.23}$$

Theorem 1.5. *A generalized metric $h^{E^{\cdot}}$ is of Hodge type if and only if*

$$E_0^{\cdot} = \ker D \tag{1.24}$$

or if

$$\dim E_0^{\cdot} = \dim H^{\cdot}(E^{\cdot}). \tag{1.25}$$

In this case,

$$E_0^{\cdot} = \ker D = \ker D^2, \tag{1.26}$$

and moreover, we have the canonical isomorphism,

$$E_0^{\cdot} \simeq H^{\cdot}\left(E^{\cdot}\right). \tag{1.27}$$

A generalized metric $h^{E^{\cdot}}$ is of Hodge type if and only if

$$E^{\cdot} = (\ker \partial \cap \ker \partial^*) \oplus \operatorname{Im} \partial \oplus \operatorname{Im} \partial^*. \tag{1.28}$$

The splitting in (1.28) *is $h^{E^{\cdot}}$ orthogonal, and moreover,*

$$E_0^{\cdot} = \ker \partial \cap \ker \partial^*, \qquad E_*^{\cdot} = \operatorname{Im} \partial \oplus \operatorname{Im} \partial^*. \tag{1.29}$$

Proof. Clearly $\ker D \subset E_0^{\cdot}$. If $f \in E_0^i$ and if $Df = 0$, then $\partial f = 0, \partial^* f = 0$. So we have shown that $h^{E^{\cdot}}$ is of Hodge type if and only if (1.24) holds. If (1.23) holds, the chain map of the complex $E_0^{\cdot}$ is 0, so that using (1.19),

$$E_0^{\cdot} = H^{\cdot}\left(E^{\cdot}\right), \tag{1.30}$$

and so (1.25) holds. Conversely if (1.25) holds, by (1.19), the restriction of ∂ to $E_0^{\cdot}$ vanishes, and so the restriction of ∂^* to $E_0^{\cdot}$ also vanishes. Moreover, for any $k \in \mathbf{N}$ large enough,

$$E_0^{\cdot} \subset \{f \in E^{\cdot}, \partial f = 0, \partial^* f = 0\} \subset \ker D \subset \ker D^2 \subset \ker \left(D^2\right)^k = E_0^{\cdot}. \tag{1.31}$$

Equation (1.26) follows from (1.31).

If $h^{E^{\cdot}}$ is of Hodge type, by (1.16) and (1.20), we get (1.28). Conversely, if (1.28) holds, the splitting in (1.28) is $h^{E^{\cdot}}$ orthogonal. In particular,

$$\ker \partial = (\ker \partial \cap \ker \partial^*) \oplus \operatorname{Im} \partial, \qquad \ker \partial^* = (\ker \partial \cap \ker \partial^*) \oplus \operatorname{Im} \partial^*. \tag{1.32}$$

By (1.32), we find that D^2 acts as an invertible operator on $\operatorname{Im} \partial \oplus \operatorname{Im} \partial^*$ and that $\ker D^2 = \ker \partial \cap \ker \partial^*$. The proof of our theorem is completed. □

1.4. **A generalized metric on** $\det E^{\cdot}$**.** Note first that if λ is a complex line, $\lambda^{-1} \otimes \overline{\lambda}^{-1}$ is a real line. Also note that there is a canonical isomorphism

$$\left(\lambda^{-1} \otimes \overline{\lambda}^{-1}\right)^* / \mathbf{R}_+^* \simeq \{-1, +1\}. \tag{1.33}$$

Equivalently, nonzero sections of $\lambda^{-1} \otimes \overline{\lambda}^{-1}$ have a sign. In particular metrics on λ have a sign equal to $+1$.

For $0 \leq i \leq n$, $\det h^{E^i}$ is a nonzero real section of the real line $\left(\det E^i\right)^{-1} \otimes \left(\det \overline{E}^i\right)^{-1}$. Let $\operatorname{sign} h^{E^i}$ be the signature of h^{E^i}. Then the sign of $\det h^{E^i}$ in $\left(\det E^i\right)^{-1} \otimes \left(\det \overline{E}^i\right)^{-1}$ is just $(-1)^{\operatorname{sign} h^{E^i}}$. Set

$$\det h^{E^{\cdot}} = \bigotimes_{i=0}^{n} \left(\det h^{E^i}\right)^{(-1)^i}. \tag{1.34}$$

Then $\det h^{E^{\cdot}}$ is a nonzero real section of $\left(\det E^{\cdot}\right)^{-1} \otimes \left(\det \overline{E}\right)^{-1}$, whose sign is $(-1)^{\sum_{i=1}^n \operatorname{sign} h^{E^i}}$. If the h^{E^i} are ordinary Hermitian products, the sign is $+1$.

By writing $h^{E^{\cdot}}$ in diagonal form, there is a Hermitian metric $g^{E^{\cdot}} = \bigoplus_{0\leq i\leq n} g^{E^i}$ on $E^{\cdot} = \bigoplus_{0\leq i\leq n} E^i$ and a $g^{E^{\cdot}}$ self-adjoint involution u of $E^{\cdot}$ preserving the $\mathbf{Z}$-grading, such that

$$h^{E^{\cdot}} = g^{E^{\cdot}} u. \tag{1.35}$$

In general, the involution u does not commute with ∂. Also observe that for any $i, 0 \leq i \leq n$, $\det u|_{E^i} = \pm 1$. More precisely,

$$\det u|_{E^i} = (-1)^{\operatorname{sign} h^{E^i}}. \tag{1.36}$$

Set

$$\det u|_E = \prod_{0\leq i\leq n} \left(\det u|_{E^i}\right)^{(-1)^i}. \tag{1.37}$$

By (1.36), $\det u = \pm 1$ depends only on $h^{E^{\cdot}}$.

By (1.35), we get

$$\det h^{E^{\cdot}} = \det u|_E \det g^{E^{\cdot}}. \tag{1.38}$$

By (1.36) or by (1.38), we find that $\det u|_E = \pm 1$ is just the sign of $\det h^E$.

If $A \in \operatorname{End}(E)$, let $A^\dagger$ be the $g^{E^{\cdot}}$ adjoint of A. Clearly

$$A^* = uA^\dagger u. \tag{1.39}$$

By (1.11) and (1.39), we get

$$D^2 = \left[\partial, u\partial^\dagger u\right]. \tag{1.40}$$

Proposition 1.6. *The operator uD^2 is $g^{E^{\cdot}}$ self-adjoint.*

Proof. By (1.40),

$$\left(D^2\right)^\dagger = \left[\partial^\dagger, u\partial u\right] = uD^2u, \tag{1.41}$$

which establishes our proposition. □

Finally, we should observe that the sign ambiguity which was mentioned at the end of subsection 1.1 has nothing to do with the question of signs which is mentioned here.

In the sequel if $a \in \det E^{\cdot}$, we will use the notation

$$\|a\|^2_{\det E^{\cdot}} = \left\langle a \otimes \overline{a}, \det h^{E^{\cdot}} \right\rangle, \tag{1.42}$$

so that we use the same notation as in the case where $\det h^{E^{\cdot}}$ is a standard Hermitian metric. Note that if a is nonzero, $\|a\|^2_{\det E^{\cdot}} \in \mathbf{R}_*$. The object $\| \, \|^2_{\det E^{\cdot}}$ will be called a generalized metric on $\det E^{\cdot}$. This notation should be used with care—the square does not imply any idea of positivity.

In the sequel we also use the notation

$$\epsilon\left(\| \, \|^2_{\det E^{\cdot}}\right) = \det u|_E, \tag{1.43}$$

which is equal to ± 1 and is just the sign of $\det h^{E^{\cdot}}$.

1.5. **Determinant of the generalized Laplacian and generalized metrics.** Let $h^{E_0^{\cdot}}$ be the Hermitian form on $E_0^{\cdot}$ induced by $h^{E^{\cdot}}$. As we saw in Theorem 1.2, this form is nondegenerate, i.e. $h^{E_0^{\cdot}}$ is a generalized metric on $E_0^{\cdot}$. Therefore we obtain a generalized metric $\det h^{E_0^{\cdot}}$ on $\det E_0^{\cdot}$.

By (1.16) and by Theorem 1.2, we have an exact sequence of complexes

$$0 \to E_0^{\cdot} \to E^{\cdot} \to E^{\cdot}/E_0^{\cdot} \to 0, \tag{1.44}$$

and moreover $E^{\cdot}/E_0^{\cdot} \simeq E_*^{\cdot}$ is acyclic. By [KM76], $(\det E_*^{\cdot})^{-1}$ has a nonzero canonical section $\det \partial|_{E_*^{\cdot}}$. Via this nonzero section, we obtain the canonical identification which is a special case of (1.5),

$$\det E_*^{\cdot} \simeq \mathbf{C}. \tag{1.45}$$

By [KM76], there is a canonical isomorphism

$$\det E^{\cdot} \simeq \det E_0^{\cdot}. \tag{1.46}$$

This canonical isomorphism is just given by

$$s \in \det E_0^{\cdot} \to s \otimes \left(\det \partial|_{E_*^{\cdot}}\right)^{-1} \in \det E^{\cdot}. \tag{1.47}$$

On the other hand by (1.5) and (1.19), we know that

$$\det E^{\cdot} \simeq \det H^{\cdot}\left(E^{\cdot}\right), \qquad \det E_0^{\cdot} \simeq \det H^{\cdot}\left(E^{\cdot}\right). \tag{1.48}$$

The canonical isomorphism in (1.46) is in fact the isomorphism one obtains from (1.48).

From (1.44), we see that the generalized metrics $\det h^{E^{\cdot}}$ and $\det h_0^{E^{\cdot}}$ are in fact generalized metrics on the same line $\det E^{\cdot}$. One can then try to compare them.

Definition 1.7. Let $S\left(E, h^{E^{\cdot}}\right) \in \mathbf{R}_*$ be given by

$$S\left(E, h^{E^{\cdot}}\right) = \prod_{i=1}^{n} \left(\det D^2|_{E_*^i}\right)^{(-1)^i i}. \tag{1.49}$$

The number $S\left(E, h^{E^{\cdot}}\right)$ will be called the generalized analytic torsion of the complex (E, ∂).

Proposition 1.8. *The following identity holds:*

$$\left\|\left(\det \partial|_{E_*^{\cdot}}\right)^{-1}\right\|^2_{\det E_*^{\cdot}} = S\left(E, h^{E^{\cdot}}\right). \tag{1.50}$$

Proof. By Theorem 1.2, for $0 \le i \le n$,

$$E_*^i = \partial E_*^{i-1} \oplus \partial^* E_*^{i+1}, \tag{1.51}$$

and the splitting in (1.51) is $h^{E_*^i}$ orthogonal. Since $(E_*^{\cdot}, \partial)$ is acyclic, $\partial|_{E_*^{\cdot}}$ is one-to-one from $\partial^* E_*^{i+1}$ into ∂E_*^i.

Take a_0 nonzero in $\det E_*^0 \ldots$, a_i nonzero in $\det \partial^* E_*^{i+1} \ldots$. By definition [KM76],

$$\det \partial|_{E_*^{\cdot}} = a_0^{-1} \otimes \partial a_0 \wedge a_1 \otimes (\partial a_1 \wedge a_2)^{-1} \ldots. \tag{1.52}$$

Since the splitting in (1.51) is $h^{E_*^i}$ orthogonal, for any i, we get

$$\|\partial a_{i-1} \wedge a_i\|^2_{\det E_*^i} = \|\partial a_{i-1}\|^2_{\det \partial E_*^{i-1}} \|a_i\|^2_{\det \partial^* E_*^{i+1}}. \tag{1.53}$$

By construction,

$$\|\partial a_{i-1}\|^2_{\det \partial E_*^{i-1}} = \det\left(\partial^* \partial|_{\partial^* E_*^i}\right) \|a_{i-1}\|^2_{\det \partial^* E_*^i}. \tag{1.54}$$

By (1.53), (1.54), we get

$$\left\|\det \partial|_{E_*^{\cdot}}\right\|^2_{(\det E_*^{\cdot})^{-1}} = \prod_{i=1}^{n} \det\left(\partial^*\partial|_{\partial^* E_*^i}\right)^{(-1)^{i+1}}. \tag{1.55}$$

Finally observe that D^2 preserves the splitting (1.20) of $E_*^{\cdot}$. More precisely, D^2 restricts to $\partial\partial^*$ on $\mathrm{Im}\partial|_{E_*^{\cdot}}$ and to $\partial^*\partial$ on $\mathrm{Im}\partial^*|_{E_*^{\cdot}}$. One then verifies easily that (1.50) is equivalent to (1.55). The proof of our proposition is completed. □

Theorem 1.9. *The following identity holds:*

$$\| \|^2_{\det E^{\cdot}} = S\left(E, h^{E^{\cdot}}\right) \| \|^2_{\det E_0^{\cdot}}. \tag{1.56}$$

Proof. By Theorem 1.2, $E_0^{\cdot}$ and $E_*^{\cdot}$ are orthogonal in $E^{\cdot}$ with respect to $h^{E^{\cdot}}$. Our theorem now follows from (1.47) and from Proposition 1.8. □

Remark 1.10. From Theorem 1.9, we deduce in particular that

$$\epsilon\left(\| \|^2_{\det E^{\cdot}}\right) = \mathrm{sign}\, S\left(E, h^{E^{\cdot}}\right) \epsilon\left(\| \|^2_{\det E_0^*}\right). \tag{1.57}$$

Note that Theorem 1.9 extends to generalized metrics on determinant lines a well-known statement on standard metrics on determinants [BGiSo88]. Also note that as we shall see, the generalized Laplacian D^2 may well be equal to 0. In this case Theorem 1.9 is a tautology.

1.6. **Truncating the spectrum of the generalized Laplacian.** Now, we will extend Theorem 1.9 when considering more general splittings of $E^{\cdot}$ than (1.16), (1.18).

Theorem 1.11. *If $\lambda, \mu \in \mathrm{Sp} D^2$, if $\mu \neq \overline{\lambda}$, then $E_\lambda^{\cdot}$ and $E_\mu^{\cdot}$ are $h^{E^{\cdot}}$ orthogonal. Also if $\lambda \in \mathrm{Sp}\, D^2$, the restriction of $h^{E^{\cdot}}$ to $E_\lambda^{\cdot} + E_{\overline{\lambda}}^{\cdot}$ is nondegenerate.*

Proof. Take λ, μ as indicated above. Let $k \in \mathbf{N}$ be large enough so that

$$E_\lambda^{\cdot} = \ker\left(D^2 - \lambda\right)^k. \tag{1.58}$$

Then if $a \in E_\lambda^{\cdot}, b \in E_\mu^{\cdot}$, since D^2 is $h^{E^{\cdot}}$ self-adjoint, using (1.58), we get

$$\left\langle a, \left(D^2 - \overline{\lambda}\right)^k b\right\rangle = 0. \tag{1.59}$$

Now since $\mu \neq \overline{\lambda}$, the restriction of $\left(D^2 - \overline{\lambda}\right)^k$ to $E_\mu^{\cdot}$ is invertible. From (1.59), we deduce that $E_\lambda^{\cdot}$ and $E_\mu^{\cdot}$ are $h^{E^{\cdot}}$ orthogonal. Since $h^{E^{\cdot}}$ is nondegenerate, the final statement in our theorem follows easily. □

Take now $r \in \mathbf{R}_+^*$ such that if $\lambda \in \mathrm{Sp}\, D^2$, then $|\lambda| \neq r$.

Definition 1.12. Put

$$E_{<r}^{\cdot} = \bigoplus_{\substack{\lambda \in \mathrm{Sp} D^2 \\ |\lambda| < r}} E_\lambda^{\cdot}, \qquad E_{>r}^{\cdot} = \bigoplus_{\substack{\lambda \in \mathrm{Sp} D^2 \\ |\lambda| > r}} E_\lambda^{\cdot}. \tag{1.60}$$

Then $E_{<r}^{\cdot}$ and $E_{>r}^{\cdot}$ are subcomplexes of $E^{\cdot}$, and moreover

$$E^{\cdot} = E_{<r}^{\cdot} \oplus E_{>r}^{\cdot}. \tag{1.61}$$

Proposition 1.13. *If $r > 0$, then*

$$H^{\cdot}\left(E^{\cdot}_{<r}\right) = H^{\cdot}\left(E^{\cdot}\right), \qquad H^{\cdot}\left(E^{\cdot}_{>r}\right) = 0. \tag{1.62}$$

Moreover $E^{\cdot}_{<r}$ and $E^{\cdot}_{>r}$ are $h^{E^{\cdot}}$ orthogonal, and the restriction of $h^{E^{\cdot}}$ to each of these complexes is nondegenerate.

Proof. Since $E^{\cdot}_{*}$ is acyclic, we get (1.62). The second part of our proposition follows from Theorem 1.11. □

By Proposition 1.13, for $r > 0$,

$$\det E^{\cdot}_{<r} \simeq \det E^{\cdot}. \tag{1.63}$$

The identification in (1.63) is given by

$$s \in \det E^{\cdot}_{<r} \to s \otimes \left(\det \partial|_{E^{\cdot}_{>r}}\right)^{-1}. \tag{1.64}$$

Let $h^{E^{\cdot}_{>r}}$ be the generalized metric induced by $h^{E^{\cdot}}$ on $E^{\cdot}_{>r}$. Let $S\left(E^{\cdot}_{>r}, h^{E^{\cdot}_{>r}}\right)$ be defined as in Definition 1.7. Now we have the following extension of Theorem 1.9.

Theorem 1.14. *The following identity holds:*

$$\| \|^2_{\det E^{\cdot}} = S\left(E^{\cdot}_{>r}, h^{E^{\cdot}_{>r}}\right) \| \|^2_{\det E^{\cdot}_{<r}}. \tag{1.65}$$

Proof. We can either use Theorem 1.9 or prove our theorem by using the same arguments as in Theorem 1.9. □

Remark 1.15. Take $0 < r < r'$ as before. Put

$$E^{\cdot}_{]r,r'[} = \bigoplus_{\substack{\lambda \in \mathrm{Sp} D^2 \\ r < |\lambda| < r'}} E^{\cdot}_{\lambda}. \tag{1.66}$$

Then the canonical identification $\det E^{\cdot}_{<r} \simeq \det E_{<r'}$ is just given by

$$s \in \det E^{\cdot}_{<r} \to s \otimes \left(\det \partial|_{E^{\cdot}_{]r,r'[}}\right)^{-1} \in \det E^{\cdot}_{<r'}. \tag{1.67}$$

1.7. **Generalized metrics on determinant bundles.** Let (E, ∂) be a smooth complex of finite dimensional complex vector bundles on a manifold S. We will use, in this more general context, the notation of the previous subsections.

Note that the dimension of the fibres of $H^{\cdot}\left(E^{\cdot}\right)$ may well jump, so that in general $H^{\cdot}\left(E^{\cdot}\right)$ is not a vector bundle on S. Still $\det E$ is a complex line bundle on S, whose fibres $\det E_s$ are canonically isomorphic to the fibres $\det H^{\cdot}\left(E^{\cdot}\right)_s$. We will sometimes call $\det E^{\cdot}$ the determinant of the cohomology.

Let $h^{E^{\cdot}} = \bigoplus_{0 \le i \le n} h^{E^i}$ be a smooth generalized metric on $E = \bigoplus_{0 \le i \le n} E^i$. We can then construct the generalized metric $\| \|^2_{\det E^{\cdot}}$ on $\det E^{\cdot}$, which is smooth. Note that its sign is locally constant on S.

We will now use the constructions of subsection 1.5 in the present context. Take $r > 0$, and set

$$V_r = \left\{ s \in S \text{ such that if } \lambda \in \mathrm{Sp} D^2_s, \text{ then } |\lambda| \neq r \right\}. \tag{1.68}$$

Then V_r is an open set in S. Moreover $\det E_{<r}$ is a smooth line bundle on V_r, which by (1.63), (1.64), is canonically isomorphic to $\det E^{\cdot}|_{V_r}$. By (1.67), if $0 < r < r'$, the identification

$$\det E_{<r}|_{V_r \cap V_{r'}} \simeq \det E_{<r'}|_{V_r \cap V_{r'}} \tag{1.69}$$

is given by (1.67). Finally by (1.65),

$$\| \|^2_{\det E^{\cdot}|_{V_r}} = S\left(E^{\cdot}_{>r}, h^{E^{\cdot}_{>r}}\right) \| \|^2_{\det E^{\cdot}_{<r}}. \tag{1.70}$$

Remark 1.16. Assume that $(E^{\cdot}, \partial)$ is acyclic or, equivalently, that $H^{\cdot}(E^{\cdot}) = 0$. Then $(\det \partial)^{-1}$ is a canonical smooth nonzero section of $\det E$. We define the open set V_0 as in (1.68), by replacing r by 0. By Theorem 1.9, on V_0,

$$\left\|(\det \partial)^{-1}\right\|^2_{\det E^{\cdot}} = S\left(E, g^E\right). \tag{1.71}$$

Still the left-hand side of (1.71) extends to a smooth nonzero function on S, so that this is also the case for the right-hand side.

Let us give an example of why this is the case. Consider the obvious acyclic complex $(E^{\cdot}, \partial)$ given by

$$0 \to F^0 \xrightarrow{\partial} F^0 \oplus F^1 \xrightarrow{\partial} F^1 \to 0. \tag{1.72}$$

In (1.72), F^0, F^1 are finite dimensional complex vector spaces. Note that $\ker \partial \cap \ker \partial^* \subset E_1$. Then on E^0, D^2 acts like $\partial^*\partial$, which is invertible if and only if $\ker \partial \cap \ker \partial^* = 0$. Similarly, D^2 acts like $\partial\partial^*$ on $E^2 = F^1$, which is invertible under exactly the same condition. Therefore $\det D^2|_{E^0}$ and $\det D^2|_{E^2}$ vanish exactly together. Now one verifies easily that on V_0,

$$S\left(E, g^E\right) = \frac{\det D^2|_{E^2}}{\det D^2|_{E^0}}. \tag{1.73}$$

We have thus found that (1.71) and (1.73) are indeed compatible. When $h^{E^{\cdot}}$ is a Hermitian metric, $V_0 = S$. When $h^{E^{\cdot}}$ is not a Hermitian metric, on $S \setminus V_0$, (1.71) does not hold anymore. Contrary to the standard theory of metrics on determinant lines, the Hermitian form $h^{E^{\cdot}}$ is a new source of difficulties, even when the complex $(E^{\cdot}, \partial)$ is acyclic. In particular, if $\dim F^0 = \dim F^1$, one can easily construct a generalized metric $h^{E^{\cdot}}$ such that $D^2 = 0$. The conclusions of traditional Hodge theory are certainly not verified in this more general setting.

1.8. **Determinants and flat superconnections.** We make the same assumptions as in subsection 1.7. We will use the superconnection formalism of Quillen [Q85a]. Let A' be a superconnection on $E^{\cdot}$ of total degree 1 in the sense of [BLo95, Definition 2.1], which is of the form

$$A' = \sum_{j \geq 0} A'_{(j)}, \tag{1.74}$$

where $A'_{(1)} = \nabla^{E^{\cdot}}$ is a connection on $E^{\cdot}$ which preserves the $\mathbf{Z}$ grading, and for $j \neq 1$, $A'_{(j)}$ is a smooth section of $\Lambda^j(T^*S) \widehat{\otimes} \mathrm{Hom}\left(E^{\cdot}, E^{\cdot+1-j}\right)$. Here $\widehat{\otimes}$ is our notation for the $\mathbf{Z}_2$-graded tensor product. We assume that

$$A'_{(0)} = \partial \tag{1.75}$$

and also that A' is flat, i.e.

$$A'^2 = 0. \tag{1.76}$$

In particular, from (1.74)–(1.76), we can deduce as in [BLo95, Proposition 2.2] that

$$\left[\nabla^{E^{\cdot}}, \partial\right] = 0, \qquad \nabla^{E^{\cdot},2} + \left[\partial, A'_{(2)}\right] = 0. \tag{1.77}$$

Let $\nabla^{\det E^{\cdot}}$ be the connection on $\det E^{\cdot}$ induced by $\nabla^{E^{\cdot}}$. Then using (1.77), we find that

$$c_1\left(\det E^{\cdot}, \nabla^{\det E^{\cdot}}\right) = 0, \tag{1.78}$$

i.e. $\nabla^{\det E^{\cdot}}$ is flat.

By [BLo95, Proposition 2.5], one can deduce from (1.76) that the fibres $H^{\cdot}(E^{\cdot})$ have locally constant rank and patch into a smooth vector bundle on S, equipped with a canonical flat connection $\nabla^{H^{\cdot}(E^{\cdot})}$. Let $\nabla^{\det H^{\cdot}(E^{\cdot})}$ be the connection induced by $\nabla^{H^{\cdot}(E^{\cdot})}$ on $\det H^{\cdot}(E^{\cdot})$.

By (1.5), the line bundles $\det E^{\cdot}$ and $\det H^{\cdot}(E^{\cdot})$ are canonically isomorphic.

Proposition 1.17. *The flat connections $\nabla^{\det E^{\cdot}}$ and $\nabla^{\det H^{\cdot}(E^{\cdot})}$ coincide.*

Proof. Consider the exact sequence of vector bundles

$$0 \to H^0(E^{\cdot}) \to E^0 \to \partial E^0 \to 0. \tag{1.79}$$

By (1.79), the connection $\nabla^{E^{\cdot}}$ induces connections $\nabla^{H^0(E)}$ and $\nabla^{\partial E^0}$ on $H^0(E^{\cdot})$ and ∂E^0, and the chain maps in (1.79) are parallel. From (1.79), we have the canonical isomorphism

$$\det E^0 \simeq \det H^0(E) \otimes \det \partial E^0. \tag{1.80}$$

Finally the above arguments show that the identification (1.80) is parallel with respect to the obvious connections.

Also we have the exact sequence

$$0 \to \partial E^0 \to \ker \partial|_{E^1} \to H^1(E^{\cdot}) \to 0. \tag{1.81}$$

By (1.81), we have the parallel canonical isomorphism

$$\det \ker \partial|_{E^1} \simeq \det \partial E^0 \otimes \det H^1(E^{\cdot}). \tag{1.82}$$

We can iterate the above arguments, and we finally get our proposition. □

1.9. **The Hodge theorem as an open condition.** We make the same assumptions as in subsections 1.7 and 1.8.

Definition 1.18. Let S_H be the subset of the $s \in S$ such that the generalized metric $h^{E_s^{\cdot}}$ is of Hodge type.

Proposition 1.19. *The set $S_H \subset S$ is open.*

Proof. Recall that the dimension of $H^{\cdot}(E^{\cdot})$ is locally constant. Moreover, by (1.19),

$$\dim E_0^{\cdot} \geq \dim H^{\cdot}(E^{\cdot}). \tag{1.83}$$

From the upper semicontinuity of the function $\dim E_{0,s}$ on S and from the first statement in Theorem 1.5, our proposition follows. □

1.10. **Generalized metrics and flat superconnections.** We make the same assumptions as in subsections 1.7 and 1.8. Put

$$\omega\left(E^{\cdot}, h^{E^{\cdot}}\right) = \left(h^{E^{\cdot}}\right)^{-1} \nabla^{E^{\cdot}} h^{E^{\cdot}}. \tag{1.84}$$

Then $\omega\left(E, h^{E^{\cdot}}\right)$ is a 1-form with values in $h^{E^{\cdot}}$ self-adjoint sections of $\mathrm{End}(E^{\cdot})$ which preserve the $\mathbf{Z}$ grading.

Definition 1.20. Let $\nabla^{E^{\cdot},u}$ be the connection on $E^{\cdot}$,

$$\nabla^{E^{\cdot},u} = \nabla^{E^{\cdot}} + \frac{1}{2}\omega\left(E^{\cdot}, \nabla^{E^{\cdot}}\right). \tag{1.85}$$

Clearly the connection $\nabla^{E^{\cdot},u}$ preserves $h^{E^{\cdot}}$. A simple computation shows that its curvature $\nabla^{E^{\cdot},u,2}$ is given by

$$\nabla^{E^{\cdot},u,2} = \frac{1}{2}\left(\nabla^{E,2} - \nabla^{E,2,*}\right) - \frac{1}{4}\omega\left(E^{\cdot}, \nabla^{E^{\cdot}}\right)^2. \tag{1.86}$$

Let $\nabla^{\det E^{\cdot},u}$ be the connection induced by $\nabla^{E^{\cdot},u}$ on $\det E^{\cdot}$. By (1.85),

$$\nabla^{\det E^{\cdot},u} = \nabla^{\det E^{\cdot}} + \frac{1}{2}\mathrm{Tr_s}\left[\omega\left(E^{\cdot}, \nabla^{E^{\cdot}}\right)\right]. \tag{1.87}$$

As the notation indicates, $\nabla^{\det E^{\cdot},u}$ is just the connection associated to the flat connection $\nabla^{\det E^{\cdot}}$ and the generalized metric $\| \|^2_{\det E^{\cdot}}$ as in (1.85). Clearly

$$c_1\left(\det E^{\cdot}, \nabla^{\det E^{\cdot},u}\right) = 0, \tag{1.88}$$

i.e. the connection $\nabla^{\det E^{\cdot},u}$ is still flat.

Using Proposition 1.17 and (1.87), we get

$$\nabla^{\det E^{\cdot},u} = \nabla^{\det H^{\cdot}(E^{\cdot})} + \frac{1}{2}\nabla^{\det E^{\cdot}} \log\left(\| \|^2_{\det E^{\cdot}}\right). \tag{1.89}$$

Now we will give another expression for $\nabla^{\det E^{\cdot},u}$ over V_r. In fact, let $\nabla^{E^{\cdot}_{<r}}$ be the connection on $E_{<r}$ obtained by projection of $\nabla^{E^{\cdot}}$ with respect to the splitting (1.61) of $E^{\cdot}|_{V_r}$. We construct the connection $\nabla^{E^{\cdot}_{<r},u}$ as in (1.84), (1.85) with respect to the generalized metric $h^{E^{\cdot}_{<r}}$. Since the splitting (1.61) is $h^{E^{\cdot}}$ orthogonal, one verifies easily that $\nabla^{E^{\cdot}_{<r},u}$ is just the projection of $\nabla^{E^{\cdot},u}$ with respect to the splitting (1.61). We denote by $\nabla^{\det E^{\cdot}_{<r}}$ and $\nabla^{\det E^{\cdot}_{<r},u}$ the connections induced on $\det E^{\cdot}_{<r}$ by $\nabla^{E^{\cdot}_{<r}}$ and $\nabla^{E^{\cdot}_{<r},u}$.

Proposition 1.21. *The following identity of connections holds on V_r:*

$$\nabla^{\det E^{\cdot}_{<r}} = \nabla^{\det H^{\cdot}(E^{\cdot})}. \tag{1.90}$$

Moreover,

$$\nabla^{\det E^{\cdot}_{<r},u} = \nabla^{H^{\cdot}(E^{\cdot})} + \frac{1}{2}\nabla^{H^{\cdot}(E^{\cdot})} \log \| \|^2_{\det E^{\cdot}_{<r}}. \tag{1.91}$$

Proof. Clearly $\partial|_{E^{\cdot}_{<r}}$ is parallel with respect to $\nabla^{E^{\cdot}_{<r}}$. By construction, the connection $\nabla^{E^{\cdot}_{<r}}$ induces on $H^{\cdot}(E^{\cdot}_{<r}) = H^{\cdot}(E^{\cdot})$ the connection $\nabla^{H^{\cdot}(E^{\cdot})}$. By proceeding as in the proof of Proposition 1.17, we get (1.90). Also as we saw before,

$$\nabla^{\det E^{\cdot}_{<r},u} = \nabla^{\det E^{\cdot}_{<r}} + \frac{1}{2}\nabla^{\det E^{\cdot}_{<r}} \log \| \|^2_{\det E^{\cdot}_{<r}}. \tag{1.92}$$

From (1.90), (1.92), we get (1.91). The proof of our proposition is completed. □

Remark 1.22. Let $\nabla^{E^{\cdot}_{>r}}$ be the projection of $\nabla^{E^{\cdot}}$ with respect to the splitting (1.61). Then $\partial|_{E^{\cdot}_{>r}}$ is parallel with respect to $\nabla^{E^{\cdot}_{>r}}$. Let $\nabla^{\det E^{\cdot}_{>r}}$ be the connection on $\det E^{\cdot}_{>r}$ induced by $\nabla^{E^{\cdot}_{>r}}$. Then $\det \partial|_{E^{\cdot}_{>r}}$ is a flat section of $\det E^{\cdot}_{>r}$, so that $\nabla^{\det E^{\cdot}_{>r}}$ is flat. This can also be proved by noting that $\nabla^{E^{\cdot}_{>r},2}$ commutes with $\partial|_{E^{\cdot}_{>r}}$ and that $E^{\cdot}_{>r}$ is acyclic. Also by (1.61),

$$\det E^{\cdot} \simeq \det E^{\cdot}_{<r} \otimes \det E^{\cdot}_{>r}. \tag{1.93}$$

Moreover, by construction,

$$\nabla^{\det E^{\cdot}} = \nabla^{\det E^{\cdot}_{<r}} \otimes 1 + 1 \otimes \nabla^{\det E^{\cdot}_{>r}}. \tag{1.94}$$

Equation (1.94) is compatible with the flatness of the three connections which appear there.

The same argument as in (1.94) shows that

$$\nabla^{\det E^{\cdot},u} = \nabla^{\det E^{\cdot}_{<r},u} \otimes 1 + 1 \otimes \nabla^{\det E^{\cdot}_{>r},u}. \tag{1.95}$$

Moreover since $\det \partial|_{E^{\cdot}_{>r}}$ is a flat section of $\det E^{\cdot}_{>r}$, using Proposition 1.8, we get

$$\nabla^{\det E^{\cdot}_{>r},u} = \nabla^{\det E^{\cdot}_{>r}} + \frac{1}{2} d \log S\left(E^{\cdot}_{>r}, h^{E^{\cdot}_{>r}}\right). \tag{1.96}$$

From (1.95), (1.96), we get

$$\nabla^{\det E,u} = \nabla^{\det E^{\cdot}_{<r},u} + \frac{1}{2} d \log S\left(E^{\cdot}_{>r}, h^{E^{\cdot}_{>r}}\right). \tag{1.97}$$

Note that (1.97) also follows from (1.70), from Propositions 1.17 and 1.21, and from (1.94)–(1.96).

1.11. **Analyticity and the Hodge condition.** Let $(E^{\cdot}, \partial)$ be a finite dimensional complex as in (1.3). Let O be a nonempty open interval in $\mathbf{R}$. Let $s \in O \to h^{E^{\cdot}}_s$ be an analytic family of generalized metrics on $E^{\cdot}$. Then we can view $(E^{\cdot}, \partial)$ as a flat complex on O, equipped with the analytic metric $h^{E^{\cdot}}$. This metric induces a corresponding analytic metric $\| \|^2_{\det E^{\cdot}}$ on $\det E^{\cdot}$.

Proposition 1.23. *Assume that* $O_H = \left\{s \in O, h^{E^{\cdot}}_s \text{ is Hodge}\right\}$ *is nonempty. Then* $O \setminus O_H$ *is a discrete subset of* O.

Proof. As we saw in Proposition 1.19, we already know that O_H is open in O. For $0 \leq i \leq n, s \in O, z \in \mathbf{C}$, put

$$f_{i,s}(z) = \det\left(D^2_{E_i,s} - z\right). \tag{1.98}$$

Then $f_{i,s}(z)$ is a polynomial in z with analytic coefficients in s. By Theorem 1.2, given $s \in O$, $f_{i,s}(z)$ has a zero in z whose order is exactly $\dim E^i_{0,s}$. Set $h_i = \dim H^i(E^{\cdot})$. By Theorem 1.5, O_H is the set of $s \in O$ such that for $1 \leq i \leq n$, $f^{(h_i+1)}_{i,s}(0) \neq 0$. Since O_H is nonempty, for $1 \leq i \leq n$, $f^{(h_i+1)}_{i,s}(0)$ is an analytic function of s which is nonidentically 0. Therefore the zero set in O of each of these functions is a discrete subset of O. The proof of our proposition is completed. □

1.12. **The equivariant determinant.** Clearly the function log is a well-defined function from $\mathbf{C}^*$ into $\mathbf{R} \oplus i\mathbf{R}/2\pi$.

Let G be a compact Lie group. Let $\widehat{G}$ be the set of equivalence classes of irreducible representations of G. If $W \in \widehat{G}$, let χ_W be the corresponding character of G.

Now we make the same assumptions as in subsections 1.1–1.6. Also we assume that G acts on $E^{\cdot}$ by isomorphisms of complexes. Namely G preserves the $\mathbf{Z}$-grading and commutes with ∂. Finally we suppose that $h^{E^{\cdot}}$ is G-invariant.

If $W \in \widehat{G}$, set

$$E^{\cdot}_W = \mathrm{Hom}_G(W, E^{\cdot}) \otimes W. \tag{1.99}$$

Let

$$E^{\cdot} = \bigoplus_{W \in \widehat{G}} E_W \tag{1.100}$$

be the isotypical decomposition of $E^{\cdot}$. Then the various $E^{\cdot}_W$ are subcomplexes of $E^{\cdot}$. Note that (1.100) can be obtained by taking an element $g \in G$ such that the closure of the group generated by g is just G and by splitting $E^{\cdot}$ according to the distinct eigenvalues of g.

Note that the splitting (1.16) is G-invariant, so that each term in (1.100) inherits a corresponding splitting.

Proposition 1.24. *The splitting* (1.100) *of* $E^{\cdot}$ *is* $h^{E^{\cdot}}$ *orthogonal, and the restriction of* $h^{E^{\cdot}}$ *to each* E_W *is nondegenerate.*

Proof. Let $g^{E^{\cdot}}$ be a G-invariant Hermitian metric on $E^{\cdot}$. Then the splitting (1.100) is $g^{E^{\cdot}}$ orthogonal. Let $v \in \operatorname{End} E^{\cdot}$ be such that

$$h^{E^{\cdot}} = g^{E^{\cdot}} v. \tag{1.101}$$

Then v commutes with G. Therefore v preserves the decomposition (1.100). Our proposition follows. □

Set

$$\det E^{\cdot} = \bigoplus_{W \in \widehat{G}} \det E^{\cdot}_W. \tag{1.102}$$

Let $\| \|^2_{\det E_W}$ be the generalized metric on $\det E_W$ induced by the restriction of $h^{E^{\cdot}}$ to $E^{\cdot}_W$.

Definition 1.25. Set

$$\log\left(\| \|^2_{\det E^{\cdot}}\right) = \sum_{W \in \widehat{G}} \log\left(\| \|^2_{\det E_W}\right) \otimes \frac{\chi_W}{\operatorname{rk} W}. \tag{1.103}$$

The formal sum (1.103) will be called the logarithm of a generalized equivariant metric on $\det E^{\cdot}$.

Note that (1.103), the interpretation of the logarithm is the one given at the beginning of the section. In particular each term $\log\left(\| \|^2_{\det E_W}\right)$ contains the information on the sign of $\| \|^2_{\det E_W}$.

Let $\det E^{\cdot}_* = \bigoplus_{W \in \widehat{G}} \det E^{\cdot}_{*W}$ be the direct sum of lines associated to $E^{\cdot}_*$ as in (1.102). Since $E^{\cdot}_{*W}$ is acyclic, each $\det E^{\cdot}_{*W}$ has a canonical nonzero section $(\det \partial|_{E_{*W}})$. Set

$$\left(\det \partial|_{E^{\cdot}_*}\right)^{-1} = \bigoplus_{W \in \widehat{G}} \left(\det \partial|_{E^{\cdot}_{*W}}\right)^{-1}. \tag{1.104}$$

Definition 1.26. For $g \in G$, set

$$T_g\left(E^{\cdot}_*, h^{E^{\cdot}_*}\right) = \frac{1}{2} \operatorname{Tr_s}^{E^{\cdot}_*}\left[gN \log\left(D^2|_{E^{\cdot}_*}\right)\right]. \tag{1.105}$$

The expression in the right-hand side of (1.105) has to be properly understood. Indeed we can split $E^{\cdot}_*$ according to the distinct eigenvalues of g. On each eigenspace of g, the spectrum of D^2 is still conjugation invariant. The eigenvalues of D^2 which are not real do not cause any difficulty, since their imaginary parts cancel out in

(1.105). So in fact only the negative eigenvalues of D^2 can cause trouble in (1.105). However these are taken care of in our definition of the log, so that the following statement does make sense.

Proposition 1.27. *The following identity holds:*

$$\log \left\| (\det \partial)^{-1} \right\|^2_{\det E^{\cdot}_*} = 2T^{\cdot}\left(E^{\cdot}_*, h^{E^{\cdot}_*}\right). \tag{1.106}$$

Proof. First consider a generic $g \in G$, and split $E^{\cdot}_*$ according to the eigenvalues of g. On each acyclic eigencomplex, we can use Proposition 1.8, and we obtain our proposition when we evaluate both sides of (1.106) on this g. The general case follows easily. □

As in (1.46), we have the canonical isomorphism

$$\det E^{\cdot} \simeq \det E^{\cdot}_0. \tag{1.107}$$

Now we have an extension of Theorem 1.9.

Theorem 1.28. *The following identity holds:*

$$\log\left(\| \, \|^2_{\det E^{\cdot}}\right) = \log\left(\| \, \|^2_{\det E^{\cdot}_0}\right) + 2T^{\cdot}\left(E^{\cdot}_*, h^{E^{\cdot}_*}\right). \tag{1.108}$$

Proof. This follows from Propositions 1.24 and 1.27. □

The results of subsections 1.6, 1.7 and 1.10 can be extended to the equivariant case. The statements and proofs are obvious and are left to the reader.

2. The adjoint of the de Rham operator on the cotangent bundle

Let X be a Riemannian manifold, and let F be a flat Hermitian vector bundle on X. In this section, we construct the adjoint of the de Rham operator d^{T^*X} with respect to a natural sesquilinear form on the de Rham complex of T^*X. Also we prove that this operator is also the adjoint of d^{T^*X} with respect to a Hermitian form on the de Rham complex, which makes in principle the theory of Section 1 applicable in the present context.

This section is organized as follows. In subsections 2.1 and 2.2, we recall elementary results on exterior algebras, Clifford algebras and bilinear forms. In subsection 2.3, if M is a manifold, we construct the adjoint of the de Rham operator d^M with respect to a bilinear form on the tangent space. In subsection 2.4, if M is symplectic, we construct the symplectic adjoint $\overline{d}^M$ of the de Rham operator, and we show it anticommutes with d^M. In subsection 2.5, we apply this construction to T^*X. In subsection 2.6, we perturb the symplectic form of T^*X, so as to produce a better adjoint $\overline{d}^{T^*X}_{\phi}$. In subsection 2.7, we show that this operator is also the adjoint of d^{T^*X} with respect to a nontrivial Hermitian form. In subsection 2.8, we introduce a Witten twist associated to a Hamiltonian $\mathcal{H} : T^*X \to \mathbf{R}$, and we produce in this way the generalized adjoint $\overline{d}^{T^*X}_{\phi,2\mathcal{H}}$. The first order operator $A_{\phi,\mathcal{H}} = \frac{1}{2}\left(\overline{d}^{T^*X}_{\phi,2\mathcal{H}} + d^{T^*X}\right)$ and the Laplacian $A^2_{\phi,\mathcal{H}}$ are our fundamental objects of study.

In subsection 2.9, we show that if $\mathcal{H}$ is invariant under the map $p \to -p$, the operator $\overline{d}^{T^*X}_{\phi,2\mathcal{H}}$ is the adjoint of d^{T^*X} with respect to a Hermitian form. In subsection 2.10, we study the behaviour of the considered operators under Poincaré duality.

In subsection 2.11, we construct an operator $\mathfrak{A}_{\phi,\mathcal{H}}$ which is naturally conjugate to $A_{\phi,\mathcal{H}}$, which is sometimes simpler to handle. In subsection 2.12, we study the effect of the scaling of the variable $p \in T^*X$.

In subsections 2.13 and 2.14, we consider the case where $\mathcal{H} = \mathcal{H}^c$, with $\mathcal{H}^c = c\left|p\right|^2/2$. Also we show that when making $c \to \pm\infty$, the operator $A_{\phi,\mathcal{H}^c}$ has the preferred matrix structure considered in particular in [BL91, Section 8].

2.1. **Clifford algebras.** Let V be a real Euclidean vector space of dimension n. We identify V and V^* by the scalar product of V. If $U \in V$, let $U^* \in V^*$ correspond to U by the metric.

If $U \in V$, set,

$$c(U) = U^* \wedge -i_U, \qquad \widehat{c}(U) = U^* \wedge +i_U. \tag{2.1}$$

Then $c(U), \widehat{c}(U)$ lie in $\mathrm{End}^{\mathrm{odd}}(\Lambda(V^*))$. Moreover if $U, U' \in V$,

$$\begin{aligned} &[c(U), c(U')] = -2\langle U, U'\rangle, \qquad [\widehat{c}(U), \widehat{c}(U')] = 2\langle U, U'\rangle, \\ &[c(U), \widehat{c}(U')] = 0. \end{aligned} \tag{2.2}$$

By (2.2),

$$U^*\wedge = \frac{1}{2}\left(\widehat{c}(U) + c(U)\right), \qquad i_U = \frac{1}{2}\left(\widehat{c}(U) - c(U)\right). \tag{2.3}$$

Let $c(V)$ be the Clifford algebra of V. Then $c(V)$ is spanned by $1, U \in V$, with the commutation relations

$$UU' + U'U = -2\langle U, U'\rangle. \tag{2.4}$$

Let N be the number operator of $\Lambda^{\cdot}(V^*)$. Then we have the identification of vector spaces $c(V) \simeq \Lambda^{\cdot}(V^*)$. Under this identification, the action of $c(U)$ on $\Lambda^{\cdot}(V^*)$ is left multiplication by U, and the action of $(-1)^N\widehat{c}(U)$ is right multiplication by U.

Let $e_1, \ldots, e_n$ be a basis of V, and let $e^1, \ldots, e^n$ be the corresponding dual basis of V^*. If $A \in \mathrm{End}(V)$, then A acts naturally on $\Lambda^{\cdot}(V^*)$ by the formula

$$A|_{\Lambda^{\cdot}(V^*)} = -\left\langle Ae_i, e^j\right\rangle e^i \wedge i_{e_j}. \tag{2.5}$$

Also, if $g = e^A \in \mathrm{Aut}(V)$, the corresponding action $g|_{\Lambda^{\cdot}(V^*)}$ of g on $\Lambda^{\cdot}(V^*)$ is just $\exp\left(A|_{\Lambda^{\cdot}(V^*)}\right)$.

In the sequel, we will assume that $e_1, \ldots, e_n$ is an orthonormal basis of V. Then if $A \in \Lambda^{\cdot}(V^*)$ is antisymmetric,

$$A|_{\Lambda^{\cdot}(V^*)} = \frac{1}{4}\langle Ae_i, e_j\rangle\left(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\right). \tag{2.6}$$

2.2. **Vector spaces and bilinear forms.** Let V be a real vector space of even dimension $n = 2\ell$. Let $\eta \in V^* \otimes V^*$ be a bilinear form on V, which we assume to be nondegenerate. Namely to η, we can associate the isomorphism $\phi : V \to V^*$, so that if $X, Y \in V$,

$$\eta(X, Y) = \langle X, \phi Y\rangle. \tag{2.7}$$

Let $\eta^* \in V \otimes V$ be obtained from η via the identification $\phi : V \to V^*$, so that if $A, B \in V^*$,

$$\eta^*(A, B) = \left\langle \phi^{-1}A, B\right\rangle. \tag{2.8}$$

Let $\widetilde{\phi} : V \to V^*$ be the transpose of ϕ. Then η is symmetric (resp. antisymmetric) if and only if $\widetilde{\phi} = \phi$ (resp. $\widetilde{\phi} = -\phi$).

Clearly $\phi : V \to V^*$ extends to an isomorphism $\phi : \Lambda^{\cdot}(V) \to \Lambda^{\cdot}(V^*)$. The bilinear forms η, η^* extend to bilinear forms on $\Lambda^{\cdot}(V), \Lambda^{\cdot}(V^*)$. In particular if $s, s' \in \Lambda^{\cdot}(V^*)$,

$$\eta^*(s, s') = \left\langle \phi^{-1} s, s' \right\rangle. \tag{2.9}$$

Observe that if $X \in V, f \in V^*$, we have the equality of operators acting on $\Lambda^{\cdot}(V^*)$,

$$\phi X \wedge \phi^{-1} = (\phi X) \wedge, \qquad \phi i_f \phi^{-1} = i_{\widetilde{\phi}^{-1} f}. \tag{2.10}$$

From (2.10), we deduce that if $s, s' \in \Lambda^{\cdot}(V^*)$,

$$\eta^*(s, i_X s') = \eta^*(\phi X \wedge s, s'), \qquad \eta^*(s, f \wedge s') = \eta^*\left(i_{\widetilde{\phi}^{-1} f} s, s'\right). \tag{2.11}$$

Equation (2.11) just asserts that $\phi X \wedge, i_{\widetilde{\phi}^{-1} f}$ are the adjoints of $i_X, f \wedge$ with respect to η^*. Since η^* is in general not symmetric, taking twice the adjoint does not give the original operator.

Let ω be a nondegenerate antisymmetric 2-form on V. Let $\phi : V \to V^*$ be the corresponding isomorphism as in (2.7). Then $\widetilde{\phi} = -\phi$. Let ω^* be the dual 2-form on V^* to ω, which is obtained as in (2.9).

Let $L : \Lambda^{\cdot}(V^*) \to \Lambda^{\cdot+2}(V^*)$ be given by

$$L\alpha = \omega \wedge \alpha. \tag{2.12}$$

Let $\Lambda : \Lambda^{\cdot}(V) \to \Lambda^{\cdot-2}(V)$ be the transpose of L. Let $e_1, \ldots, e_n$ be a basis of V, and let $e^1, \ldots, e^n$ be the corresponding dual basis of V^*. Then

$$L = \frac{1}{2}\omega(e_i, e_j)\, e^i e^j, \qquad \Lambda = -\frac{1}{2}\omega(e_i, e_j)\, i_{e^i} i_{e^j}. \tag{2.13}$$

Also observe that $\omega^{n/2}/(n/2)! \in \Lambda(V^*)$ is nonzero, so that we have a canonical isomorphism

$$\Lambda^{\cdot}(V) \simeq \Lambda^{n-\cdot}(V^*). \tag{2.14}$$

Under the canonical isomorphism in (2.14), Λ is just equal to L. Also by identifying V with V^* by ϕ, we derive from (2.14) that there is a canonical isomorphism

$$\Lambda^{\cdot}(V^*) \simeq \Lambda^{n-\cdot}(V^*). \tag{2.15}$$

From now on, we will identify V with V^* by the map ϕ, so that ω and ω^* are identified. In particular L and Λ both act on $\Lambda(V^*)$, L increases the degree by 2, and Λ decreases the degree by 2. Using the identification (2.15), the action of Λ on the left-hand side of (2.15) just corresponds to the action of L on the right-hand side.

With this identification, equation (2.13) now becomes

$$L = \frac{1}{2}\omega(e_i, e_j)\, e^i e^j, \qquad \Lambda = -\frac{1}{2}\omega^*\left(e^i, e^j\right) i_{e_i} i_{e_j}. \tag{2.16}$$

Let N be the number operator of $\Lambda(V^*)$, so that N acts by multiplication by k on $\Lambda^k(V^*)$. Put

$$H = \frac{1}{2}(N - n/2). \tag{2.17}$$

Then one verifies easily the following identities:

$$[H, L] = L, \qquad [H, \Lambda] = -\Lambda, \qquad [L, \Lambda] = 2H. \tag{2.18}$$

We recover the well-known sl_2 commutation relations in complex Hodge theory. The main point above the exposition is that we have made no choice of a scalar product on V.

Set

$$M = L - \Lambda. \tag{2.19}$$

Now we reproduce an easy computation given in [B04a, Proposition 1.2].

Proposition 2.1. *If $X \in V, f \in V^*$, then*

$$[M, i_X] = \phi X \wedge, \qquad [M, f\wedge] = i_{\widetilde{\phi}^{-1} f}. \tag{2.20}$$

In particular, if $\vartheta \in \mathbf{R}$,

$$\begin{aligned} e^{\vartheta M} i_X e^{-\vartheta M} &= \cos(\vartheta)\, i_X + \sin(\vartheta)\, \phi X \wedge, \\ e^{\vartheta M} f \wedge e^{-\vartheta M} &= \cos(\vartheta)\, f \wedge + \sin(\vartheta)\, i_{\widetilde{\phi}^{-1} f}. \end{aligned} \tag{2.21}$$

Proof. The identities in (2.20) follow from (2.16). From (2.20), we get (2.21). □

2.3. **The adjoint of the de Rham operator with respect to a nondegenerate bilinear form.** Let M be a smooth manifold. Let dv_M be a volume form on M.

Let $\left(\Omega^{\cdot}(M), d^M\right)$ be the de Rham complex of smooth sections of $\Lambda^{\cdot}(T^*M)$, equipped with the de Rham operator d^M. Let $\Omega^{*\cdot}(M)$ be the vector space of compactly supported smooth sections of $\Lambda^{\cdot}(TM)$. If $s \in \Omega^{\cdot}(M), t^* \in \Omega^{*\cdot}(M)$, put

$$\langle s, t^* \rangle = \int_M \langle s, t^* \rangle\, dv_M. \tag{2.22}$$

Then $\Omega^{\cdot}(M)$ and $\Omega^{*\cdot}(M)$ are naturally dual to each other by (2.22). Let $\widetilde{d}^M : \Omega^{*\cdot}(M) \to \Omega^{\cdot-1*}(M)$ be the transpose of d^M. Then $\left(\Omega^{*\cdot}(M), \widetilde{d}^M\right)$ is a complex.

Let η be a smooth section of $T^*M \otimes T^*M$, i.e. a smooth bilinear form on TM. Equivalently, η defines a smooth morphism $\phi : TM \to T^*M$, so that as in (2.7), if $U, V \in TM$,

$$\eta(U, V) = \langle U, \phi V \rangle. \tag{2.23}$$

We assume that the form is nondegenerate, i.e. $\phi : TM \to T^*M$ is invertible. We still denote by ϕ the corresponding isomorphism $\Lambda^{\cdot}(TM) \to \Lambda^{\cdot}(T^*M)$. In particular ϕ provides us with the identification

$$\Omega^{\cdot}(M) \simeq \Omega^{*\cdot}(M). \tag{2.24}$$

Let $\overline{d}^M$ correspond to $\widetilde{d}^M$ via the isomorphism (2.24). More precisely, $\overline{d}^M$ is given by the formula

$$\overline{d}^M = \phi \widetilde{d}^M \phi^{-1}. \tag{2.25}$$

Then $\left(\Omega^{\cdot}(M), \overline{d}^M\right)$ is a complex, and the operator $\overline{d}^M$ decreases the degree by 1. Note that $\overline{d}^M$ depends on dv_M and on η.

We can reformulate the above construction as follows. As we saw in subsection 2.2, η defines a bilinear form η^* on T^*M, which extends into a nondegenerate bilinear form on $\Lambda^{\cdot}(T^*M)$. We equip $\Omega^{\cdot}(M)$ with the nondegenerate bilinear form

$$\langle s, s' \rangle_\phi = \int_M \eta^*(s, s')\, dv_M. \tag{2.26}$$

Then by construction, if $s, s' \in \Omega^{\cdot}(M)$,

$$\left\langle s, d^M s' \right\rangle_\phi = \left\langle \overline{d}^M s, s' \right\rangle_\phi . \tag{2.27}$$

A case where the above construction is well known is the case where η is symmetric and positive definite, i.e. η is a metric g^{TM} on TM, and dv_M is the associated volume form. In this case, $\overline{d}^M$ is just the formal adjoint d^{M*} of d^M. In the next subsection, we will consider the case where η is associated to a symplectic form.

2.4. **The symplectic adjoint of the de Rham operator.** Let M be a compact manifold of even dimension $n = 2\ell$, and let ω be a symplectic form on M. We will use the formalism of subsections 2.2 and 2.3 applied to the fibres of TM, which are equipped with the symplectic form ω. In particular, L maps $\Lambda^{\cdot}(T^*M)$ into $\Lambda^{\cdot+2}(T^*M)$, and Λ maps $\Lambda^{\cdot}(T^*M)$ into $\Lambda^{\cdot-2}(T^*M)$. Let dv_M be the symplectic volume form on M.

As in subsections 2.3 and 2.2, we identify TM and T^*M by the map ϕ. Therefore $\Omega^{\cdot}(M)$ and $\Omega^{*\cdot}(M)$ are identified as in (2.24).

Let $\overline{d}^M$ be the operator $\Omega^{\cdot}(M) \to \Omega^{\cdot-1}(M)$ which corresponds to $\widetilde{d}^M$ by this identification as in (2.25). The operator $\overline{d}^M$ will be called the symplectic adjoint of d^M. More generally, one can define the symplectic adjoint $\overline{A}$ of a differential operator acting on $\Omega^{\cdot}(M)$ in the same way.

Then $\Omega^{\cdot}(M)$ is now equipped with the chain maps d^M and $\overline{d}^M$ which, respectively, increase and decrease the degree by 1.

Theorem 2.2. *The following identities hold:*

$$\overline{d}^M = -\left[d^M, \Lambda\right], \qquad d^M = -\left[\overline{d}^M, L\right], \qquad \left[d^M, \overline{d}^M\right] = 0. \tag{2.28}$$

Proof. By Darboux's theorem, if $x \in M$, there is a coordinate system

$$\left(p_1, \ldots, p_\ell, q^1, \ldots, q^\ell\right)$$

such that x is represented by $0 \in \mathbf{R}^n$, and moreover, in these coordinates,

$$\omega = \sum_1^\ell dp_i \wedge dq^i. \tag{2.29}$$

By (2.16), we find that in these coordinates,

$$L = \sum_1^\ell dp_i \wedge dq^i, \qquad \Lambda = -\sum_1^\ell i_{\partial/\partial p_i} i_{\partial/\partial q^i}. \tag{2.30}$$

Using (2.10), (2.29), we find easily that

$$\overline{d}^M = i_{\partial/\partial q_i} \partial/\partial p_i - i_{\partial/\partial p_i} \partial/\partial q^i. \tag{2.31}$$

By (2.30) and (2.31), we get the first two identities in (2.28). Note that the second identity is just the ω-transpose of the first one. The third identity follows from (2.31) or from the sequence of identities

$$\left[d^M, \left[d^M, \Lambda\right]\right] = 2\left[d^{M,2}, \Lambda\right] - \left[d^M, \left[d^M, \Lambda\right]\right] \tag{2.32}$$

and from the fact that $d^{M,2} = 0$. The proof of our theorem is completed. $\square$

Let $\mathcal{H} : M \to \mathbf{R}$ be a smooth function. Let $Y^{\mathcal{H}}$ be the associated Hamiltonian vector field, so that

$$d^M \mathcal{H} + i_{Y^{\mathcal{H}}} \omega = 0. \tag{2.33}$$

The symplectic adjoint of (2.33) is just

$$\left[\overline{d}^M, \mathcal{H}\right] = i_{Y^{\mathcal{H}}}. \tag{2.34}$$

Proposition 2.3. *The following identities hold:*

$$e^{-\mathcal{H}} d^M e^{\mathcal{H}} = d^M + d^M \mathcal{H} \wedge, \qquad e^{\mathcal{H}} \overline{d}^M e^{-\mathcal{H}} = \overline{d}^M - i_{Y^{\mathcal{H}}}. \tag{2.35}$$

Proof. The first identity in (2.35) is trivial. By taking the symplectic adjoint of this identity, we get the second identity. □

2.5. **The de Rham operator on T^*X and its symplectic adjoint.** Let X be a smooth compact manifold of dimension n, and let $\pi : T^*X \to X$ be its cotangent bundle. If $x \in X$, p will denote the generic element in $T^*_x X$. Then T^*X is equipped with the canonical 1-form $\theta = \pi^* p$, and $\omega = d^{T^*X}\theta$ is the canonical symplectic form of T^*X.

Let g^{TX} be a Riemannian metric on TX, and let g^{T^*X} be the corresponding metric on T^*X. Let ∇^{TX} be the Levi-Civita connection on $\left(TX, g^{TX}\right)$, and let R^{TX} be its curvature. Let ∇^{T^*X} be the corresponding connection on T^*X, and let R^{T^*X} be its curvature. We will often identify TX and T^*X by the metric g^{TX}. In particular the notation R^{TX} will sometimes be used instead of R^{T^*X}.

Recall that if $A, B, C, D \in TX$,

$$\begin{aligned} &R^{TX}(A,B)\,C + R^{TX}(B,C)\,A + R^{TX}(C,A)\,B = 0, \\ &\left\langle R^{TX}(A,B)\,C, D\right\rangle = \left\langle R^{TX}(C,D)\,A, B\right\rangle. \end{aligned} \tag{2.36}$$

The first identity in (2.36) will be referred to as the circular identity, the second one as the $(2,2)$ symmetry property of R^{TX}. Finally if $A, B, C \in TX$, the Bianchi identity can be written in the form

$$\nabla^{TX}_A R^{TX}(B,C) + \nabla^{TX}_B R^{TX}(C,A) + \nabla^{TX}_C R^{TX}(A,B) = 0. \tag{2.37}$$

Then the connection ∇^{T^*X} induces a horizontal subbundle $T^H T^*X \simeq \pi^* TX$ of TT^*X, so that we have the splittings

$$TT^*X = \pi^*\left(TX \oplus T^*X\right), \qquad T^*T^*X = \pi^*\left(T^*X \oplus TX\right). \tag{2.38}$$

If $U \in TX$, let $U^H \in T^H T^*X \simeq TX$ be the lift of U. By (2.38), we obtain the isomorphism of $\mathbf{Z}$-graded bundles of algebras

$$\Lambda^{\cdot}\left(T^*T^*X\right) = \pi^*\left(\Lambda^{\cdot}\left(T^*X\right) \widehat{\otimes} \Lambda^{\cdot}\left(TX\right)\right). \tag{2.39}$$

As we said before, it will often be convenient to identify the fibres TX and T^*X by the metric g^{TX}. This may lead to notational confusion. For this reason, in (2.39), the Grassmann variables in $\Lambda^{\cdot}(TX)$ will be hatted. In particular if $U \in TX$, $\widehat{U}$ will denote the corresponding element in $\Lambda^{\cdot}(TX)$.

Let F be a complex vector bundle on X, and let ∇^F be a flat connection on F. Let $\left(\Omega^{\cdot}(X,F), d^X\right)$ be the de Rham complex of smooth sections of F on X, equipped with the de Rham operator d^X. Let $\left(\Omega^{\cdot}(T^*X, \pi^*F), d^{T^*X}\right)$ be the corresponding de Rham complex on T^*X of smooth sections of $\Lambda^{\cdot}(T^*T^*X) \widehat{\otimes} \pi^*F$ which have compact support.

If $x \in X$, let I_x be the vector space of smooth sections with compact support of $\left(\Lambda^{\cdot}(TX)\widehat{\otimes}F\right)_x$ along the fibre T_x^*X. Then the I_x are the fibres of an infinite dimensional $\mathbf{Z}$-graded vector bundle I on X. Using obvious notation, by (2.39),

$$\Omega^{\cdot}(T^*X, \pi^*F) = \Omega^{\cdot}(X, I). \tag{2.40}$$

Let $\widehat{d}^{T^*X}$ be the fibrewise de Rham operator acting on I. Then $\widehat{d}^{T^*X}$ acts on $\Omega^{\cdot}(T^*X, \pi^*F)$. Let $\nabla^{\Lambda^{\cdot}(TX)\otimes F}$ be the connection on $\Lambda^{\cdot}(TX)\otimes F$ induced by ∇^{TX} and ∇^F on $\Lambda^{\cdot}(TX)\otimes F$. Let ∇^I be the connection on I, such that if s is a smooth section of I, and $U \in TX$,

$$\nabla^I_U s = \nabla^{\Lambda^{\cdot}(TX)\otimes F}_{U^H} s. \tag{2.41}$$

The connection ∇^I also acts naturally as a differential operator on $\Omega^{\cdot}(T^*X, \pi^*F)$. Finally the operator $i_{\widehat{R^{TX}p}}$ acts on $\Omega^{\cdot}(T^*X, \pi^*F)$, it increases the total degree by 1, it increases the degree in $\Lambda^{\cdot}(T^*X)$ by 2 and it decreases the degree in $\Lambda^{\cdot}(TX)$ by 1.

Proposition 2.4. *The following identity of operators acting on $\Omega^{\cdot}(T^*X, \pi^*F)$ holds:*

$$d^{T^*X} = \widehat{d}^{T^*X} + \nabla^I + i_{\widehat{R^{TX}p}}. \tag{2.42}$$

Proof. Let ∇^{TT^*X} be the tautological connection on TT^*X , which is associated to the splitting (2.38). One verifies easily that its torsion is just the tensor $\pi^*\widehat{R^{TX}p}$, which implies the proposition. □

As in the proof of Proposition 2.4, we denote by $\nabla^{TT^*X} = \pi^*\left(\nabla^{TX} \oplus \nabla^{T^*X}\right)$ the tautological connection on $TT^*X = \pi^*(TX \oplus T^*X)$ associated to the Levi-Civita connection ∇^{TX}. Let $\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}$ be the induced connection on $\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F$.

Let $e_1, \ldots, e_n$ be a basis of TX, and let $e^1, \ldots, e^n$ be the corresponding dual basis of T^*X. We denote by $\widehat{e}_1, \ldots, \widehat{e}_n$ and $\widehat{e}^1, \ldots, \widehat{e}^n$ other copies of these two bases. By (2.38), $e_1, \ldots, e_n, \widehat{e}^1, \ldots, \widehat{e}^n$ is a basis of TT^*X, and $e^1, \ldots, e^n, \widehat{e}_1, \ldots, \widehat{e}_n$ is the corresponding dual basis of T^*T^*X.

From now on, we will use Einstein's summation conventions.

Proposition 2.5. *The following identity holds:*

$$d^{T^*X} = e^i \wedge \nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{e_i} + \widehat{e}_i \wedge \nabla_{\widehat{e}^i} + i_{\widehat{R^{TX}p}}. \tag{2.43}$$

Proof. Clearly,

$$\widehat{d}^{T^*X} = \widehat{e}_i \nabla_{\widehat{e}^i}, \qquad \nabla^I = e^i \wedge \nabla^F_{e_i}. \tag{2.44}$$

By (2.42), (2.44), we get (2.43). □

Proposition 2.6. *The following identities hold:*

$$\theta = \langle p, e_i \rangle e^i, \qquad \omega = \widehat{e}_i \wedge e^i. \tag{2.45}$$

Proof. The first identity is obvious. Using (2.43), we get the second identity. □

Remark 2.7. By combining the fact that $d\omega = 0$ with (2.42), (2.45), we recover the circular symmetry of the Levi-Civita curvature R^{TX}. Also note that by (2.45), we get the identities

$$L = \widehat{e}_i \wedge e^i, \qquad \Lambda = -i_{\widehat{e}^i} i_{e_i}. \tag{2.46}$$

Let $g^{TT^*X} = g^{TX} \oplus g^{T^*X}$ be the obvious metric on $TT^*X = TX \oplus T^*X$. Let J be the almost complex structure on TT^*X given in matrix form by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.47}$$

By (2.45), if $A, B \in TT^*X$,

$$\omega(X, Y) = \langle A, JB \rangle_{g^{TT^*X}}, \tag{2.48}$$

and moreover J is antisymmetric with respect to g^{TT^*X}, so that J polarizes ω.

Let $\overline{F}^*$ be the vector bundle antidual to F, and let $\nabla^{\overline{F}^*}$ be the corresponding flat connection. Let g^F be a Hermitian metric on F. We do not assume here that g^F is flat. By proceeding as in subsections 2.3 and 2.4, using (ω, g^F), we find that there is a natural duality between $\Omega^{\cdot}(T^*X, \pi^*F)$ and $\Omega^{\cdot}\left(T^*X, \pi^*\overline{F}^*\right)$. Let $\overline{d}^{T^*X}$ be the symplectic adjoint of the operator d^{T^*X} with respect to g^F. If $F = \mathbf{C}$ is the trivial Hermitian flat vector bundle, the operator $\overline{d}^{T^*X}$ is defined as in subsection 2.4. More generally, if (F, ∇^F, g^F) is an arbitrary flat vector bundle equipped with the metric g^F, we define $\overline{d}^{T^*X}$ as in subsection 2.4. In particular, the definition of $\overline{d}^{T^*X}$ does not involve the choice of any metric on TX.

Set

$$\omega\left(\nabla^F, g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F. \tag{2.49}$$

Then $\omega\left(\nabla^F, g^F\right)$ is a 1-form on X with values in self-adjoint elements of $\operatorname{End}(F)$. Let ∇^{F*} be the adjoint connection to ∇^F with respect to g^F, which is still a flat connection. Then

$$\nabla^{F*} = \nabla^F + \omega\left(\nabla^F, g^F\right). \tag{2.50}$$

Let $\nabla^{F,u}$ be the unitary connection on F,

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega\left(\nabla^F, g^F\right). \tag{2.51}$$

The curvature $R^{F,u}$ of $\nabla^{F,u}$ is given by

$$R^{F,u} = -\frac{1}{4}\omega^2\left(\nabla^F, g^F\right). \tag{2.52}$$

Moreover, one has the trivial

$$\nabla^F \omega\left(\nabla^F, g^F\right) = -\omega\left(\nabla^F, g^F\right)^2, \qquad \nabla^{F,u}_{\cdot}\omega\left(\nabla^F, g^F\right) = 0. \tag{2.53}$$

By (2.53), we find that if $A, B \in TX$,

$$\nabla^{F,u}_A \omega\left(\nabla^F, g^F\right)(B) = \nabla^{F,u}_B \omega\left(\nabla^F, g^F\right)(A). \tag{2.54}$$

Let g^{TX} be a Riemannian metric on TX. We now use the same notation as in subsection 2.5. In particular, we take $e_1, \ldots, e_n$ as in that subsection. Set

$$R^{TX} p\wedge = \frac{1}{2} i_{\widehat{e}^i} i_{\widehat{e}^j} R^{TX}(e_i, e_j)\, p \wedge. \tag{2.55}$$

First we will give a formula for the operator $\overline{d}^{T^*X}$.

Proposition 2.8. *The following identity holds:*

$$\overline{d}^{T^*X} = -i_{\widehat{e}^i}\left(\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{e_i} + \omega\left(\nabla^F, g^F\right)(e_i)\right) + i_{e_i}\nabla_{\widehat{e}^i} + R^{TX} p \wedge. \tag{2.56}$$

Proof. This follows from (2.10), (2.43) and from (2.45). □

2.6. **A bilinear form on T^*X and the adjoint of d^{T^*X}.** We fix once and for all a metric g^{TX} on TX, and we identify TX and T^*X by the metric g^{TX}. Also, we use the identifications in (2.38).

Set

$$\phi = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.57}$$

Observe that if

$$\psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.58}$$

then

$$\phi = \psi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.59}$$

and moreover,

$$\psi = \exp \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.60}$$

Clearly,

$$\phi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{2.61}$$

Using the identifications in (2.38), ϕ will be considered as an isomorphism $TT^*X \to T^*T^*X$. In more intrinsic terms, ϕ and ϕ^{-1} are given in matrix form by

$$\phi = \begin{pmatrix} g^{TX} & -1_{|T^*X} \\ 1_{|TX} & 0 \end{pmatrix}, \qquad \phi^{-1} = \begin{pmatrix} 0 & 1_{|TX} \\ -1_{|T^*X} & g^{TX} \end{pmatrix}. \tag{2.62}$$

Let η be the bilinear form on TT^*X which is associated to ϕ as in (2.23). If $U, V \in TT^*X$, we find that

$$\eta(U,V) = \langle \pi_* U, \pi_* V \rangle_{g^{TX}} + \omega(U,V). \tag{2.63}$$

The right-hand side of (2.63) gives the splitting of η into its symmetric and anti-symmetric parts.

Let dv_{T^*X} be the symplectic volume form on T^*X. We equip $\Omega^{\cdot}(T^*X, \pi^*F)$ with the nondegenerate sesquilinear form which one obtains as in (2.26), i.e. if $s, s' \in \Omega^{\cdot}(T^*X, \pi^*F)$,

$$\langle s, s' \rangle_\phi = \int_{T^*X} \eta^* (s, s')_{g^F} \, dv_{T^*X}. \tag{2.64}$$

Let $e_1, \ldots, e_n$ be a basis of TX, and let $e^1, \ldots, e^n$ be the corresponding dual basis.

Definition 2.9. Let $\overline{d}_\phi^{T^*X}$ be the adjoint of d^{T^*X} with respect to $\langle\ \rangle_\phi$.

Put

$$\lambda_0 = \left\langle g^{TX} e_i, e_j \right\rangle e^i \wedge i_{\widehat{e}^j}, \qquad \delta^{T^*X,V} = -\left\langle g^{TX} e_i, e_j \right\rangle i_{\widehat{e}^i} \nabla_{\widehat{e}^j}. \tag{2.65}$$

In the sequel, we will assume that the basis $e_1, \ldots, e_n$ is orthonormal, so that

$$\lambda_0 = e^i \wedge i_{\widehat{e}^i}, \qquad \delta^{T^*X,V} = -i_{\widehat{e}^i} \nabla_{\widehat{e}^i}. \tag{2.66}$$

Proposition 2.10. *The following identity holds:*

$$\overline{d}_{\phi}^{T^*X} = e^{\lambda_0}\overline{d}^{T^*X}e^{-\lambda_0}. \tag{2.67}$$

Equivalently,

$$\overline{d}_{\phi}^{T^*X} = \overline{d}^{T^*X} - \left[\overline{d}^{T^*X}, \lambda_0\right]. \tag{2.68}$$

Also,

$$\begin{aligned} \left[\overline{d}^{T^*X}, \lambda_0\right] &= -\delta^{T^*X,V}, \\ \overline{d}_{\phi}^{T^*X} &= -i_{\widehat{e}^i}\left(\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{e_i} + \omega\left(\nabla^F, g^F\right)(e_i)\right) \\ &\quad + i_{e_i}\nabla_{\widehat{e}^i} + R^{TX}p \wedge -i_{\widehat{e}^i}\nabla_{\widehat{e}^i}. \end{aligned} \tag{2.69}$$

Proof. By (2.25), (2.59) and (2.60), we get (2.67). Using (2.56), we get the first equation in (2.69), and so

$$\left[\left[\overline{d}^{T^*X}, \lambda_0\right], \lambda_0\right] = 0. \tag{2.70}$$

From (2.56), (2.67) and (2.70), we get (2.68) and the second equation in (2.69). The proof of our proposition is completed. □

Remark 2.11. By (2.57), (2.61), we find that for $1 \leq i \leq n$,

$$\widetilde{\phi}^{-1}e^i = \widehat{e}^i, \qquad \widetilde{\phi}^{-1}\widehat{e}_i = -e_i + \widehat{e}^i, \qquad \phi\widehat{e}^i = -e^i. \tag{2.71}$$

Using (2.11), (2.71), we get another proof of the second identity in (2.69).

2.7. **A fundamental symmetry.** Set

$$f = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{2.72}$$

Then f defines a scalar product on $\mathbf{R}^2$, and F is an involution of $\mathbf{R}^2$, which is an isometry with respect to f. Its $+1$ eigenspace is spanned by $(1,0)$, and the -1 eigenspace is spanned by the $(1,-1)$. Finally, the volume form on $\mathbf{R}^2$ which is attached to f is just the original volume form of $\mathbf{R}^2$.

Set

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.73}$$

Then one verifies easily that

$$fFh = \phi. \tag{2.74}$$

Using the identifications in (2.38), we observe that f defines a metric $\mathfrak{g}^{TT^*X}$ on TT^*X given by

$$\mathfrak{g}^{TT^*X} = \begin{pmatrix} g^{TX} & 1|_{T^*X} \\ 1|_{TX} & 2g^{T^*X} \end{pmatrix}. \tag{2.75}$$

Let $\mathfrak{p} : TT^*X \to T^*X$ be the obvious projection with respect to the splitting (2.38) of TT^*X. Then if $U \in TT^*X$,

$$\langle U, U\rangle_{\mathfrak{g}^{TT^*X}} = \langle \pi_* U, \pi_* U\rangle_{g^{TX}} + 2\langle \pi_* U, \mathfrak{p}U\rangle + 2\langle \mathfrak{p}U, \mathfrak{p}U\rangle_{g^{T^*X}}. \tag{2.76}$$

Then the volume form on T^*X which is attached to $\mathfrak{g}^{TT^*X}$ is the symplectic volume form.

Similarly, we will identify F with the $\mathfrak{g}^{TT^*X}$ isometric involution of TT^*X,

$$F = \begin{pmatrix} 1|_{TX} & 2\left(g^{TX}\right)^{-1} \\ 0 & -1|_{T^*X} \end{pmatrix}. \tag{2.77}$$

Then F acts as $\widetilde{F}^{-1} = \widetilde{F}$ on $\Lambda^{\cdot}\left(T^*T^*X\right)$.

Let $r : T^*X \to T^*X$ be the involution $(x,p) \to (x,-p)$.

Definition 2.12. Let $\langle\ \rangle_{\mathfrak{g}^{\Omega^{\cdot}(T^*X,\pi^*F)}}$ be the Hermitian product on $\Omega^{\cdot}\left(T^*X,\pi^*F\right)$ which is naturally associated to the metrics $\mathfrak{g}^{TT^*X}$ and g^F. Let u be the isometric involution of $\Omega^{\cdot}\left(T^*X,\pi^*F\right)$ with respect to $\langle\ \rangle_{\mathfrak{g}^{\Omega^{\cdot}(T^*X,\pi^*F)}}$ such that if $s \in \Omega^{\cdot}\left(T^*X,\pi^*F\right)$,

$$us\left(x,p\right) = Fs(x,-p). \tag{2.78}$$

Let $\langle\ \rangle_{\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}}$ be the Hermitian form on $\Omega^{\cdot}\left(T^*X,\pi^*F\right)$,

$$\langle s,s'\rangle_{\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}} = \langle us,s'\rangle_{\mathfrak{g}^{\Omega^{\cdot}(T^*X,\pi^*F)}}. \tag{2.79}$$

It should be pointed out that in (2.78), the change of variable $p \to -p$ is not made on the form part of s. So this action does not incorporate the full action of r^*.

Theorem 2.13. *The operator $\overline{d}_{\phi}^{T^*X}$ is the $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ adjoint of d^{T^*X}.*

Proof. The proof of our theorem can easily be done by a direct computation, using in particular (2.43) and (2.69). Here, we will give a more conceptual proof. Since F is a $\mathfrak{g}^{TT^*X}$ isometry, by (2.74), we get

$$\widetilde{F}^{-1} fh = \phi. \tag{2.80}$$

Moreover,

$$r^* d^{T^*X} r^{*-1} = d^{T^*X}. \tag{2.81}$$

By (2.80), (2.81), we get our theorem easily. □

Remark 2.14. The main point about Theorem 2.13 is that the form in (2.64) is neither Hermitian nor skew-adjoint, while the form (2.79) is Hermitian, and nondegenerate. We will then be able to apply the formalism of Section 1 to the present situation.

2.8. **A Hamiltonian function.** Let $\mathcal{H} : T^*X \to \mathbf{R}$ be a smooth function. Let $Y^{\mathcal{H}}$ be the corresponding Hamiltonian vector field. Observe that by the second identity in (2.45),

$$Y^{\mathcal{H}} = \left(\nabla_{\widehat{e}^i}\mathcal{H}\right) e_i - \left(\nabla_{e_i}\mathcal{H}\right)\widehat{e}^i. \tag{2.82}$$

In the sequel, we will also use the notation

$$\widehat{\nabla^V}\mathcal{H} = \left(\nabla_{\widehat{e}^i}\mathcal{H}\right)\widehat{e}^i. \tag{2.83}$$

Then $\widehat{\nabla^V}\mathcal{H}$ is just the fibrewise gradient field of $\mathcal{H}$.

Definition 2.15. Set

$$d_{\mathcal{H}}^{T^*X} = e^{-\mathcal{H}} d^{T^*X} e^{\mathcal{H}}, \qquad \overline{d}_{\phi,\mathcal{H}}^{T^*X} = e^{\mathcal{H}} \overline{d}_{\phi}^{T^*X} e^{-\mathcal{H}}. \tag{2.84}$$

Observe that $\overline{d}^{T^*X}_{\phi,\mathcal{H}}$ is the adjoint of $d^{T^*X}_{\mathcal{H}}$ with respect to the Hermitian form $\langle\,\rangle_\phi$ in (2.64). Also, if $s, s' \in \Omega^{\cdot}(T^*X, \pi^*F)$, put

$$\langle s, s'\rangle_{\phi,\mathcal{H}} = \int_{T^*X} \eta^* (s, s')_{g^F} e^{-2\mathcal{H}} dv_{T^*X}. \tag{2.85}$$

Then $\overline{d}^{T^*X}_{\phi,2\mathcal{H}}$ is the adjoint of d^{T^*X} with respect to $\langle\,\rangle_{\phi,\mathcal{H}}$. The map $s \to e^{-\mathcal{H}}s$ induces an isomorphism of complexes $\left(\Omega^{\cdot}(T^*X, \pi^*F), d^{T^*X}\right) \to \left(\Omega^{\cdot}(T^*X, \pi^*F), d^{T^*X}_{\mathcal{H}}\right)$, which maps $\langle\,\rangle_{\phi,\mathcal{H}}$ into $\langle\,\rangle_\phi$ and $\overline{d}^{T^*X}_{\phi,2\mathcal{H}}$ into $\overline{d}^{T^*X}_{\phi,\mathcal{H}}$.

Proposition 2.16. *The following identities hold:*

$$\begin{aligned} d^{T^*X}_{\mathcal{H}} &= d^{T^*X} + d^{T^*X}\mathcal{H}, \\ \overline{d}^{T^*X}_{\phi,\mathcal{H}} &= \overline{d}^{T^*X} - i_{Y^{\mathcal{H}}} - \left[\overline{d}^{T^*X} - i_{Y^{\mathcal{H}}}, \lambda_0\right] \\ &= \overline{d}^{T^*X} - i_{Y^{\mathcal{H}} - \widehat{\nabla^V}\mathcal{H}} + \delta^{T^*X,V}. \end{aligned} \tag{2.86}$$

More precisely,

$$\begin{aligned} d^{T^*X}_{\mathcal{H}} =& e^i \wedge \left(\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{e_i} + \nabla_{e_i}\mathcal{H}\right) + \widehat{e}_i \wedge \left(\nabla_{\widehat{e}^i} + \nabla_{\widehat{e}^i}\mathcal{H}\right) + i_{\widehat{R^{TX}p}}, \\ \overline{d}^{T^*X}_{\phi,\mathcal{H}} =& - i_{\widehat{e}^i}\left(\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{e_i} + \omega\left(\nabla^F, g^F\right)(e_i) - \nabla_{e_i}\mathcal{H}\right) \\ &+ i_{e_i}\left(\nabla_{\widehat{e}^i} - \nabla_{\widehat{e}^i}\mathcal{H}\right) + R^{TX}p \wedge -i_{\widehat{e}^i}\left(\nabla_{\widehat{e}^i} - \nabla_{\widehat{e}^i}\mathcal{H}\right). \end{aligned} \tag{2.87}$$

Proof. This follows from Propositions 2.3, 2.5 and 2.10. □

Definition 2.17. Set

$$\begin{aligned} A_{\phi,\mathcal{H}} &= \frac{1}{2}\left(\overline{d}^{T^*X}_{\phi,2\mathcal{H}} + d^{T^*X}\right), & B_{\phi,\mathcal{H}} &= \frac{1}{2}\left(\overline{d}^{T^*X}_{\phi,2\mathcal{H}} - d^{T^*X}\right), \\ \mathfrak{A}_{\phi,\mathcal{H}} &= \frac{1}{2}\left(\overline{d}^{T^*X}_{\phi,\mathcal{H}} + d^{T^*X}_{\mathcal{H}}\right), & \mathfrak{B}_{\phi,\mathcal{H}} &= \frac{1}{2}\left(\overline{d}^{T^*X}_{\phi,\mathcal{H}} - d^{T^*X}_{\mathcal{H}}\right). \end{aligned} \tag{2.88}$$

Clearly,

$$\mathfrak{A}_{\phi,\mathcal{H}} = e^{-\mathcal{H}} A_{\phi,\mathcal{H}} e^{\mathcal{H}}, \qquad \mathfrak{B}_{\phi,\mathcal{H}} = e^{-\mathcal{H}} B_{\phi,\mathcal{H}} e^{\mathcal{H}}. \tag{2.89}$$

In the sequel, we will use the two couples $A_{\phi,\mathcal{H}}, B_{\phi,\mathcal{H}}$ or $\mathfrak{A}_{\phi,\mathcal{H}}, \mathfrak{B}_{\phi,\mathcal{H}}$ indiscriminately. We will also establish algebraic properties of one of the couples, while freely using these properties for the other couple.

We have the identities

$$d^{T^*X,2} = 0, \qquad \overline{d}^{T^*X,2}_{\phi,\mathcal{H}} = 0. \tag{2.90}$$

From (2.90), we deduce that

$$\begin{aligned} A^2_{\phi,\mathcal{H}} = -B^2_{\phi,\mathcal{H}} &= \frac{1}{4}\left[d^{T^*X}, \overline{d}^{T^*X}_{\phi,2\mathcal{H}}\right], & \left[A_{\phi,\mathcal{H}}, B_{\phi,\mathcal{H}}\right] &= 0, \\ \left[d^{T^*X}, A^2_{\phi,\mathcal{H}}\right] &= 0, & \left[\overline{d}^{T^*X}_{\phi,2\mathcal{H}}, A^2_{\phi,\mathcal{H}}\right] &= 0. \end{aligned} \tag{2.91}$$

Let $\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}$ be the connection on $\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F$ which is associated to ∇^{TT^*X} and $\nabla^{F,u}$.

Proposition 2.18. *The following identities hold:*

$$
\begin{aligned}
A_{\phi,\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} - \frac{1}{4}\left(e^i + i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i) \\
&\quad + \frac{1}{2}\left(\widehat{e}_i + i_{e_i - \widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p \wedge + i_{\widehat{R^{TX}p}}\right) \\
&\quad + i_{\widehat{e}^i}\nabla_{e_i}\mathcal{H} + i_{\widehat{e}^i - e_i}\nabla_{\widehat{e}^i}\mathcal{H}, \\
B_{\phi,\mathcal{H}} &= -\frac{1}{2}\left(e^i + i_{\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} + \frac{1}{4}\left(e^i - i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i) \\
&\quad - \frac{1}{2}\left(\widehat{e}_i + i_{\widehat{e}^i - e_i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p \wedge - i_{\widehat{R^{TX}p}}\right) \\
&\quad + i_{\widehat{e}^i}\nabla_{e_i}\mathcal{H} + i_{\widehat{e}^i - e_i}\nabla_{\widehat{e}^i}\mathcal{H}, \\
\mathfrak{A}_{\phi,\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} - \frac{1}{4}\left(e^i + i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i) \\
&\quad + \frac{1}{2}\left(\widehat{e}_i + i_{e_i - \widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p \wedge + i_{\widehat{R^{TX}p}}\right) \\
&\quad + \frac{1}{2}\left(e^i + i_{\widehat{e}^i}\right)\nabla_{e_i}\mathcal{H} + \frac{1}{2}\left(\widehat{e}_i + i_{\widehat{e}^i - e_i}\right)\nabla_{\widehat{e}^i}\mathcal{H}, \\
\mathfrak{B}_{\phi,\mathcal{H}} &= -\frac{1}{2}\left(e^i + i_{\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} + \frac{1}{4}\left(e^i - i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i) \\
&\quad - \frac{1}{2}\left(\widehat{e}_i + i_{\widehat{e}^i - e_i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p \wedge - i_{\widehat{R^{TX}p}}\right) \\
&\quad - \frac{1}{2}\left(e^i - i_{\widehat{e}^i}\right)\nabla_{e_i}\mathcal{H} - \frac{1}{2}\left(\widehat{e}_i + i_{e_i - \widehat{e}^i}\right)\nabla_{\widehat{e}^i}\mathcal{H}.
\end{aligned}
\tag{2.92}
$$

Proof. This follows from Proposition 2.16. □

Remark 2.19. Let $\xi = \xi^H + \xi^V$ be the generic element of $T^*T^*X = T^*X \oplus TX$. Let $\sigma(A)$ denote the principal symbol of a differential operator A. By Proposition 2.18, we get

$$
\sigma\left(A_{\phi,\mathcal{H}}\right) = \frac{i}{2}\left(\xi^H + \widehat{\xi^V} + i_{\xi^V - \widehat{\xi^H} - \widehat{\xi^V}}\right), \tag{2.93}
$$

and so

$$
\sigma\left(A_{\phi,\mathcal{H}}^2\right) = \frac{1}{4}\left|\xi^V\right|^2. \tag{2.94}
$$

Therefore the operator $A_{\phi,\mathcal{H}}$ is certainly not elliptic. Incidentally, note that if $F = \mathbf{C}$,

$$
\sigma\left(\frac{1}{2}\left(d^{T^*X} + \overline{d}^{T^*X}\right)\right) = \frac{i}{2}\left(\xi^H + \widehat{\xi^V} + i_{\xi^V - \widehat{\xi^H}}\right), \tag{2.95}
$$

so that

$$
\sigma\left(\frac{1}{4}\left(d^{T^*X} + \overline{d}^{T^*X}\right)^2\right) = 0, \tag{2.96}
$$

which fits with Theorem 2.2, which asserts that the operator in (2.96) vanishes identically.

2.9. **The symmetry in the case where $\mathcal{H}$ is r-invariant.** Put

$$\mathcal{H}_{-}(x,p)=\mathcal{H}(x,-p). \tag{2.97}$$

Recall that the Hermitian product $\mathfrak{g}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ on $\Omega^{\cdot}(T^*X,\pi^*F)$ was defined in Definition 2.12. Let $\langle\,\rangle_{\mathfrak{h}_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}}$ be the sesquilinear form on $\Omega^{\cdot}(T^*X,\pi^*F)$,

$$\langle s,s'\rangle_{\mathfrak{h}_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}}=\left\langle ue^{-2\mathcal{H}}s,s'\right\rangle_{\mathfrak{g}^{\Omega^{\cdot}(T^*X,\pi^*F)}}. \tag{2.98}$$

Note that $\langle\,\rangle_{h_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}}$ is a Hermitian form if and only if $\mathcal{H}$ is r-invariant.

Proposition 2.20. *The operator $\overline{d}_{\phi,2\mathcal{H}}^{T^*X}$ is the $\mathfrak{h}_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ adjoint of d^{T^*X}, and the operator $\overline{d}_{\phi,\mathcal{H}_{-}}^{T^*X}$ is the $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ adjoint of $d_{\mathcal{H}}^{T^*X}$.*

Proof. This is a trivial consequence of Theorem 2.13. □

Theorem 2.21. *If $\mathcal{H}$ is r-invariant, then $A_{\phi,\mathcal{H}}$ (resp. $B_{\phi,\mathcal{H}}$) is $\mathfrak{h}_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ self-adjoint (resp. skew-adjoint), and $\mathfrak{A}_{\phi,\mathcal{H}}$ (resp. $\mathfrak{B}_{\phi,\mathcal{H}}$) is $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ self-adjoint (resp. skew-adjoint).*

Proof. This is a trivial consequence of Proposition 2.20. □

2.10. **Poincaré duality.** In classical Hodge theory, the operator d^{X*} is conjugate to d^X by a conjugation involving the Hodge $*$ operator. This gives a Hodge theoretic version of Poincaré duality. Here we give a corresponding version of Poincaré duality.

Let $*^{T^*X}$ be the Hodge operator which is associated to the metric $\mathfrak{g}^{T^*X}$ and to the orientation of T^*X by the symplectic form ω. Let $\kappa^F:\Omega^{\cdot}(T^*X,\pi^*F)\to\Omega^{2n-\cdot}\left(T^*X,\pi^*\overline{F}^*\right)$ be the linear map such that if $s\in\Omega^i(T^*X,\pi^*F)$, then

$$\kappa^F s=(-1)^{i(i+1)/2}\,u*^{T^*X}g^F s. \tag{2.99}$$

Then

$$\kappa^{\overline{F}^*}\kappa^F=1. \tag{2.100}$$

Set

$$\kappa_{\mathcal{H}}^F=\kappa^F e^{-2\mathcal{H}}. \tag{2.101}$$

By (2.100), we get

$$\kappa_{\mathcal{H}}^{F,-1}=\kappa_{-\mathcal{H}_{-}}^{\overline{F}^*}. \tag{2.102}$$

We use temporarily the notation $d^{T^*X,F}$ instead of d^{T^*X} and corresponding notation for the other operators. As a consequence of Proposition 2.20, we easily get

$$\overline{d}_{\phi,2\mathcal{H}}^{T^*X,F}=\kappa_{\mathcal{H}}^{F,-1}d^{T^*X,\overline{F}^*}\kappa_{\mathcal{H}}^F,\qquad \overline{d}_{\phi,\mathcal{H}}^{T^*X,F}=\kappa^{F,-1}d_{-\mathcal{H}_{-}}^{T^*X,\overline{F}^*}\kappa^F. \tag{2.103}$$

From (2.103), we find that if $\mathcal{H}$ is r-invariant,

$$A^F_{\phi,\mathcal{H}} = \kappa^{F,-1}_{\mathcal{H}} A^{\overline{F}^*}_{\phi,-\mathcal{H}} \kappa^F_{\mathcal{H}}, \qquad B^F_{\phi,\mathcal{H}} = -\kappa^{F,-1}_{\mathcal{H}} B^{\overline{F}^*}_{\phi,-\mathcal{H}} \kappa^F_{\mathcal{H}},$$
$$\mathfrak{A}^F_{\phi,\mathcal{H}} = \kappa^{F,-1} \mathfrak{A}^{\overline{F}^*}_{\phi,-\mathcal{H}} \kappa^F, \qquad \mathfrak{B}^F_{\phi,\mathcal{H}} = -\kappa^{F,-1} \mathfrak{B}^{\overline{F}^*}_{\phi,-\mathcal{H}} \kappa^F. \tag{2.104}$$

Note here that because of (2.102), (2.104) is indeed symmetric in F and $\overline{F}^*$.

2.11. **A conjugation of the de Rham operator.** Recall that $g^{TT^*X} = g^{TX} \oplus g^{T^*X}$. Let $\langle\,\rangle_{g^{\Omega^\cdot(T^*X,\pi^*F)}}$ be the Hermitian product on $\Omega^\cdot(T^*X,\pi^*F)$, which is associated to g^{TT^*X}, g^F. In this subsection, we identify TX and T^*X by the metric g^{TX}. In particular ϕ will be viewed here as the element of $\mathrm{End}\,(TX \oplus T^*X)$ whose matrix is (2.57). This will also be the case for $\widetilde{\phi}^{-1}$.

Let d^{T^*X*} be the ordinary adjoint of d^{T^*X} with respect to $g^{\Omega^\cdot(T^*X,\pi^*F)}$.

Proposition 2.22. *The following identity holds:*

$$d^{T^*X*} = -i_{e_i}\left(\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F}_{e_i} + \omega\left(\nabla^F, g^F\right)(e_i)\right) - i_{\widehat{e}^i}\nabla_{\widehat{e}^i} - \frac{1}{2} i_{e_i} i_{e_j} \widehat{e}_k \left\langle R^{TX}(e_i,e_j)\,p, e_k\right\rangle. \tag{2.105}$$

Proof. This follows from (2.43). □

Definition 2.23. Put

$$d^{T^*X*}_{\mathcal{H}} = e^{\mathcal{H}} d^{T^*X*} e^{-\mathcal{H}}. \tag{2.106}$$

Definition 2.24. Set

$$\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2.107}$$

Note that

$$\widetilde{\rho}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \widetilde{\sigma}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{2.108}$$

Recall that ϕ was defined in (2.57). Then using (2.61), we also get the formula

$$\sigma = \rho\widetilde{\phi}^{-1}. \tag{2.109}$$

Again we will consider ρ, σ as automorphisms of $TT^*X \simeq TX \oplus T^*X$, so that (2.109) is an identity of automorphisms of TT^*X.

The corresponding actions of ρ, σ on $T^*T^*X \simeq T^*X \oplus TX$ are given by $\widetilde{\rho}^{-1}, \widetilde{\sigma}^{-1}$. They induce obvious actions on $\Omega^\cdot(T^*X,\pi^*F)$.

Recall that λ_0 was defined in (2.65), (2.66). Set

$$\mu_0 = \widehat{e}_i \wedge i_{e_i}. \tag{2.110}$$

Then μ_0 is the standard adjoint of λ_0.

By (2.107), (2.108), we have the identities of operators acting on $\Omega^\cdot(T^*X,\pi^*F)$,

$$\rho = e^{-\mu_0}, \qquad \sigma = e^{-\lambda_0}. \tag{2.111}$$

By (2.111), conjugating operators acting on $\Omega^{\cdot}\left(T^*X,\pi^*F\right)$ by ρ is equivalent to replacing e^i by $e^i-\widehat{e}_i$, $i_{\widehat{e}^i}$ by $i_{\widehat{e}^i+e_i}$, while leaving the other creation or annihilation operators unchanged, and conjugating by σ is equivalent to replacing $\widehat{e}_i$ by $\widehat{e}_i-e^i$, i_{e_i} by $i_{e_i+\widehat{e}^i}$, while leaving the other operators unchanged.

Definition 2.25. Put

$$d_{\phi,\mathcal{H}}^{T^*X\prime}=\rho d_{\mathcal{H}}^{T^*X}\rho^{-1},\qquad \overline{d}_{\phi,\mathcal{H}}^{T^*X\prime *}=\sigma d_{\mathcal{H}}^{T^*X*}\sigma^{-1}. \tag{2.112}$$

Using (2.109), we get

$$\overline{d}_{\phi,\mathcal{H}}^{T^*X\prime *}=\rho\overline{d}_{\phi,\mathcal{H}}^{T^*X}\rho^{-1}. \tag{2.113}$$

Theorem 2.26. *The following identities hold:*

$$d_{\phi,\mathcal{H}}^{T^*X\prime}=e^{-\mu_0}d_{\mathcal{H}}^{T^*X}e^{\mu_0},\qquad \overline{d}_{\phi,\mathcal{H}}^{T^*X\prime *}=e^{-\lambda_0}d_{\mathcal{H}}^{T^*X*}e^{\lambda_0}. \tag{2.114}$$

Proof. Equation (2.114) is a trivial consequence of (2.112) and (2.111). The proof of our theorem is completed. □

Definition 2.27. Put

$$\mathfrak{A}'_{\phi,\mathcal{H}}=\frac{1}{2}\left(\overline{d}_{\phi,\mathcal{H}}^{T^*X\prime *}+d_{\phi,\mathcal{H}}^{T^*X\prime}\right),\qquad \mathfrak{B}'_{\phi,\mathcal{H}}=\frac{1}{2}\left(\overline{d}_{\phi,\mathcal{H}}^{T^*X\prime *}-d_{\phi,\mathcal{H}}^{T^*X\prime}\right). \tag{2.115}$$

By (2.112), (2.113), we get

$$\mathfrak{A}'_{\phi,\mathcal{H}}=\rho\mathfrak{A}_{\phi,\mathcal{H}}\rho^{-1},\qquad \mathfrak{B}'_{\phi,\mathcal{H}}=\rho\mathfrak{B}_{\phi,\mathcal{H}}\rho^{-1}. \tag{2.116}$$

Proposition 2.28.

(2.117)

$$\begin{aligned}
\mathfrak{A}'_{\phi,\mathcal{H}}&=\frac{1}{2}\left(c(e_i)-\widehat{c}(\widehat{e}^i)\right)\nabla_{e_i}^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}\\
&\quad-\frac{1}{4}\left(\widehat{c}(e_i)-c(\widehat{e}^i)\right)\omega\left(\nabla^F,g^F\right)(e_i)+\frac{1}{2}c(\widehat{e}^i)\nabla_{\widehat{e}^i}\\
&\quad+\frac{1}{4}\left(\left(e^i-\widehat{e}_i\right)\left(e^j-\widehat{e}_j\right)i_{\widehat{e}^k+e_k}-i_{e_i+\widehat{e}^i}i_{e_j+\widehat{e}^j}\left(\widehat{e}_k-e^k\right)\right)\left\langle R^{TX}\left(e_i,e_j\right)p,e_k\right\rangle\\
&\quad+\frac{1}{2}\left(\widehat{c}(e_i)-c(\widehat{e}^i)\right)\nabla_{e_i}\mathcal{H}+\frac{1}{2}\widehat{c}\left(\widehat{\nabla^V}\mathcal{H}\right),\\
\mathfrak{B}'_{\phi,\mathcal{H}}&=-\frac{1}{2}\left(\widehat{c}\left(e_i\right)-c\left(\widehat{e}^i\right)\right)\nabla_{e_i}^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}\\
&\quad+\frac{1}{4}\left(c\left(e_i\right)-\widehat{c}\left(\widehat{e}^i\right)\right)\omega\left(\nabla^F,g^F\right)(e_i)-\frac{1}{2}\widehat{c}\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i}\\
&\quad-\frac{1}{4}\left(\left(e^i-\widehat{e}_i\right)\left(e^j-\widehat{e}_j\right)i_{\widehat{e}^k+e_k}+i_{e_i+\widehat{e}^i}i_{e_j+\widehat{e}^j}\left(\widehat{e}_k-e^k\right)\right)\left\langle R^{TX}\left(e_i,e_j\right)p,e_k\right\rangle\\
&\quad-\frac{1}{2}\left(c\left(e_i\right)-\widehat{c}\left(\widehat{e}^i\right)\right)\nabla_{e_i}\mathcal{H}-\frac{1}{2}c\left(\widehat{\nabla^V}\mathcal{H}\right).
\end{aligned}$$

Proof. Using Proposition 2.18, (2.111) and (2.116), we get (2.117). □

We will now interpret the symmetries of subsection 2.7 for these new conjugate operators. Set

$$G=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{2.118}$$

Observe that

$$\widetilde{\rho}^{-1} f \rho^{-1} = 1, \qquad \rho F \rho^{-1} = G. \tag{2.119}$$

We can rewrite (2.119) in terms of metrics and endomorphisms of $TT^*X = TX \oplus T^*X$. Namely, from (2.119), we get

$$\rho_* \mathfrak{g}^{TT^*X} = g^{TT^*X}, \qquad \rho_* F = G. \tag{2.120}$$

The main point of (2.120) is that g^{TT^*X} is the standard metric on TT^*X and that G is just the standard involution on TT^*X. Let $h^{\Omega^\cdot(T^*X,\pi^*F)}$ be the Hermitian form on $\Omega^\cdot(T^*X,\pi^*F)$,

$$\langle s, s' \rangle_{h^{\Omega^\cdot(T^*X,\pi^*F)}} = \langle r^* s, s' \rangle_{g^{\Omega^\cdot(T^*X,\pi^*F)}}. \tag{2.121}$$

Proposition 2.29. *The operator $\overline{d}^{T^*X\prime *}_{\phi,\mathcal{H}_-}$ is the $h^{\Omega^\cdot(T^*X,\pi^*F)}$ adjoint of $d^{T^*X\prime}_{\phi,\mathcal{H}}$.*

Proof. This is an obvious consequence of Proposition 2.20 and of (2.120). □

Theorem 2.30. *If $\mathcal{H}$ is r-invariant, then $\mathfrak{A}'_{\phi,\mathcal{H}}$ (resp. $\mathfrak{B}'_{\phi,\mathcal{H}}$) is $h^{\Omega^\cdot(T^*X,\pi^*F)}$ self-adjoint (resp. skew-adjoint).*

Proof. This is an obvious consequence of Proposition 2.29. □

Remark 2.31. Needless to say, Theorem 2.30 can also be very simply proved by using the explicit formulas of Proposition 2.28. Also note that Theorem 2.30 has a much simpler form than Theorem 2.21. Indeed the metric g^{TT^*X} and the involution r are natural, while the metric $\mathfrak{g}^{TT^*X}$ and the involution F are more exotic.

2.12. **The scaling of the variable p.** For $b \in \mathbf{R}_*$, set

$$r_b(x,p) = (x, bp). \tag{2.122}$$

Then $r = r_{-1}$. Let $\eta(U,V)$ be the bilinear form on TT^*X given by (2.63). Put

$$\phi_b = \begin{pmatrix} 1 & -b \\ b & 0 \end{pmatrix}, \qquad \eta_b = r_b^* \eta, \qquad \mathcal{H}_b = r_b^* \mathcal{H}. \tag{2.123}$$

Then

$$\eta_b(U,V) = \langle \pi_* U, \pi_* V \rangle_{g^{TX}} + b\omega(U,V). \tag{2.124}$$

Moreover ϕ_b is associated to η_b as in (2.23). As in (2.62), we identify ϕ_b with an endomorphism of TT^*X. Note that

$$r_b^* \eta(U,V) = b\left(\langle \pi_* U, \pi_* V \rangle_{g^{TX}/b} + \omega(U,V) \right). \tag{2.125}$$

Incidentally, observe that if $b < 0$, g^{TX}/b is not a metric. However this fact does not play any role in the sequel. Also note that

$$\phi_b^{-1} = \begin{pmatrix} 0 & 1/b \\ -1/b & 1/b^2 \end{pmatrix}. \tag{2.126}$$

We can now consider the objects of the previous sections, which are now associated to ϕ_b. We will denote them by the subscript ϕ_b.

To ϕ_b, instead of f, F, we should now associate f_b, F_b given by

$$f_b = \begin{pmatrix} 1 & b \\ b & 2b^2 \end{pmatrix}, \qquad F_b = \begin{pmatrix} 1 & 2b \\ 0 & -1 \end{pmatrix}. \tag{2.127}$$

Instead of (2.107), we should now have

$$\rho_b = \begin{pmatrix} 1 & b \\ 0 & b \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 1 & 0 \\ 1 & 1/b \end{pmatrix}. \tag{2.128}$$

The obvious analogues of (2.74), (2.109) and (2.119) still hold, with h and G remaining unchanged. Moreover the conclusions of Theorem 2.30 remain unchanged, by simply replacing ϕ by ϕ_b. Observe that this last fact also follows from (2.125). As in (2.72), (2.75), to f_b, we can associate a metric $\mathfrak{g}_b^{TT^*X}$ on TT^*X. Note that

$$\mathfrak{g}_b^{TT^*X} = r_b^* \mathfrak{g}^{TT^*X}. \tag{2.129}$$

For $a \in \mathbf{R}^*$, let K_a be the transformation of $\Omega^{\cdot}(T^*X, \pi^*F)$ which is given by $f(x,p) \to f(x,ap)$. Note that K_a does not scale the forms on T^*X.

Proposition 2.32. *The following identities hold:*

$$\begin{aligned}
&\overline{d}_{\phi_b}^{T^*X} = e^{\lambda_0/b}\overline{d}^{T^*X}/be^{-\lambda_0/b}, && r_b^* d^{T^*X} r_b^{*-1} = d^{T^*X}, \\
&r_b^* \overline{d}^{T^*X} r_b^{*-1} = \overline{d}^{T^*X}/b, && r_b^* \overline{d}_{\phi}^{T^*X} r_b^{*-1} = \overline{d}_{\phi_b}^{T^*X}, \\
&r_b^* d_{\mathcal{H}_{1/b}}^{T^*X} r_b^{*-1} = d_{\mathcal{H}}^{T^*X}, && r_b^* \overline{d}_{\phi,\mathcal{H}_{1/b}}^{T^*X} r_b^{*-1} = \overline{d}_{\phi_b,\mathcal{H}}^{T^*X}, \\
&r_b^* A_{\phi,\mathcal{H}_{1/b}} r_b^{*-1} = A_{\phi_b,\mathcal{H}}, && r_b^* B_{\phi,\mathcal{H}_{1/b}} r_b^{*-1} = B_{\phi_b,\mathcal{H}}, \\
&r_b^* \mathfrak{A}_{\phi,\mathcal{H}_{1/b}} r_b^{*-1} = \mathfrak{A}_{\phi_b,\mathcal{H}}, && r_b^* \mathfrak{B}_{\phi,\mathcal{H}_{1/b}} r_b^{*-1} = \mathfrak{B}_{\phi_b,\mathcal{H}}, \\
&K_b \mathfrak{A}'_{\phi,\mathcal{H}_{1/b}} K_b^{-1} = \mathfrak{A}'_{\phi_b,\mathcal{H}}, && K_b \mathfrak{B}'_{\phi,\mathcal{H}_{1/b}} K_b^{-1} = \mathfrak{B}'_{\phi_b,\mathcal{H}}.
\end{aligned} \tag{2.130}$$

Proof. This is an easy consequence of (2.67), (2.124) and (2.125). Note in particular that the last two identities follow from (2.116), from the previous identities and from the matrix identity,

$$\begin{pmatrix} 1 & b \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.131}$$

□

Set

$$\beta = 1/b. \tag{2.132}$$

Proposition 2.33. *The following identity holds:*

$$\overline{d}_{\phi_b,\mathcal{H}}^{T^*X} = \beta\left(\overline{d}^{T^*X} - i_{Y^{\mathcal{H}}}\right) - \beta^2\left[\overline{d}^{T^*X} - i_{Y^{\mathcal{H}}}, \lambda_0\right]. \tag{2.133}$$

Equivalently,

$$\begin{aligned}
\overline{d}_{\phi_b,\mathcal{H}}^{T^*X} = \beta\Big(&-i_{\widehat{e}^i}\left(\nabla_{e_i}^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F} + \omega\left(\nabla^F, g^F\right)(e_i) - \nabla_{e_i}\mathcal{H}\right) \\
&+ i_{e_i}\left(\nabla_{\widehat{e}^i} - \nabla_{\widehat{e}^i}\mathcal{H}\right) + R^{TX}p\wedge\Big) - \beta^2 i_{\widehat{e}^i}\left(\nabla_{\widehat{e}^i} - \nabla_{\widehat{e}^i}\mathcal{H}\right).
\end{aligned} \tag{2.134}$$

Finally,

$$
\begin{aligned}
A_{\phi_b,\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\beta\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} - \frac{1}{4}\left(e^i + i_{\beta\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad + \frac{1}{2}\left(\widehat{e}_i + i_{\beta e_i - \beta^2\widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(\beta R^{TX}p\wedge + i_{\widehat{R^{TX}p}}\right)\\
&\quad + i_{\beta\widehat{e}^i}\nabla_{e_i}\mathcal{H} + i_{\beta^2\widehat{e}^i - \beta e_i}\nabla_{\widehat{e}^i}\mathcal{H},\\
B_{\phi_b,\mathcal{H}} &= -\frac{1}{2}\left(e^i + i_{\beta\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} + \frac{1}{4}\left(e^i - \beta i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad - \frac{1}{2}\left(\widehat{e}_i + i_{\beta^2\widehat{e}^i - \beta e_i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(\beta R^{TX}p\wedge - i_{\widehat{R^{TX}p}}\right)\\
&\quad + i_{\beta\widehat{e}^i}\nabla_{e_i}\mathcal{H} + i_{\beta^2\widehat{e}^i - \beta e_i}\nabla_{\widehat{e}^i}\mathcal{H},\\
\mathfrak{A}_{\phi_b,\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\beta\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} - \frac{1}{4}\left(e^i + i_{\beta\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad + \frac{1}{2}\left(\widehat{e}_i + i_{\beta e_i - \beta^2\widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(\beta R^{TX}p\wedge + i_{\widehat{R^{TX}p}}\right)\\
&\quad + \frac{1}{2}\left(e^i + i_{\beta\widehat{e}^i}\right)\nabla_{e_i}\mathcal{H} + \frac{1}{2}\left(\widehat{e}_i + i_{\beta^2\widehat{e}^i - \beta e_i}\right)\nabla_{\widehat{e}^i}\mathcal{H},\\
\mathfrak{B}_{\phi_b,\mathcal{H}} &= -\frac{1}{2}\left(e^i + i_{\beta\widehat{e}^i}\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u} + \frac{1}{4}\left(e^i - i_{\beta\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad - \frac{1}{2}\left(\widehat{e}_i + i_{\beta^2\widehat{e}^i - \beta e_i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(\beta R^{TX}p\wedge - i_{\widehat{R^{TX}p}}\right)\\
&\quad - \frac{1}{2}\left(e^i - i_{\beta\widehat{e}^i}\right)\nabla_{e_i}\mathcal{H} - \frac{1}{2}\left(\widehat{e}^i + i_{\beta e_i - \beta^2\widehat{e}^i}\right)\nabla_{\widehat{e}^i}\mathcal{H}.
\end{aligned}
\tag{2.135}
$$

Proof. This follows from Propositions 2.16 and 2.32, or from an easy direct computation. □

In the same way, using Propositions 2.28 and 2.32, we obtain the following result.

Proposition 2.34. *The following identities hold:*

(2.136)

$$
\begin{aligned}
\mathfrak{A}'_{\phi_b,\mathcal{H}} &= \frac{1}{2}\left(c(e_i) - \widehat{c}(\widehat{e}^i)\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}\\
&\quad - \frac{1}{4}\left(\widehat{c}(e_i) - c(\widehat{e}^i)\right)\omega\left(\nabla^F, g^F\right)(e_i) + \frac{\beta}{2}c(\widehat{e}^i)\nabla_{\widehat{e}^i}\\
&\quad + \frac{1}{4\beta}\left(\left(e^i - \widehat{e}_i\right)\left(e^j - \widehat{e}_j\right)i_{\widehat{e}^k + e_k} - i_{e_i + \widehat{e}^i}i_{e_j + \widehat{e}^j}\left(\widehat{e}_k - e^k\right)\right)\left\langle R^{TX}\left(e_i, e_j\right)p, e_k\right\rangle\\
&\quad + \frac{1}{2}\left(\widehat{c}(e_i) - c(\widehat{e}^i)\right)\nabla_{e_i}\mathcal{H} + \frac{\beta}{2}\widehat{c}\left(\widehat{\nabla^V}\mathcal{H}\right),\\
\mathfrak{B}'_{\phi_b,\mathcal{H}} &= -\frac{1}{2}\left(\widehat{c}\left(e_i\right) - c\left(\widehat{e}^i\right)\right)\nabla_{e_i}^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}\\
&\quad + \frac{1}{4}\left(c\left(e_i\right) - \widehat{c}\left(\widehat{e}^i\right)\right)\omega\left(\nabla^F, g^F\right)(e_i) - \frac{\beta}{2}\widehat{c}\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i}\\
&\quad - \frac{1}{4\beta}\left(\left(e^i - \widehat{e}_i\right)\left(e^j - \widehat{e}_j\right)i_{\widehat{e}^k + e_k} + i_{e_i + \widehat{e}^i}i_{e_j + \widehat{e}^j}\left(\widehat{e}_k - e^k\right)\right)\left\langle R^{TX}\left(e_i, e_j\right)p, e_k\right\rangle\\
&\quad - \frac{1}{2}\left(c\left(e_i\right) - \widehat{c}\left(\widehat{e}^i\right)\right)\nabla_{e_i}\mathcal{H} - \frac{\beta}{2}c\left(\widehat{\nabla^V}\mathcal{H}\right).
\end{aligned}
$$

Let N^{T^*X} be the number operator of $\Omega^{\cdot}(T^*X, \pi^*F)$. Let $\widehat{p}$ be the canonical radial vector field on T^*X. Also we identify p with the corresponding horizontal 1-form π^*p on T^*X. Let $L_{\widehat{p}}$ be the corresponding Lie derivative operator. Clearly

$$L_{\widehat{p}} = \nabla_{\widehat{p}} + \widehat{e}_i i_{\widehat{e}^i}. \tag{2.137}$$

Proposition 2.35. *The following identities hold:*

$$\left[d^{T^*X}, i_{\widehat{p}}\right] = L_{\widehat{p}}, \tag{2.138}$$

$$\left[\overline{d}^{T^*X}_{\phi,2\mathcal{H}}, p\right] = L_{\widehat{p}} - 2\nabla^V_{\widehat{p}}\mathcal{H} + n - N^{T^*X} + \lambda_0.$$

Proof. The first identity in (2.138) is trivial. Using (2.137) and taking the symplectic adjoint of this identity with $\mathcal{H} = 0$, we get

$$\left[\overline{d}^{T^*X}, p\right] = L_{\widehat{p}} + n - N^{T^*X}. \tag{2.139}$$

Note that (2.139) can also be proved directly using (2.56). By (2.67), (2.137) and (2.139), we get the second identity in (2.138) with $\mathcal{H} = 0$. The general case is now obvious. □

2.13. **The classical Hamiltonians.** For $c \in \mathbf{R}$, let $\mathcal{H}^c : T^*X \to \mathbf{R}$ be given by

$$\mathcal{H}^c = \frac{c}{2}|p|^2. \tag{2.140}$$

When $c = 1$, we will use the notation $\mathcal{H}$ instead of $\mathcal{H}^1$.

By (2.82) and (2.140), we get

$$Y^{\mathcal{H}^c} = c\langle p, e_i\rangle e_i. \tag{2.141}$$

Proposition 2.36. *The following identities hold:*

(2.142)

$$\begin{aligned}
A_{\phi,\mathcal{H}^c} &= \frac{1}{2}\left(e^i - i_{\widehat{e}^i}\right)\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} - \frac{1}{4}\left(e^i + i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad + \frac{1}{2}\left(\widehat{e}_i + i_{e_i-\widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p\wedge + i_{\widehat{R^{TX}p}}\right) + c i_{\widehat{e}^i - e_i}\langle p, e_i\rangle,\\
B_{\phi,\mathcal{H}^c} &= -\frac{1}{2}\left(e^i + i_{\widehat{e}^i}\right)\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} + \frac{1}{4}\left(e^i - i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad - \frac{1}{2}\left(\widehat{e}_i - i_{e_i-\widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p\wedge - i_{\widehat{R^{TX}p}}\right) + c i_{\widehat{e}^i - e_i}\langle p, e_i\rangle,\\
\mathfrak{A}_{\phi,\mathcal{H}^c} &= \frac{1}{2}\left(e^i - i_{\widehat{e}^i}\right)\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} - \frac{1}{4}\left(e^i + i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad + \frac{1}{2}\left(\widehat{e}_i + i_{e_i-\widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p\wedge + i_{\widehat{R^{TX}p}}\right) + \frac{c}{2}\left(\widehat{e}_i + i_{\widehat{e}^i - e_i}\right)\langle p, e_i\rangle,\\
\mathfrak{B}_{\phi,\mathcal{H}^c} &= -\frac{1}{2}\left(e^i + i_{\widehat{e}^i}\right)\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} + \frac{1}{4}\left(e^i - i_{\widehat{e}^i}\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad - \frac{1}{2}\left(\widehat{e}_i - i_{e_i-\widehat{e}^i}\right)\nabla_{\widehat{e}^i} + \frac{1}{2}\left(R^{TX}p\wedge - i_{\widehat{R^{TX}p}}\right) + \frac{c}{2}\left(-\widehat{e}_i + i_{\widehat{e}^i - e_i}\right)\langle p, e_i\rangle,\\
\mathfrak{A}'_{\phi,\mathcal{H}^c} &= \frac{1}{2}\left(c(e_i) - \widehat{c}(\widehat{e}^i)\right)\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} - \frac{1}{4}\left(\widehat{c}(e_i) - c(\widehat{e}^i)\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad + \frac{1}{2}c(\widehat{e}^i)\nabla_{\widehat{e}^i} + \frac{1}{4}\left(\left(e^i - \widehat{e}_i\right)\left(e^j - \widehat{e}_j\right)i_{\widehat{e}^k + e_k} + i_{e_i+\widehat{e}^i}i_{e_j+\widehat{e}^j}\left(e^k - \widehat{e}_k\right)\right)\\
&\quad \left\langle R^{TX}\left(e_i, e_j\right)p, e_k\right\rangle + \frac{c}{2}\widehat{c}\left(\widehat{p}\right),\\
\mathfrak{B}'_{\phi,\mathcal{H}^c} &= -\frac{1}{2}\left(\widehat{c}\left(e_i\right) - c\left(\widehat{e}^i\right)\right)\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} + \frac{1}{4}\left(c\left(e_i\right) - \widehat{c}\left(\widehat{e}^i\right)\right)\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad - \frac{1}{2}\widehat{c}\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i} - \frac{1}{4}\left(\left(e^i - \widehat{e}_i\right)\left(e^j - \widehat{e}_j\right)i_{\widehat{e}^k + e_k} - i_{e_i+\widehat{e}^i}i_{e_j+\widehat{e}^j}\left(e^k - \widehat{e}_k\right)\right)\\
&\quad \left\langle R^{TX}\left(e_i, e_j\right)p, e_k\right\rangle - \frac{c}{2}c\left(\widehat{p}\right).
\end{aligned}$$

Proof. Our proposition follows from Propositions 2.18 and 2.28. □

Remark 2.37. Let $f : X \to \mathbf{R}$ be a smooth function. We could have considered more general Hamiltonian functions of the type

$$\mathcal{K} = \frac{c}{2}|p|^2 + f. \tag{2.143}$$

However this is equivalent to replacing g^F by $e^{-2f}g^F$, while still working with $\mathcal{H}^c$. This is why we chose not to introduce the function f.

In Remark 2.19, we noted that the operator $A_{\phi,\mathcal{H}^c}$ is not elliptic. However as we shall see in Section 3, for $c \neq 0$, the generalized Laplacian $A^2_{\phi,\mathcal{H}^c}$ is such that $\frac{\partial}{\partial u} - A^2_{\phi,\mathcal{H}^c}$ is hypoelliptic.

2.14. **A rescaling on the variable p for classical Hamiltonians.** Recall that d^{T^*X} commutes with any $r^*_c, c \in \mathbf{R}^*$.

Proposition 2.38. *For $c > 0$, the following identity holds:*

$$r^*_{1/\sqrt{c}}\overline{d}^{T^*X}_{\phi,2\mathcal{H}^c}r^*_{\sqrt{c}} = \overline{d}^{T^*X}_{\phi_{1/\sqrt{c}},2\mathcal{H}} = c\left(\delta^{T^*X,V} + 2i_{\widehat{p}}\right) + \sqrt{c}\left(\overline{d}^{T^*X} - 2i_{Y^{\mathcal{H}}}\right). \tag{2.144}$$

Moreover the operator $\delta^{T^*X,V}+2i_{\widehat{p}}$ *commutes with* r^**, and* $\overline{d}^{T^*X}-2i_p$ *anticommutes with* r^**.*

For $c<0$*, an identity similar to* (2.144) *holds, with* c *replaced by* $-c$*,* $2\mathcal{H}$ *replaced by* $-2\mathcal{H}$*, and* $i_{Y^{\mathcal{H}}}, i_{\widehat{p}}$ *replaced by* $-i_{Y^{\mathcal{H}}}, -i_{\widehat{p}}$*.*

Proof. The first part of our proposition follows from Proposition 2.16 or from Proposition 2.32 and 2.33. The remainder of our proposition is trivial. □

Proposition 2.39. *For* $c>0$*, the following identity holds:*

$$\begin{aligned}(2.145)\quad K_{1/\sqrt{c}}\mathfrak{A}'_{\phi,\mathcal{H}^c}K_{\sqrt{c}} &= \frac{\sqrt{c}}{2}\left(c\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i}+\widehat{c}\left(\widehat{p}\right)\right)\\ &+\frac{1}{2}\left(c(e_i)-\widehat{c}(\widehat{e}^i)\right)\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}_{e_i}-\frac{1}{4}\left(\widehat{c}(e_i)-c(\widehat{e}^i)\right)\omega\left(\nabla^F,g^F\right)(e_i)\\ &+\frac{1}{4\sqrt{c}}\left(\left(e^i-\widehat{e}_i\right)\left(e^j-\widehat{e}_j\right)i_{\widehat{e}^k+e_k}+i_{e_i+\widehat{e}^i}i_{e_j+\widehat{e}^j}\left(e^k-\widehat{e}_k\right)\right)\left\langle R^{TX}\left(e_i,e_j\right)p,e_k\right\rangle.\end{aligned}$$

If $c<0$*, a similar identity holds, in which* $\sqrt{c}$ *is replaced by* $\sqrt{-c}$ *and* $\widehat{c}\left(\widehat{p}\right)$ *is replaced by* $-\widehat{c}\left(\widehat{p}\right)$*.*

Proof. This is a trivial consequence of equation (2.142) in Proposition 2.36. □

Let η be a unit volume form along the fibres of T^*X.

Proposition 2.40. *The kernel of the operator* $c\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i}+\widehat{c}\left(\widehat{p}\right)$ *(resp. of* $c\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i}-\widehat{c}\left(\widehat{p}\right)$*) is* 1*-dimensional and spanned by* $\exp\left(-\left|p\right|^2/2\right)$ *(resp.* $\exp\left(-\left|p\right|^2/2\right)\eta$*).*

Proof. Each of the considered 1-dimensional vector spaces lies in the corresponding kernel. Let Δ^V be the Laplacian along the fibres of T^*X. One has the obvious formulas

$$\begin{aligned}(2.146)\qquad \left(c\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i}+\widehat{c}\left(\widehat{p}\right)\right)^2 &= -\Delta^V+\left|p\right|^2+2\widehat{e}_i i_{\widehat{e}^i}-n,\\ \left(c\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i}-\widehat{c}\left(\widehat{p}\right)\right)^2 &= -\Delta^V+\left|p\right|^2-2\widehat{e}_i i_{\widehat{e}^i}+n.\end{aligned}$$

One recognizes in the right-hand side the harmonic oscillator which appears in Witten [W82]. Our result now follows from (2.146) and from the fact that the kernel of the scalar harmonic oscillator $-\Delta^V+\left|p\right|^2-n$ is spanned by the function $\exp\left(-\left|p\right|^2/2\right)$. □

Since the operators which appear in Proposition 2.40 are self-adjoint, they are semisimple. Let P_+ (resp. P_-) be the corresponding fibrewise orthogonal projection operators on the corresponding kernels.

Now let $\Omega^{\cdot}\left(T^*X,\pi^*F\right)$ denote the space of smooth sections of $\Lambda^{\cdot}\left(T^*T^*X\right)\widehat{\otimes}F$ on T^*X which lie in the corresponding Schwartz space. Let $o\left(TX\right)$ be the orientation bundle of TX. We identify $\Omega^{\cdot}\left(X,F\right)$ (resp. $\Omega^{\cdot}\left(X,F\otimes o\left(TX\right)\right)$) with its image in $\Omega^{\cdot}\left(T^*X,\pi^*F\right)$ by the embedding

$$\alpha\to\pi^*\alpha\exp\left(-\left|p\right|^2/2\right)/\pi^{n/4},$$

(resp.

$$\alpha\to\pi^*\alpha\exp\left(-\left|p\right|^2/2\right)\eta/\pi^{n/4}).$$

Let d^X be the de Rham operator acting on $\Omega^{\cdot}\left(X,F\right)$ or on $\Omega^{\cdot}\left(X,F\otimes o\left(TX\right)\right)$, and let d^{X*} be its adjoint with respect to the obvious L_2 Hermitian product.

Proposition 2.41. *The following identity of operators acting on* $\Omega^{\cdot}(X,F)$ *and on* $\Omega^{\cdot}(X,F\otimes o(TX))$ *holds:*

$$P_{\pm}\Big(\big(c(e_i)-\widehat{c}(\widehat{e}^{\,i})\big)\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}_{e_i} - \frac{1}{2}\big(\widehat{c}(e_i)-c(\widehat{e}^{\,i})\big)\,\omega\left(\nabla^F,g^F\right)(e_i)\Big)P_{\pm} = d^X + d^{X*}. \tag{2.147}$$

Proof. Recall that the kernel of the considered operator is concentrated fibrewise in a single degree. Since the $c\left(\widehat{e}^{\,i}\right),\widehat{c}\left(\widehat{e}^{\,i}\right)$ are fibrewise odd operators, we get

$$\begin{aligned} P_{+}\Big(\big(c(e_i)-\widehat{c}(\widehat{e}^{\,i})\big)\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}_{e_i} - \frac{1}{2}\big(\widehat{c}(e_i)-c(\widehat{e}^{\,i})\big)\,\omega\left(\nabla^F,g^F\right)(e_i)\Big)P_{+} \\ = c\,(e_i)\,\nabla^{\Lambda^{\cdot}(T^*X)\otimes F,u} - \frac{1}{2}\widehat{c}\,(e_i)\,\omega\left(\nabla^F,g^F\right)(e_i)\,, \end{aligned} \tag{2.148}$$

which is just (2.147). The same argument also works when replacing P_+ by P_-. □

Remark 2.42. Similar considerations apply to the operators $A_{\phi,\mathcal{H}^c},\mathfrak{B}_{\phi,\mathcal{H}^c}$, which are conjugate to the operator $\mathfrak{A}'_{\phi,\mathcal{H}^c}$. With obvious changes, it applies to all the other operators in Proposition 2.36. Propositions 2.39–2.41 show that the asymptotics as $c\to\pm\infty$ of the operator $\mathfrak{A}'_{\phi,\mathcal{H}^c}$ is very similar to the asymptotics of the Dirac operator considered by Bismut and Lebeau in [BL91, Theorems 8.18 and 8.21] and in [BerB94, Theorem 5.17]. The main difference with the above references is that the operators which are considered here are not elliptic and are not self-adjoint.

3. The hypoelliptic Laplacian

In this section, we give a Weitzenböck formula for $A^2_{\phi,\mathcal{H}}$. We prove that if $\mathcal{H}^c = c\,|p|^2/2$, with $c\neq 0$, then $\frac{\partial}{\partial u}-A^2_{\phi,\mathcal{H}^c}$ is hypoelliptic, this fact being a consequence of a theorem of Hörmander [Hö67]. Also we study the behaviour of this operator as $c\to\pm\infty$ and $c\to 0$, giving evidence to the fact that $2A^2_{\phi,\mathcal{H}^c}$ interpolates between the scaled Laplacian $\square^X/2$ of X and the generator of the geodesic flow $L_{Y^{\mathcal{H}}}$ on T^*X. We elaborate on the functional integral interpretation of these operators and of the corresponding stochastic processes, and we relate these processes to stochastic mechanics in the sense of [B81]. Finally we consider the special case where $X=S_1$.

This section is organized as follows. In subsection 3.1, we establish a Weitzenböck formula for $A^2_{\phi,\mathcal{H}}$. In subsection 3.2, we specialize this formula to the case $\mathcal{H}^c = c\,|p|^2/2$. In subsection 3.3, we show that for $c\neq 0$, the operator $\frac{\partial}{\partial u}-A^2_{\phi,\mathcal{H}^c}$ is hypoelliptic. In subsection 3.4, we establish the corresponding Weitzenböck formulas when ϕ is replaced by ϕ_b. In subsection 3.5, we consider the effect of a rescaling of p.

In subsection 3.6, we show that the Thom form of Mathai and Quillen [MaQ86] lies in the kernel of the vertical part of our Weitzenböck formula. In subsection 3.7, we give a formula relating $A^2_{\phi,\mathcal{H}^c}$ to the standard Laplacian Δ^X of X. This formula shows that as $c\to\pm\infty$, the operator $A^2_{\phi,\mathcal{H}^c}$ has a matrix structure similar to the

one already encountered in earlier work with Lebeau in [BL05]. It will lead to the proof in [BL05] that as $c \to \pm\infty$, our Laplacian converges in the appropriate sense to the scaled Laplacian $\frac{1}{4}\square^X$.

In subsection 3.8, we show that as $c \to 0$, in the appropriate sense, the operator $A^2_{\phi,\mathcal{H}^c}$ converges to the generator of the geodesic flow on T^*X.

In subsection 3.9 we allow ourselves more heuristic considerations, inspired by path integrals. First we relate the construction of $A^2_{\phi,\mathcal{H}}$ to the stochastic mechanics constructions of [B81]. Also we give a nonrigorous path integral formula for $\mathrm{Tr_s}\left[\exp\left(-A^2_{\phi,\mathcal{H}}\right)\right]$ which fits in part with the idea explained in the introduction that it can be expressed as an integral on the loop space LX, involving the pullback by the section ∇I of the equivariant Mathai-Quillen Thom form of TLX. We relate the non-self-adjointness of $A^2_{\phi,\mathcal{H}}$ to the behaviour of a C^1-path in X under time reversal. Finally we show how to give instead a construction of an operator $A^2_{\phi^\epsilon,\mathcal{H}}$ which is elliptic (and still non-self-adjoint) for $\epsilon > 0$, which coincides with $A^2_{\phi,\mathcal{H}}$, for $\epsilon = 0$, this last construction being most natural from the point of view of the Legendre transform.

Finally, in subsection 3.10, we consider the case where $X = S_1$. In this case, all the operators can be computed explicitly. The idea that our family of operators interpolates between the Laplacian and the geodesic flow leads to a geometric proof of the Poisson formula.

In the entire section, we use the notation of Section 2.

3.1. **The Weitzenböck formulas.** Let $L_{Y^{\mathcal{H}}}$ be the Lie derivative which is attached to the vector field $Y^{\mathcal{H}}$ on T^*X. Then $L_{Y^{\mathcal{H}}}$ acts naturally on $\Omega^{\cdot}\left(T^*X, \pi^*F\right)$.

Proposition 3.1. *The following identities hold:*

$$\left[d^{T^*X}, \overline{d}^{T^*X}\right] = -e^i i_{\widehat{e}^j} \nabla^F_{e_i} \omega\left(\nabla^F, g^F\right)(e_j) - \omega\left(\nabla^F, g^F\right)(e_i) \nabla_{\widehat{e}^i}, \tag{3.1}$$

$$\left[d^{T^*X}, i_{Y^{\mathcal{H}}}\right] = L_{Y^{\mathcal{H}}}, \qquad \left[\overline{d}^{T^*X}, d^{T^*X}\mathcal{H}\right] = -L_{Y^{\mathcal{H}}} - \omega\left(\nabla^F, g^F\right)\left(Y^{\mathcal{H}}\right).$$

Proof. Using Theorem 2.2 and Proposition 2.8, we get

$$\left[d^{T^*X}, \overline{d}^{T^*X}\right] = \left[d^{T^*X}, -i_{\widehat{e}^i}\omega\left(\nabla^F, g^F\right)(e_i)\right]. \tag{3.2}$$

The first identity in (3.1) follows from (2.43) and (3.2). The second identity in (3.1) is trivial, and the third identity is just its symplectic adjoint. □

We extend the notation of (2.83) to

$$\begin{aligned} \nabla^V\mathcal{H} &= \nabla_{\widehat{e}^i}\mathcal{H}e_i, & \nabla^H\mathcal{H} &= \nabla_{e_i}\mathcal{H}e_i, \\ \widehat{\nabla^V}\mathcal{H} &= \nabla_{\widehat{e}^i}\mathcal{H}\widehat{e}^i, & \widehat{\nabla^H}\mathcal{H} &= \nabla_{e_i}\mathcal{H}\widehat{e}^i. \end{aligned} \tag{3.3}$$

Let Δ^V be the fibrewise Laplacian along the fibres of T^*X, i.e.

$$\Delta^V = \nabla^2_{\widehat{e}^i}. \tag{3.4}$$

Proposition 3.2. *The following identities hold:*

$$\begin{aligned}
\left[d^{T^*X}, \overline{d}^{T^*X}_{\phi,2\mathcal{H}}\right] &= \left[d^{T^*X}, \overline{d}^{T^*X} - 2i_{Y^{\mathcal{H}}}\right] - \left[d^{T^*X}, \left[\overline{d}^{T^*X} - 2i_{Y^{\mathcal{H}}}, \lambda_0\right]\right] \\
&= \left[d^{T^*X}, \overline{d}^{T^*X}\right] - 2L_{Y^{\mathcal{H}} - \widehat{\nabla^V}\mathcal{H}} + \left[d^{T^*X}, \delta^{T^*X,V}\right], \\
\left[d^{T^*X}_{\mathcal{H}}, \overline{d}^{T^*X}_{\phi,\mathcal{H}}\right] &= \left[d^{T^*X} + d^{T^*X}\mathcal{H}, \overline{d}^{T^*X} - i_{Y^{\mathcal{H}}}\right] \\
&\quad - \left[d^{T^*X} + d^{T^*X}\mathcal{H}, \left[\overline{d}^{T^*X} - i_{Y^{\mathcal{H}}}, \lambda_0\right]\right] \\
&= \left[d^{T^*X}, \overline{d}^{T^*X}\right] - 2L_{Y^{\mathcal{H}}} - \omega\left(\nabla^F, g^F\right)\left(Y^{\mathcal{H}}\right) \\
&\quad + \left[d^{T^*X} + d^{T^*X}\mathcal{H}, \delta^{T^*X,V} + i_{\widehat{\nabla^V}\mathcal{H}}\right].
\end{aligned} \tag{3.5}$$

Moreover,

$$\begin{aligned}
&\left[d^{T^*X}, \delta^{T^*X,V}\right] = -\Delta^V - \frac{1}{2}\left\langle R^{TX}\left(e_i, e_j\right)e_k, e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l}, \\
&\left[d^{T^*X} + d^{T^*X}\mathcal{H}, \delta^{T^*X,V} + i_{\widehat{\nabla^V}\mathcal{H}}\right] = -\Delta^V - \frac{1}{2}\left\langle R^{TX}\left(e_i, e_j\right)e_k, e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} \\
&\qquad + \left|\nabla^V\mathcal{H}\right|^2 + L_{\widehat{\nabla^V}\mathcal{H}} - \nabla_{\widehat{\nabla^V}\mathcal{H}} - i_{\widehat{e}^i}\nabla_{\widehat{e}^i}\left(d^{T^*X}\mathcal{H}\right).
\end{aligned} \tag{3.6}$$

Finally,

$$\begin{aligned}
L_{\widehat{\nabla^V}\mathcal{H}} - \nabla_{\widehat{\nabla^V}\mathcal{H}} - i_{\widehat{e}^i}\nabla_{\widehat{e}^i} d^{T^*X}\mathcal{H} &= -\Delta^V\mathcal{H} + 2\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H}\widehat{e}_i i_{\widehat{e}^j} \\
&\quad + 2\nabla_{\widehat{e}^i}\nabla_{e_j}\mathcal{H}e^j i_{\widehat{e}^i}.
\end{aligned} \tag{3.7}$$

Proof. The identities in (3.5) and (3.6) follow from Propositions 2.5, 2.10, 2.16 and 3.1. By the argument given in the proof of Proposition 2.4, the torsion of the tautological connection ∇^{TT^*X} on TT^*X is a horizontal 2-form. It follows that if U is a smooth section of TT^*X, then

$$\left[\widehat{\nabla^V}\mathcal{H}, U\right] = \nabla^{TT^*X}_{\widehat{\nabla^V}\mathcal{H}} U - \nabla^{TT^*X}_U \widehat{\nabla^V}\mathcal{H}. \tag{3.8}$$

By (3.8), we deduce that

$$L_{\widehat{\nabla^V}\mathcal{H}} = \nabla_{\widehat{\nabla^V}\mathcal{H}} + \left(\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H}\right)\widehat{e}_i i_{\widehat{e}^j} + \left(\nabla^{TX}_{e_i}\nabla_{\widehat{e}^j}\mathcal{H}\right)e^i i_{\widehat{e}^j}. \tag{3.9}$$

Moreover

$$i_{\widehat{e}^i}\nabla_{\widehat{e}^i}d^{T^*X}\mathcal{H}=\left(\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H}\right)i_{\widehat{e}^i}\widehat{e}_j+\left(\nabla_{\widehat{e}^i}\nabla_{e_j}\mathcal{H}\right)i_{\widehat{e}^i}e^j. \tag{3.10}$$

From (3.9), (3.10), we get (3.7). The proof of our proposition is completed. □

Theorem 3.3. *The following identities hold:*

$$\begin{aligned}
A^2_{\phi,\mathcal{H}}&=\frac{1}{4}\left(-\Delta^V-\frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l}+2L_{\widehat{\nabla^V}\mathcal{H}}\right)\\
&-\frac{1}{2}\left(L_{Y^{\mathcal{H}}}+\frac{1}{2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)+\frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right),\\
\mathfrak{A}^2_{\phi,\mathcal{H}}&=\frac{1}{4}\Bigg(-\Delta^V-\frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l}+\left|\nabla^V\mathcal{H}\right|^2\\
&\qquad-\Delta^V\mathcal{H}+2\left(\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H}\right)\widehat{e}_ii_{\widehat{e}^j}+2\left(\nabla_{\widehat{e}^i}\nabla_{e_j}\mathcal{H}\right)e^ji_{\widehat{e}^i}\Bigg)\\
&-\frac{1}{2}\Bigg(L_{Y^{\mathcal{H}}}+\frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}}\right)+\frac{1}{2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\\
&\qquad+\frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\Bigg).
\end{aligned}\tag{3.11}$$

Proof. This follows from Propositions 3.1 and 3.2. □

3.2. **The Weitzenböck formulas for the classical Hamiltonians.** Now we use the notation of subsection 2.13.

Theorem 3.4. *The following identity holds:*

$$L_{Y^{\mathcal{H}^c}}=\nabla^{\Lambda^{\cdot}(T^*T^*X)\otimes F}_{Y^{\mathcal{H}^c}}+c\widehat{e}_ii_{e_i}+c\left\langle R^{TX}\left(p,e_i\right)p,e_j\right\rangle e^ii_{\widehat{e}^j}. \tag{3.12}$$

Moreover,

(3.13)

$$
\begin{aligned}
A^2_{\phi,\mathcal{H}^c} &= \frac{1}{4}\left(-\Delta^V + 2cL_{\widehat{p}} - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l}\right)\\
&- \frac{1}{2}\left(L_{Y^{\mathcal{H}^c}} + \frac{1}{2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right) + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right),\\
\mathfrak{A}^2_{\phi,\mathcal{H}^c} &= \frac{1}{4}\left(-\Delta^V + c^2\left|p\right|^2 + c\left(2\widehat{e}_ii_{\widehat{e}^i} - n\right) - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l}\right)\\
&- \frac{1}{2}\Bigg(L_{Y^{\mathcal{H}^c}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}^c}\right) + \frac{1}{2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\\
&\qquad + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\Bigg),\\
\mathfrak{A}'^2_{\phi,\mathcal{H}^c} &= \frac{1}{4}\Bigg(-\Delta^V + c^2\left|p\right|^2 + c\left(2\widehat{e}_ii_{\widehat{e}^i} - n\right)\\
&- \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle\left(e^i - \widehat{e}_i\right)\left(e^j - \widehat{e}_j\right)i_{e_k+\widehat{e}^k}i_{e_\ell+\widehat{e}^l}\Bigg)\\
&- \frac{1}{2}\Bigg(\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{Y^{\mathcal{H}^c}} + \Bigg(c\left\langle R^{TX}\left(p,e_i\right)p,e_j\right\rangle\\
&+ \frac{1}{2}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\Bigg)\left(e^i - \widehat{e}_i\right)i_{e_j+\widehat{e}^j} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\Bigg).
\end{aligned}
$$

Proof. As we saw in Proposition 2.4, the torsion of the fibrewise connection ∇^{TT^*X} is just $\pi^*\widehat{R^{TX}p}$. Therefore, if U is a smooth section of TT^*X,

$$
\left[Y^{\mathcal{H}^c},U\right] = \nabla^{TT^*X}_{Y^{\mathcal{H}^c}}U - \nabla^{TT^*X}_{U}Y^{\mathcal{H}^c} - cR^{TX}\widehat{(p,\pi_*U)}\,p. \tag{3.14}
$$

From (3.14), we get (3.12).

The first two equations in (3.13) follow from Theorem 3.3. By (2.111), (2.116), (3.12) and the second equation in (3.13), we get the last equation in (3.13). The proof of our theorem is completed. □

Proposition 3.5. *The operators*

$$
\begin{aligned}
\mathfrak{a}_c &= \frac{1}{2}\left(-\Delta^V + 2cL_{\widehat{p}} - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l}\right),\\
\mathfrak{b}_c &= -\left(L_{Y^{\mathcal{H}^c}} + \frac{1}{2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right) + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right)
\end{aligned} \tag{3.15}
$$

commute and anticommute, respectively, with r^*. Moreover,

$$
\mathfrak{a}_c = \frac{1}{2}\left[d^{T^*X},\delta^{T^*X,V} + 2ci_{\widehat{p}}\right], \qquad \mathfrak{b}_c = \frac{1}{2}\left[d^{T^*X},\overline{d}^{T^*X} - 2ci_{Y^{\mathcal{H}}}\right]. \tag{3.16}
$$

In particular the operators $\mathfrak{a}_c,\mathfrak{b}_c$ commute with d^{T^*X}.

Proof. The first part of our proposition is obvious. The second part of our proposition follows in particular from Propositions 3.1 and 3.2. As a consequence of (3.16), $\mathfrak{a}_c, \mathfrak{b}_c$ commute with d^{T^*X}. □

3.3. **Hypoellipticity of the operator $\frac{\partial}{\partial u} - A^2_{\phi,\mathcal{H}^c}$.** Let $u \in \mathbf{R}$ be an extra variable.

Theorem 3.6. *For $c \in \mathbf{R}^*$, the differential operator $\frac{\partial}{\partial u} - A^2_{\phi,\mathcal{H}^c}$ is hypoelliptic.*

Proof. By Theorem 3.4, up to an operator of order 0, the operator $A^2_{\phi,\mathcal{H}^c}$ coincides with the operator $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{4}\Delta^V - \frac{c}{2}\nabla_{p-\widehat{p}} - \frac{1}{4}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i}. \tag{3.17}$$

Using (2.141) and (3.17), one finds easily that the operators $\frac{\partial}{\partial u} - \mathcal{L}$ and $\frac{\partial}{\partial u} - A^2_{\phi,c\mathcal{H}}$ verify Hörmander's hypoellipticity conditions in [Hö67]. In fact these operators are elliptic along the fibres T^*X, and by (2.141), the Hörmander Lie brackets of vertical vector fields with $Y^{\mathcal{H}^c}$ span the whole TT^*X. The proof of our theorem is completed. □

3.4. **The Weitzenböck formulas associated to ϕ_b.** Now we use the notation of subsection 2.12. Take $b \in \mathbf{R}^*$. Recall that $\beta = 1/b$.

Theorem 3.7. *The following identities hold:*

$$\begin{aligned} A^2_{\phi_b,\mathcal{H}} &= \frac{\beta^2}{4}\left(-\Delta^V - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k, e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} + 2L_{\widehat{\nabla^V}\mathcal{H}}\right) \\ &- \frac{\beta}{2}\left(L_{Y^{\mathcal{H}}} + \frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F, g^F\right)(e_j) + \frac{1}{2}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i}\right), \\ \mathfrak{A}^2_{\phi_b,\mathcal{H}} &= \frac{\beta^2}{4}\Bigg(-\Delta^V - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k, e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} + \left|\nabla^V\mathcal{H}\right|^2 \\ &\qquad - \Delta^V\mathcal{H} + 2\widehat{e}_i i_{\widehat{e}^j}\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H} + 2e^j i_{\widehat{e}^i}\nabla_{\widehat{e}^i}\nabla_{e_j}\mathcal{H}\Bigg) \\ &- \frac{\beta}{2}\Bigg(L_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(Y^{\mathcal{H}}\right) + \frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F, g^F\right)(e_j) \\ &\qquad + \frac{1}{2}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i}\Bigg). \end{aligned} \tag{3.18}$$

Proof. We can use Proposition 2.32 and Theorem 3.3, or we can also use Propositions 2.33 and 2.34 and proceed directly. □

3.5. **The rescaling of the p coordinate.** In this subsection, we take $\mathcal{H} = |p|^2/2$.

Theorem 3.8. *For $c>0$, the following identities hold:*

$$
\begin{aligned}
A^2_{\phi_{1/\sqrt{c}},\mathcal{H}} &= r^*_{1/\sqrt{c}} A^2_{\phi,\mathcal{H}^c} r^*_{\sqrt{c}} \\
&= \frac{c}{4}\left(-\Delta^V + 2L_{\widehat{p}} - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l}\right) \\
&- \frac{\sqrt{c}}{2}\left(L_{Y^{\mathcal{H}}} + \frac{1}{2} e^i i_{\widehat{e}^j} \nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right) + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right), \\
\mathfrak{A}^2_{\phi_{1/\sqrt{c}},\mathcal{H}} &= r^*_{1/\sqrt{c}} \mathfrak{A}^2_{\phi,\mathcal{H}^c} r^*_{\sqrt{c}} \\
&= \frac{c}{4}\left(-\Delta^V + |p|^2 + 2\widehat{e}_i i_{\widehat{e}^i} - n - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l}\right) \\
&- \frac{\sqrt{c}}{2}\Bigg(L_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}}\right) + \frac{1}{2} e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right) \\
&\qquad + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\Bigg), \\
\mathfrak{A}'^2_{\phi_{1/\sqrt{c}},\mathcal{H}} &= K_{1/\sqrt{c}}\mathfrak{A}'^2_{\phi,\mathcal{H}^c} K_{\sqrt{c}} = \frac{c}{4}\left(-\Delta^V + |p|^2 + 2\widehat{e}_i i_{\widehat{e}^i} - n\right) \\
&- \frac{\sqrt{c}}{2}\left(\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right) \\
&- \frac{1}{8}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle\left(e^i-\widehat{e}_i\right)\left(e^j-\widehat{e}_j\right) i_{e_k+\widehat{e}^k} i_{e_\ell+\widehat{e}^l} \\
&- \frac{1}{2}\left(\left\langle R^{TX}\left(p,e_i\right)p,e_j\right\rangle + \frac{1}{2}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\right)\left(e^i-\widehat{e}_i\right) i_{e_j+\widehat{e}^j}.
\end{aligned}
\tag{3.19}
$$

For $c<0$, similar identities hold, in which c is replaced by $-c$, $\mathcal{H}$ by $-\mathcal{H}$, $L_{\widehat{p}}$ by $-L_{\widehat{p}}$, $R^{TX}\left(p,e_i\right)p$ by $-R^{TX}\left(p,e_i\right)p$, and $\widehat{e}_i i_{\widehat{e}^i}$ by $n-\widehat{e}_i i_{\widehat{e}^i}$.

Proof. This is a trivial consequence of Proposition 2.32 and of Theorems 3.4 and 3.7. □

Remark 3.9. Observe that the first identity in (3.19) is also a consequence of Propositions 2.38 and 3.5. Also the fact that for $c>0$, with $c=1/b^2$, the right-hand sides of the first equations in (3.18) and (3.19) coincide should be taken for granted, because of Proposition 2.32. Similar considerations apply for $c<0$, with $c=-1/b^2$.

3.6. **Harmonic oscillator and the Thom form.** Let S be a smooth manifold, let E be a real vector bundle of dimension n on S, let g^E be a smooth metric on E, let ∇^E be a Euclidean connection on E, and let R^E be the curvature of ∇^E. Let $o\left(E\right)$ be the orientation bundle of E, and let $\mathcal{M}^{E^*}$ be the total space of E^*. Let $d^{\mathcal{M}^{E^*}}$ be the de Rham operator of $\mathcal{M}^{E^*}$, and let d^{E^*} be the fibrewise de Rham operator. Let p be the generic element in E^*. By using notation similar to the notation in Propositions 2.4 and 2.5, we get

$$
d^{\mathcal{M}^{E^*}} = d^{E^*} + \nabla^{\Omega^{\cdot}(E^*)} + i_{\widehat{R^{E^*}p}}. \tag{3.20}
$$

As in (2.65), (2.66), set

$$
\delta^{E^*} = -i_{\widehat{e}^i}\nabla_{\widehat{e}^i}. \tag{3.21}
$$

We still define the functions $\mathcal{H}, \mathcal{H}^c$ on $\mathcal{M}^{E^*}$ as in (2.140), i.e.

$$\mathcal{H} = \frac{1}{2}|p|^2, \qquad \mathcal{H}^c = \frac{c}{2}|p|^2. \tag{3.22}$$

Set

$$d^{\mathcal{M}^{E^*}}_{\mathcal{H}^c} = e^{-\mathcal{H}^c} d^{\mathcal{M}^{E^*}} e^{\mathcal{H}^c}, \qquad \delta^{E^*}_{\mathcal{H}^c} = e^{\mathcal{H}^c} \delta^{E^*} e^{-\mathcal{H}^c}. \tag{3.23}$$

By (3.20), (3.21),

$$d^{\mathcal{M}^{E^*}}_{\mathcal{H}^c} = d^{E^*} + c\widehat{p} \wedge + \nabla^{\Omega^\cdot(E^*)} + i_{R^{E^*}p}, \qquad \delta^{E^*}_{\mathcal{H}^c} = \delta^{E^*} + c i_{\widehat{p}}. \tag{3.24}$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of E.

Proposition 3.10. *The following identities hold:*

$$\begin{aligned} \left[d^{\mathcal{M}^{E^*}}, \delta^{E^*}_{2\mathcal{H}^c}\right] &= -\Delta^V + 2cL_{\widehat{p}} - \left\langle R^E e_k, e_l \right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}, \\ \left[d^{\mathcal{M}^{E^*}}_{\mathcal{H}^c}, \delta^{E^*}_{\mathcal{H}^c}\right] &= -\Delta^V + c^2|p|^2 + c\left(2\widehat{e}_i i_{\widehat{e}^i} - n\right) - \left\langle R^E e_k, e_l \right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}. \end{aligned} \tag{3.25}$$

For $c \neq 0$,

(3.26)

$$\begin{aligned} \left[d^{\mathcal{M}^{E^*}}, \delta^{E^*}_{2\mathcal{H}^c}\right] &= \exp\left(-\frac{1}{4c}\left\langle R^E e_k, e_l \right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\left(-\Delta^V + 2cL_{\widehat{p}}\right) \\ &\quad \times \exp\left(\frac{1}{4c}\left\langle R^E e_k, e_l \right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right), \\ \left[d^{\mathcal{M}^{E^*}}_{\mathcal{H}^c}, \delta^{E^*}_{\mathcal{H}^c}\right] &= \exp\left(-\frac{1}{4c}\left\langle R^E e_k, e_l \right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\left(-\Delta^V + c^2|p|^2 + c\left(2\widehat{e}_i i_{\widehat{e}^i} - n\right)\right) \\ &\quad \times \exp\left(\frac{1}{4c}\left\langle R^E e_k, e_l \right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right). \end{aligned}$$

Proof. The identities in (3.25) follow from (3.24). The second identity in (3.26) is trivial, and the first follows by conjugation by $e^{\mathcal{H}^c}$. □

The operators which appear in Proposition 3.10 commute with the left and right actions of $\Lambda^\cdot(T^*S)$, and they are all conjugate to each other. The operators which appear in the right-hand side of (3.26) are self-adjoint, and they have finite dimensional kernel. Since they are semisimple, if A is any of the operators appearing in (3.26), then the total space $\Lambda^\cdot(T^*S)\widehat{\otimes}\Omega^\cdot(E^*)$ on which they act splits as

$$\Lambda^\cdot(T^*S)\widehat{\otimes}\Omega^\cdot(E^*) = \ker A \oplus \operatorname{Im} A. \tag{3.27}$$

Since all these operators are conjugate to each other, the splittings in (3.27) are conjugate to each other. We can then consider the projection operator Q on $\ker A$ with respect to the splitting (3.27). All these operators commute with the left action of $\Lambda^\cdot(T^*S)$.

For $c \neq 0$, we will be concerned with the operators $\left[d^{\mathcal{M}^{E^*}}, \delta^{E^*}_{2\mathcal{H}^c}\right]$ which are more geometric. Let $Q^{E^*}_c$ be the corresponding projection on $\ker\left[d^{\mathcal{M}^{E^*}}, \delta^{E^*}_{2\mathcal{H}^c}\right]$. A trivial conjugation shows that it is enough to study the two cases where $c = \pm 1$. In this case, we will write $Q^{E^*}_{\pm}$ instead of $Q^{E^*}_{\pm 1}$. We will denote by π_* the operator of integration along the fibre E^* of integrable differential forms. Our sign conventions

are such that if α is a smooth form on S and β is an integrable differential form on $\mathcal{M}^{E^*}$, then

$$\pi_* \left(\left(\pi^* \alpha \right) \beta \right) = \alpha \pi_* \beta. \tag{3.28}$$

Recall that in [MaQ86], Mathai and Quillen gave an explicit construction of a Thom form on the total space of E^*. To explain their construction, we recall briefly the definition of the Berezin integral. Let $\overline{e}^1, \ldots, \overline{e}^n$ be still another copy of the dual basis to the basis $e_1, \ldots, e_n$. Let $\Lambda^{\cdot}\left(\overline{E}^*\right)$ be the exterior algebra generated by the $\overline{e}^i$.

We denote by $\int^{\overline{B}}$ the functional defined on $\Lambda^{\cdot}\left(\overline{E}^*\right)$, with values in $o\left(E\right)$, which vanishes in degree strictly less than n and is such that

$$\int^{\overline{B}} \overline{e}^1 \wedge \ldots \wedge \overline{e}^n = \left(-1\right)^{n(n+1)/2}. \tag{3.29}$$

We extend $\int^{\overline{B}}$ to a functional defined on $\Lambda^{\cdot}\left(T^*\mathcal{M}^{E^*}\right) \widehat{\otimes} \Lambda^{\cdot}\left(\overline{E}^*\right)$ with values in $\Lambda^{\cdot}\left(T^*\mathcal{M}^{E^*}\right)$, with the convention that if $\alpha \in \Lambda^{\cdot}\left(T^*\mathcal{M}^{E^*}\right), \beta \in \Lambda^{\cdot}\left(\overline{E}^*\right)$, then

$$\int^{\overline{B}} \alpha\beta = \alpha \int^{\overline{B}} \beta. \tag{3.30}$$

Now consider the Berezin integral in the variables $\overline{e}_i, 1 \leq i \leq n$,

$$\Phi^{E^*} = \frac{1}{\pi^{n/2}} \int^{\overline{B}} \exp\left(-\frac{1}{4} \left\langle e_k, R^E e_l \right\rangle \overline{e}^k \overline{e}^l - \widehat{e}_i \overline{e}^i - \left|p\right|^2 \right). \tag{3.31}$$

Then the basic result of [MaQ86] is that Φ^{E^*} is a closed form of degree n on $\mathcal{M}^{E^*}$ with coefficients in $o\left(E\right)$, and moreover

$$\pi_* \Phi^{E^*} = 1, \tag{3.32}$$

so that Φ^{E^*} is indeed a Thom form on $\mathcal{M}^{E^*}$.

Theorem 3.11. *The following identities hold:*

$$d^{\mathcal{M}^{E^*}} \Phi^{E^*} = 0, \qquad \left(\delta^{E^*} - 2i_{\widehat{p}}\right) \Phi^{E^*} = 0. \tag{3.33}$$

The kernel of $\left[d^{\mathcal{M}^{E^*}}, \delta^{E^*}_{\mathcal{H}}\right]$ *is spanned over* $\Lambda^{\cdot}\left(T^*S\right)$ *by the zero form* 1. *The corresponding projection operator* $Q^{E^*}_+$ *is given by*

$$\alpha \to Q^{E^*}_+ \alpha = \pi^* \pi_* \left(\alpha \Phi^{E^*}\right). \tag{3.34}$$

The kernel of $\left[d^{\mathcal{M}^{E^*}}, \delta^{E^*}_{-\mathcal{H}}\right]$ *is spanned over* $\Lambda^{\cdot}\left(T^*S\right)$ *by* Φ^{E^*}, *and the corresponding projection operator* $Q^{E^*}_-$ *is given by*

$$\alpha \to Q^{E^*}_- \alpha = \left(\pi^* \pi_* \alpha\right) \wedge \Phi^{E^*}. \tag{3.35}$$

Proof. We already know that Φ^{E^*} is closed. The second identity in (3.33) is trivial. Using basic properties of the harmonic oscillator as in the proof of Proposition 2.40, we know that for $c = 1$, the fibrewise kernel of the harmonic oscillator which appears in the right-hand side of the second identity in (3.26) is spanned by the function $\exp\left(-\left|p\right|^2/2\right)$, and for $c = -1$, if η is a unit fibrewise volume form in

E^*, it is spanned by $\exp\left(-|p|^2/2\right)\eta$. The associated projectors are given by the obvious fibrewise L_2 scalar product with a unit element of this kernel. If α is a form on $\mathcal{M}^{E^*}$, let $\alpha^{(0)}$ be the component of α which is degree 0 in the variables $\widehat{e}_i$. Using the conjugations in (3.26), we get the formulas

$$Q_+^{E^*}\alpha = \frac{1}{\pi^{n/2}}\int_{E^*}\left(\exp\left(\frac{1}{4}\left\langle R^E e_k, e_l\right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\alpha\right)^{(0)}\exp\left(-|p|^2\right)dp, \tag{3.36}$$
$$Q_-^{E^*}\alpha = \frac{1}{\pi^{n/2}}\left(\pi_*\alpha\right)\exp\left(-|p|^2\right)\exp\left(\frac{1}{4}\left\langle R^E e_k, e_l\right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\eta.$$

Note that in (3.36), we were careful to respect the sign conventions in (3.28), (3.30), and also the fact that the projectors $Q_\pm^{E^*}$ commute with left or right multiplication by $\Lambda^\cdot(T^*S)$.

Let $\widehat{*}$ be the fibrewise Hodge operator. Clearly

$$\int_{E^*}\left(\exp\left(\frac{1}{4}\left\langle R^E e_k, e_l\right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\alpha\right)^{(0)}\exp\left(-|p|^2\right)dp \tag{3.37}$$
$$= \int_{E^*}\alpha\wedge\widehat{*}\exp\left(\frac{1}{4}\left\langle e_k, R^E e_l\right\rangle \widehat{e}_k\widehat{e}_l\right)\exp\left(-|p|^2\right).$$

Let ρ be the automorphism of $\Lambda^\cdot(E)$ such that $\rho 1 = 1$, that if $\alpha\in E, \rho\alpha = -\alpha$, and if $\beta,\gamma\in\Lambda^\cdot(E), \rho(\alpha\beta) = (\rho\beta)(\rho\alpha)$. If $\alpha\in\Lambda^\cdot(E)$, let $\overline{\alpha}$ be the corresponding element in $\Lambda^\cdot(E^*)$. One then verifies easily that if the $\widehat{e}_i$ span $\Lambda^\cdot(E)$, if $\alpha\in\Lambda^\cdot(E)$,

$$\widehat{*}\alpha = \int^{\overline{B}}\overline{\rho\alpha}\exp\left(-\widehat{e}_i\overline{e}^i\right). \tag{3.38}$$

An application of (3.38) shows that

$$\widehat{*}\exp\left(\frac{1}{4}\left\langle e_k, R^E e_l\right\rangle \widehat{e}_k\widehat{e}_l\right) = \int^{\overline{B}}\exp\left(-\frac{1}{4}\left\langle e_k, R^E e_l\right\rangle \overline{e}^k\overline{e}^l - \widehat{e}_i\overline{e}^i\right). \tag{3.39}$$

From (3.31), (3.36)–(3.38), we get (3.34). Similarly, since $\eta = \widehat{*}1$,

$$\exp\left(\frac{1}{4}\left\langle R^E e_k, e_l\right\rangle i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\eta = \widehat{*}\exp\left(\frac{1}{4}\left\langle e_k, R^E e_l\right\rangle \widehat{e}_k\widehat{e}_l\right). \tag{3.40}$$

By (3.31), (3.36)–(3.40), we get (3.35). The proof of our theorem is completed. □

Remark 3.12. By a simple conjugation argument, one verifies that more generally

(3.41)
$$Q_c^{E^*}\alpha = \pi^*\pi_*\left(\left(r^*_{1/\sqrt{c}}\alpha\right)\Phi^{E^*}\right) \text{ if } c>0, \quad Q_c^{E^*}\alpha = \left(\pi^*\pi_*\alpha\right) r^*_{\sqrt{-c}}\Phi^{E^*} \text{ if } c<0.$$

Let $i: S\to\mathcal{M}^{E^*}$ be the embedding of S as the zero section of E^*. Note that δ_S, the current of integration on S, is a current on $\mathcal{M}^{E^*}$ with values in $o(E)$. By (3.32), (3.41), we find that we have the convergence of currents

$$Q_c^{E^*}\alpha\to\pi^* i^*\alpha \text{ if } c\to+\infty, \qquad Q_c^{E^*}\alpha\to\left(\pi^*\pi_*\alpha\right)\delta_S \text{ if } c\to-\infty. \tag{3.42}$$

3.7. **The hypoelliptic Laplacian and the standard Laplacian.** We use the notation of subsection 3.6 in the case where $E = TX$ over $S = X$. In particular, we assume that $\mathcal{H} = \frac{1}{2}|p|^2$. Also we will write $\mathfrak{a}_\pm, \mathfrak{b}_\pm$ instead of $\mathfrak{a}_{\pm 1}, \mathfrak{b}_{\pm 1}$. By Theorem 3.11, $\ker \mathfrak{a}_+$ is included in the r^*-invariant part of $\Omega^{\cdot}(T^*X, \pi^*F)$, and $\ker \mathfrak{a}_-$ is included in the invariant part of $\Omega^{\cdot}(T^*X, \pi^*F)$ if n is even, in the antiinvariant part if n is odd.

We denote by $\mathfrak{a}_\pm^{-1}$ the inverse of $\mathfrak{a}_\pm$ acting on $\operatorname{Im} \mathfrak{a}_\pm$. By Proposition 3.5, $\mathfrak{b}_\pm$ exchanges the invariant and the antiinvariant parts of $\Omega^{\cdot}(T^*X, \pi^*F)$, and so it maps $\ker \mathfrak{a}_\pm$ into $\operatorname{Im} \mathfrak{a}_\pm$.

The first identity in (3.19) can be written in the form

$$2A^2_{\phi_{1/\sqrt{c}},\mathcal{H}} = 2r^*_{1/\sqrt{c}} A^2_{\phi,\mathcal{H}^c} r^*_{\sqrt{c}} = c\mathfrak{a}_+ + \sqrt{c}\mathfrak{b}_+. \tag{3.43}$$

A similar identity holds for $c < 0$.

Now we use the notation in subsection 2.14. Temporarily $\Omega^{\cdot}(T^*X, \pi^*F)$ denotes the vector space of smooth sections s of $\Lambda^{\cdot}(T^*T^*X) \widehat{\otimes} F$ which are such that $e^{\mp|p|^2/2}s$ lies in the corresponding Schwartz space on T^*X. In the $+$ case (resp. the $-$ case), we identify $\Omega^{\cdot}(X, F)$ (resp. $\Omega^{\cdot}(X, F \otimes o(TX))$) with its image in $\Omega^{\cdot}(T^*X, \pi^*F)$ by the map $\alpha \to \pi^*\alpha$ (resp. $\alpha \to \pi^*\alpha \wedge \Phi^{T^*X}$). Note that $Q_+^{T^*X}$ (resp. $Q_-^{T^*X}$) is a projector on the corresponding image.

Recall that d^X is the de Rham operator of $\Omega^{\cdot}(X, F)$ or $\Omega^{\cdot}(X, F \otimes o(TX))$ and that d^{X*} is its formal adjoint. The Hodge Laplacian $\square^X$ is given by the formula

$$\square^X = \left(d^X + d^{X*}\right)^2 = \left[d^X, d^{X*}\right]. \tag{3.44}$$

Let Δ^H be the horizontal Laplacian acting on any of these spaces. Namely let $e_1, \ldots, e_n$ be a locally defined smooth orthonormal basis of TX. Then when acting on $\Omega^{\cdot}(X, F)$,

$$\Delta^H = \sum_{i=1}^n \nabla^{\Lambda^{\cdot}(T^*X)\otimes F,2}_{e_i} - \nabla^{\Lambda^{\cdot}(T^*X)\otimes F}_{\sum_{i=1}^n \nabla^{TX}_{e_i} e_i}. \tag{3.45}$$

The action of Δ^H on $\Omega^{\cdot}(X, F \otimes o(TX))$ is obtained by replacing F by $F \otimes o(TX)$. Note that Δ^H is not self-adjoint, except when g^F is flat.

The Weitzenböck formula says that we have the identity

$$\begin{aligned} \square^X = -\Delta^H &+ \left\langle R^{TX}(e_i, e_j) e_k, e_l \right\rangle e^i i_{e_j} e^k i_{e_l} \\ &- \omega\left(\nabla^F, g^F\right)(e_i) \nabla^{\Lambda^{\cdot}(T^*X)\otimes F}_{e_i} - e^i i_{e_j} \nabla^F_{e_i} \omega\left(\nabla^F, g^F\right)(e_j). \end{aligned} \tag{3.46}$$

Let S^{TX} be the Ricci tensor associated to g^{TX}. We can rewrite (3.46) in the form

$$\begin{aligned} \square^X = -\Delta^H &+ \left\langle S^{TX} e_i, e_j \right\rangle e^i i_{e_j} - \left\langle R^{TX}(e_i, e_j) e_k, e_l \right\rangle e^i e^k i_{e_j} i_{e_l} \\ &- \omega\left(\nabla^F, g^F\right)(e_i) \nabla^{\Lambda^{\cdot}(T^*X)\otimes F}_{e_i} - e^i i_{e_j} \nabla^F_{e_i} \omega\left(\nabla^F, g^F\right)(e_j). \end{aligned} \tag{3.47}$$

Using the circular symmetry of R^{TX}, we get

$$\begin{aligned} \square^X = -\Delta^H &+ \left\langle S^{TX} e_i, e_j \right\rangle e^i i_{e_j} - \frac{1}{2}\left\langle R^{TX}(e_i, e_j) e_k, e_l \right\rangle e^i e^j i_{e_k} i_{e_l} \\ &- \omega\left(\nabla^F, g^F\right)(e_i) \nabla^{\Lambda^{\cdot}(T^*X)\otimes F}_{e_i} - e^i i_{e_j} \nabla^F_{e_i} \omega\left(\nabla^F, g^F\right)(e_j). \end{aligned} \tag{3.48}$$

Note that there is an obvious analogue of the above formulas when replacing F by $F \otimes o(TX)$.

Theorem 3.13. *The following identity holds:*

$$-Q_{\pm}^{T^*X}\mathfrak{b}_{\pm}\mathfrak{a}_{\pm}^{-1}\mathfrak{b}_{\pm}Q_{\pm}^{T^*X}=\frac{1}{2}\left(d^X+d^{X*}\right)^2. \tag{3.49}$$

Proof. First, we establish our theorem for the + case. By (3.26), we know that $\mathfrak{a}_+$ is conjugate to $H+\widehat{N}$, where H is a standard harmonic oscillator and $\widehat{N}$ is the vertical number operator. Moreover $p_i=\langle p,e_i\rangle$ is proportional to the half sum of a bosonic creation and annihilation operator, and so it maps $\ker\mathfrak{a}_+$ into the corresponding $+1$ eigenspace of $\mathfrak{a}_+$. Using (3.12), (3.15), (3.16), (3.25), (3.26), (3.31), we get

$$\begin{aligned}-Q_+^{T^*X}\mathfrak{b}_+\mathfrak{a}_+^{-1}\mathfrak{b}_+Q_+^{T^*X}&=-Q_+^{T^*X}\nabla_p^{\Lambda^\cdot(T^*T^*X)\otimes F,2}Q_+^{T^*X}-Q_+^{T^*X}\widehat{e}_ii_{e_i}\widehat{e}_ji_{e_j}Q_+^{T^*X}\\&\quad-Q_+^{T^*X}\left\langle R^{TX}(p,e_i)p,e_j\right\rangle e^ii_{\widehat{e}^j}\widehat{e}_ki_{e_k}Q_+^{T^*X}\\&\quad-Q_+^{T^*X}\frac{1}{2}e^ii_{\widehat{e}^j}\nabla_{e_i}^{TX}\omega\left(\nabla^F,g^F\right)(e_j)\,\widehat{e}_ki_{e_k}Q_+^{T^*X}\\&\quad-Q_+^{T^*X}\frac{1}{2}\omega\left(\nabla^F,g^F\right)(e_i)\,\nabla_{\widehat{e}^i}\nabla_p^{\Lambda^\cdot(T^*T^*X)\otimes F}Q_+^{T^*X}.\end{aligned} \tag{3.50}$$

Moreover we have the trivial equality,

$$\int_{\mathbf{R}}x^2\exp\left(-x^2\right)\frac{dx}{\sqrt{\pi}}=\frac{1}{2}. \tag{3.51}$$

Using (3.31), (3.51), we obtain

$$-Q_+^{T^*X}\nabla_p^{\Lambda^\cdot(T^*T^*X)\otimes F,2}Q_+^{T^*X}=-\frac{1}{2}\Delta^H. \tag{3.52}$$

Moreover, one can check easily the following formula:

$$\pi_*\left[\widehat{e}_i\widehat{e}_j\Phi^{T^*X}\right]=-\frac{1}{2}\left\langle R^{TX}e_i,e_j\right\rangle. \tag{3.53}$$

By (3.53), we get

$$-Q_+^{T^*X}\widehat{e}_ii_{e_i}\widehat{e}_ji_{e_j}Q_+^{T^*X}=-\frac{1}{2}\left\langle R^{TX}e_i,e_j\right\rangle i_{e_i}i_{e_j}, \tag{3.54}$$

which is equivalent to

$$-Q_+^{T^*X}\widehat{e}_ii_{e_i}\widehat{e}_ji_{e_j}Q_+^{T^*X}=-\frac{1}{4}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^ie^ji_{e_k}i_{e_l}. \tag{3.55}$$

Moreover using (3.51), we get

$$-Q_+^{T^*X}\left\langle R^{TX}(p,e_i)p,e_j\right\rangle e^ii_{\widehat{e}^j}\widehat{e}_ki_{e_k}Q_+^{T^*X}=\frac{1}{2}\left\langle S^{TX}e_i,e_j\right\rangle e^ii_{e_j}. \tag{3.56}$$

Finally, using (3.51) again, one has the trivial identities

$$\begin{aligned}&-Q_+^{T^*X}\frac{1}{2}e^ii_{\widehat{e}^j}\nabla_{e_i}^{TX}\omega\left(\nabla^F,g^F\right)(e_j)\,\widehat{e}_ki_{e_k}Q_+^{T^*X}=-\frac{1}{2}\nabla_{e_i}^{TX}\omega\left(\nabla^F,g^F\right)(e_j)\,e^ii_{e_j},\\&-Q_+^{T^*X}\frac{1}{2}\omega\left(\nabla^F,g^F\right)(e_i)\nabla_{\widehat{e}^i}\nabla_p^{\Lambda^\cdot(T^*T^*X)\otimes F}Q_+^{T^*X}=-\frac{1}{2}\omega\left(\nabla^F,g^F\right)(e_i)\nabla_{e_i}^{\Lambda^\cdot(T^*X)\otimes F}.\end{aligned} \tag{3.57}$$

Comparing (3.48) with (3.50)–(3.57), we get (3.49) in the + case.

Now, we consider the $-$ case. As we said before, we identify $\Omega^{\cdot}(X, F\otimes(TX))$ with its image in $\Omega^{\cdot+n}(T^*X,\pi^*F)$ by the map $\alpha\to\pi^*\alpha\wedge\Phi^{T^*X}$. Recall that if $\beta\in\Omega^{\cdot}(T^*X,\pi^*F)$ has compact support, for $1\le i\le n$,

$$\pi_*\left(i_{\widehat{e}^i}\beta\right)=0,\quad \pi_* i_{e_i}\beta=i_{e_i}\pi_*\beta,\qquad \pi_*\nabla^{\Lambda^{\cdot}(T^*T^*X)\otimes F}_{e_i}\beta=\nabla^{\Lambda^{\cdot}(T^*X)\otimes F}_{e_i}\pi_*\beta. \tag{3.58}$$

Using (3.32), we deduce from (3.58) that

$$\pi_*\nabla^{\Lambda^{\cdot}(T^*T^*X)}_{e_i}\Phi^{T^*X}=0. \tag{3.59}$$

Let $[A,B]_+$ denote the anticommutator of A and B. By (3.26), (3.58), we find that the analogue of (3.50) is now

$$\begin{aligned}-Q_-^{T^*X}\mathfrak{b}_-\mathfrak{a}_-^{-1}\mathfrak{b}_-Q_-^{T^*X}\alpha&=-\pi_*\left[\nabla_p^{\Lambda^{\cdot}(T^*T^*X)\otimes F\otimes o(TX),2}\left(\pi^*\alpha\wedge\Phi^{T^*X}\right)\right]\\ &+i_{e_i}i_{e_j}\left(\alpha\pi_*\left[\widehat{e}_i\widehat{e}_j\wedge\Phi^{T^*X}\right]\right)-\pi_*\left[\left\langle R^{TX}(p,e_i)p,e_j\right\rangle\widehat{e}_k i_{e_k}e^i i_{\widehat{e}^j}\left(\pi^*\alpha\wedge\Phi^{T^*X}\right)\right]\\ &+\pi_*\left[\widehat{e}_k i_{e_k}\frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)(e_j)\left(\pi_*\alpha\wedge\Phi^{T^*X}\right)\right]\\ &+\pi_*\left[\left[\frac{1}{2}\omega\left(\nabla^F,g^F\right)(e_i)\nabla_{\widehat{e}^i},\nabla_p^{\Lambda^{\cdot}(T^*T^*X)\otimes F\otimes o(TX)}\right]_+\left(\pi^*\alpha\wedge\Phi^{T^*X}\right)\right].\end{aligned} \tag{3.60}$$

By proceeding as in (3.52) and using (3.59), we get

$$-\pi_*\left[\nabla_p^{\Lambda^{\cdot}(T^*T^*X)\otimes F\otimes o(TX),2}\left(\pi^*\alpha\wedge\Phi^{T^*X}\right)\right]=-\frac{1}{2}\Delta^H\alpha. \tag{3.61}$$

Let K denote the scalar curvature of X. By (3.53), we obtain by a simple computation

$$\begin{aligned}i_{e_i}i_{e_j}\left(\alpha\pi_*\left[\widehat{e}_i\widehat{e}_j\Phi^{T^*X}\right]\right)=\Big(&-\frac{1}{2}K+\left\langle S^{TX}e_i,e_j\right\rangle e^i i_{e_j}\\ &-\frac{1}{4}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^i e^j i_{e_k}i_{e_l}\Big)\alpha.\end{aligned} \tag{3.62}$$

Still using (3.51), (3.58), we get

$$\begin{aligned}-\pi_*\left[\left\langle R^{TX}(p,e_i)p,e_j\right\rangle\widehat{e}_k i_{e_k}e^i i_{\widehat{e}^j}\left(\pi^*\alpha\wedge\Phi^{T^*X}\right)\right]&\\ =\frac{1}{2}\left(K-\left\langle S^{TX}e_i,e_j\right\rangle e^i i_{e_j}\right)\alpha.&\end{aligned} \tag{3.63}$$

The same arguments as before also show that

$$\begin{aligned}&\pi_*\left[\widehat{e}_k i_{e_k}\frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)(e_j)\left(\pi_*\alpha\wedge\Phi^{T^*X}\right)\right]\\ &\qquad=\frac{1}{2}\left(-e^i i_{e_j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)(e_j)+\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)(e_i)\right)\alpha,\\ &\pi_*\left[\left[\frac{1}{2}\omega\left(\nabla^F,g^F\right)(e_i)\nabla_{\widehat{e}^i},\nabla_p^{\Lambda^{\cdot}(T^*T^*X)\otimes F}\right]_+\left(\pi^*\alpha\wedge\Phi^{T^*X}\right)\right]\\ &\qquad=-\frac{1}{2}\left(\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)(e_i)+\omega\left(\nabla^F,g^F\right)(e_i)\nabla^{\Lambda^{\cdot}(T^*X)\otimes F}_{e_i}\right)\alpha.\end{aligned} \tag{3.64}$$

By comparing again (3.48) with (3.60)–(3.64), we also get (3.49) in the $-$ case. The proof of our theorem is completed. $\square$

We rewrite the third formula for $c > 0$ in (3.19) in the form

$$K_{1/\sqrt{c}} 2\mathfrak{A}'^2_{\phi,\mathcal{H}^c} K_{\sqrt{c}} = c\alpha_+ + \sqrt{c}\beta_+ + \gamma_+. \tag{3.65}$$

There is a corresponding formula in the case $c < 0$, with $-$ indices. Let $P_\pm$ be the orthogonal fibrewise projection operator on $\ker \alpha_\pm$. This projector was constructed in subsection 2.14. Also we use the notation of that subsection. Another version of Theorem 3.13 is as follows.

Theorem 3.14. *The following identity holds:*

$$P_\pm \left(\gamma_\pm - \beta_\pm \alpha_\pm^{-1} \beta_\pm\right) P_\pm = \frac{1}{2}\square^X. \tag{3.66}$$

Proof. The operator α_+ is again of the form $H + \widehat{N}$. Since $\ker a_+$ is concentrated in fibrewise degree 0, by proceeding as in the proof of Theorem 3.13, we get

$$\begin{aligned} P_+\gamma_+P_+ = &-\frac{1}{4}\left\langle R^{TX}\left(e_i, e_j\right) e_k, e_l\right\rangle e^i e^j i_{e_k} i_{e_l} \\ &+ \frac{1}{2}\left(\left\langle S^{TX} e_i, e_j\right\rangle - \nabla^F_{e_i}\omega\left(\nabla^F, g^F\right)\left(e_j\right)\right) e^i i_{e_j}. \end{aligned} \tag{3.67}$$

Similar arguments show that

$$-P_+\beta_+\alpha_+^{-1}\beta_+P_+ = -\frac{1}{2}\Delta^H - \frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(e_i\right)\nabla^{\Lambda^\cdot(T^*X)\otimes F}_{e_i}. \tag{3.68}$$

Using (3.48) and (3.67), (3.68), we get (3.66) in the $+$ case. The proof in the $-$ case is exactly the same. $\square$

Remark 3.15. Needless to say, Theorems 3.13 and 3.14 can be deduced from each other. Moreover Theorem 3.14 also follows from Propositions 2.39–2.41. Indeed, for $c > 0$, we can write (2.145) in the form

$$K_{1/\sqrt{c}}\sqrt{2}\mathfrak{A}'_{\phi,\mathcal{H}^c} K_{1/\sqrt{c}} = \sqrt{c}\mathcal{A}_+ + \mathcal{B}_+ + \frac{1}{\sqrt{c}}\mathcal{C}_+, \tag{3.69}$$

so that comparing with (3.65), we get

$$\alpha_+ = \mathcal{A}_+^2, \qquad \beta_+ = \left[\mathcal{A}_+, \mathcal{B}_+\right], \qquad \gamma_+ = \mathcal{B}_+^2 + \left[\mathcal{A}_+, \mathcal{C}_+\right]. \tag{3.70}$$

We deduce from (3.70) that

$$P_+\left(\gamma_+ - \beta_+\alpha_+^{-1}\beta_+\right)P_+ = P_+\left(\mathcal{B}_+^2 - \left[\mathcal{B}_+, \mathcal{A}_+\right]\mathcal{A}_+^{-2}\left[\mathcal{A}_+, \mathcal{B}_+\right]\right)P_+. \tag{3.71}$$

By (3.71), we get

$$P_+\left(\gamma_+ - \beta_+\alpha_+^{-1}\beta_+\right)P_+ = \left(P_+\mathcal{B}_+P_+\right)^2. \tag{3.72}$$

Now by Proposition 2.41,

$$P_+\mathcal{B}_+P_+ = \frac{1}{\sqrt{2}}\left(d^X + d^{X*}\right). \tag{3.73}$$

By (3.71)–(3.73), we get (3.66), in the $+$ case. The $-$ case can be handled in the same way.

Still, it does not seem possible to give a direct easy proof of Theorem 3.13 using just the identity (2.144) in Proposition 2.38. We saw in Remark 3.9 that the first identity in (3.19) follows from Propositions 2.38 and 3.5. However this does not lead trivially to Theorem 3.13.

3.8. **Interpolating between the Laplacian and the geodesic flow.** Recall that $\mathcal{H}_b$ was defined in (2.123). Observe that if $\mathcal{H} = \frac{1}{2}|p|^2$, then

$$b^2\mathcal{H}_{1/b} = \mathcal{H}. \tag{3.74}$$

Here we consider the case of an arbitrary $\mathcal{H}$. Observe that

$$\widehat{\nabla^V} b^2\mathcal{H}_{1/b} = r^*_{1/b}\widehat{\nabla^V}\mathcal{H}, \qquad Y^{b^2\mathcal{H}_{1/b}} = br^*_{1/b}Y^{\mathcal{H}}. \tag{3.75}$$

Consider the operator $A^2_{\phi,b^2\mathcal{H}_{1/b^2}}$, which by Proposition 2.32 is conjugate to $A^2_{\phi_b,b^2\mathcal{H}_{1/b}}$. By Theorem 3.7, we get

$$\begin{aligned} A^2_{\phi_b,b^2\mathcal{H}_{1/b}} &= \frac{1}{4b^2}\left(-\Delta^V - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l} + 2L_{r^*_{1/b}\widehat{\nabla^V}\mathcal{H}}\right) \\ &\quad - \frac{1}{2b}\left(L_{br^*_{1/b}Y^{\mathcal{H}}} + \frac{1}{2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)(e_j) + \frac{1}{2}\omega\left(\nabla^F,g^F\right)(e_i)\nabla_{\widehat{e}^i}\right). \end{aligned} \tag{3.76}$$

We use temporarily the notation $\overline{\mathcal{H}} = \frac{1}{2}|p|^2$. Suppose that as $|p| \to +\infty$, $\mathcal{H} \simeq \overline{\mathcal{H}}$. Then as $b \to 0$,

$$b^2\mathcal{H}_{1/b} \simeq \overline{\mathcal{H}}. \tag{3.77}$$

Under the appropriate conditions on $\mathcal{H}$, we find that as $b \to 0$,

$$r^*_{1/b}\widehat{\nabla^V}\mathcal{H} \simeq \widehat{p}, \qquad br^*_{1/b}Y^{\mathcal{H}} \simeq Y^{\overline{\mathcal{H}}}. \tag{3.78}$$

Now we will argue only in the case $\mathcal{H} = \frac{1}{2}|p|^2$, although (3.76)–(3.78) suggest that what follows holds under more general conditions. In this case, using Theorem 3.13 and proceeding formally as in [BL91, sections 11 and 12], we have a hint that as $b \to 0$, in the proper sense, $A^2_{\phi_b,\mathcal{H}}$ converges to the operator $\frac{1}{4}\square^X$ acting on $\Omega^{\cdot}(X,F)$. Incidentally, note that if $\mathcal{H}$ is replaced by $-\mathcal{H}$, a similar result holds, with F replaced by $F \otimes o(TX)$.

Now consider the case where $b \to +\infty$. Again we assume $\mathcal{H}$ to be arbitrary. By Proposition 2.32, the operator $\mathfrak{A}^2_{\phi,b^2\mathcal{H}_{1/b^2}}$ is conjugate to $\mathfrak{A}^2_{\phi_b,b^2\mathcal{H}_{1/b}}$. By Theorem 3.7 and by (3.75),

$$\begin{aligned} r^*_b\mathfrak{A}^2_{\phi,b^2\mathcal{H}_{1/b}}r^*_{1/b} &= \frac{1}{4b^4}\left(-\Delta^V - \frac{1}{2}\left\langle R^{TX}(e_i,e_j)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l}\right) \\ &\quad + \frac{1}{4}\left|\nabla^V\mathcal{H}\right|^2 + \frac{1}{4b^2}\left(-\Delta^V\mathcal{H} + 2\widehat{e}_ii_{\widehat{e}^j}\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H} + 2e^ji_{\widehat{e}^i}\nabla_{\widehat{e}^i}\nabla_{e_j}\mathcal{H}\right) \\ &\quad - \frac{1}{2}\Bigg(L_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}}\right) + \frac{1}{2b^2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)(e_j) \\ &\qquad\qquad + \frac{1}{2b^2}\omega\left(\nabla^F,g^F\right)(e_i)\nabla_{\widehat{e}^i}\Bigg). \end{aligned} \tag{3.79}$$

By (3.78), we find that as $b \to +\infty$,

$$r^*_{b^2}2\mathfrak{A}^2_{\phi,b^2\mathcal{H}_{1/b^2}}r^*_{1/b^2} = \frac{1}{2}\left|\nabla^V\mathcal{H}\right|^2 - \left(L_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}}\right)\right) + \mathcal{O}(1/b). \tag{3.80}$$

To simplify our argument, assume that $\omega\left(\nabla^F,g^F\right) = 0$. Then (3.80) indicates that as $b \to +\infty$, the operator $2\mathfrak{A}^2_{\phi,b^2\mathcal{H}_{1/b^2}}$ behaves more and more like a simple perturbation of the generator of the Hamiltonian flow on T^*X.

Finally note that by (3.79), if $\omega\left(\nabla^F, g^F\right) = 0$,

$$r_b^* \mathfrak{A}^2_{\phi, b^2 \mathcal{H}_{1/b^2}} r^*_{1/b^2}|_{\Omega^0(X,F)} = -\frac{1}{4}\left(\frac{1}{b^4}\Delta^V + \left|\nabla^V \mathcal{H}\right|^2 - \frac{1}{b^2}\Delta^V \mathcal{H}\right) - \frac{1}{2}\nabla_{Y^{\mathcal{H}}}. \tag{3.81}$$

Equation (3.81) is especially simple. Also the first operator in the right-hand side of (3.81) is self-adjoint and nonnegative, while the operator $\nabla_{Y^{\mathcal{H}}}$ is skew-adjoint.

3.9. **Stochastic mechanics and the hypoelliptic Laplacian.** Let us now briefly explain how to construct the heat kernel associated to $\exp\left(-2A^2_{\phi,\mathcal{H}}\right)$. Here we skip all details concerning the effective existence of this kernel. Indeed inspection of (3.11) shows that to evaluate this kernel, we need to consider a stochastic differential equation on T^*X. Indeed take $x_0 \in X$, and let $s \in]0,1] \to w_s^* \in T^*_{x_0}X$ be a Brownian motion. Given $p_0 \in T^*_{x_0}X$, consider the stochastic differential equation,

$$\frac{dx}{ds} = \frac{\partial \mathcal{H}}{\partial p}(x,p), \qquad dp = -\frac{\partial \mathcal{H}}{\partial x}(x,p)\, ds + \tau_s^0 dw^*. \tag{3.82}$$

In (3.82), τ_s^0 denotes parallel transport with respect to the connection ∇^{T^*X}. Equations like (3.82) were explicitly considered in our earlier work [B81] on random mechanics, as generalized Hamiltonian equations under random perturbations.

Then elementary arguments on Itô equations show that at least formally,

$$\begin{aligned} &\mathrm{Tr_s}\left[\exp\left(-2A^2_{\phi,\mathcal{H}}\right)\right] \\ &= \int_{LX} \exp\left(-\frac{1}{2}\int_0^1 \left(\left|\frac{\partial \mathcal{H}}{\partial p}(x_s,p_s)\right|^2 + \left|\dot{p}_s + \frac{\partial \mathcal{H}}{\partial x}(x_s,p_s)\right|^2\right) ds + \dots\right). \end{aligned} \tag{3.83}$$

Using (3.82), we can rewrite (3.83) in the form

$$\begin{aligned} &\mathrm{Tr_s}\left[\exp\left(-2A^2_{\phi,\mathcal{H}}\right)\right] \\ &= \int_{LX} \exp\left(-\frac{1}{2}\int_0^1 \left(|\dot{x}_s|^2 + \left|\dot{p}_s + \frac{\partial \mathcal{H}}{\partial x}(x_s,p_s)\right|^2\right) ds + \dots\right). \end{aligned} \tag{3.84}$$

Assume now that $\mathcal{H}$ is the Legendre transform of a well-defined Lagrangian $L(x,\dot{x})$, so that

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p}(x,p). \tag{3.85}$$

Let I be the functional on LX,

$$I(x) = \int_0^1 L(x,\dot{x})\, ds. \tag{3.86}$$

Clearly,

$$\nabla I(x) = -\frac{D}{Dt}\frac{\partial L}{\partial \dot{x}}(x,\dot{x}) + \frac{\partial L}{\partial x}(x,\dot{x}). \tag{3.87}$$

We can then rewrite (3.84) in the form

$$\mathrm{Tr_s}\left[\exp\left(-2A^2_{\phi,\mathcal{H}}\right)\right] = \int_{LX} \exp\left(-\frac{1}{2}\int_0^1 \left(|\dot{x}_s|^2 + |\nabla I(x)|^2\right) ds + \dots\right). \tag{3.88}$$

When replacing $\mathcal{H}(x,p)$ by $\mathcal{H}_{1/b}(x,p) = \mathcal{H}(x,p/b)$, $L(x,\dot{x})$ is replaced by $L_b(x,\dot{x})$ given by

$$L_b(x,\dot{x}) = L(x,b\dot{x}). \tag{3.89}$$

If $\mathcal{H}$ is replaced by $b^2\mathcal{H}_{1/b^2}$, then $L(x,\dot{x})$ is replaced by $b^2L(x,\dot{x})$. In this last case, by (3.88), (3.89), we get

(3.90)
$$\mathrm{Tr_s}\left[\exp\left(-2A^2_{\phi,b^2\mathcal{H}_{1/b^2}}\right)\right]=\int_{LX}\exp\left(-\frac{1}{2}\int_0^1\left(|\dot{x}_s|^2+b^4\left|\nabla I(x)\right|^2\right)ds+\ldots\right).$$

Formula (3.90) clearly indicates that as $b\to 0$, the right-hand side becomes an integral with respect to the classical Brownian motion on X, and as $b\to+\infty$, the integral should localize on the critical points of the functional I. Now these critical points correspond exactly to the closed integral trajectories of the Hamiltonian vector field $Y^{\mathcal{H}}$ in T^*X.

The pictures which emerge from subsections 3.8 and 3.9 are then remarkably consistent. Either at the level of operators or at the level of functional integration, we see that when considering $2A^2_{\phi,b^2\mathcal{H}_{1/b^2}}$, as $b\to 0$, we should recover the scaled Laplacian $\square^X/2$ on X and the classical Brownian integral on LX, and as $b\to+\infty$, we obtain the generator of the Hamiltonian vector field $Y^{\mathcal{H}}$, and the functional integral localizes on the corresponding closed trajectories.

Let $\Phi\left(TLX,g^{TLX},\nabla^{TLX}\right)$ be the Thom form of Mathai and Quillen [MaQ86] on the total space of TLX. Recall that in the introduction, we explained that when filling in the $\ldots$ in the right-hand sides of (3.88), we would obtain a natural cohomological expression, in which the form $(\nabla I)^*\,\Phi^{LTX}\left(TLX,g^{TLX},\nabla^{TLX}\right)$ would appear. Since this argument is essentially formal, we leave it to the inspired reader. Let us just mention that Witten and Atiyah [A85] gave the idea that equivariant cohomology with respect to the action of S_1 on LX is important, understanding basic formal aspects of the heat equation method in index theory, and that this line of argument was further developed in [B85, B92a, B92b].

Let us make two final observations. A first observation is that our formalism works starting from a Hamiltonian function $\mathcal{H}$, that is, even in the case where $\mathcal{H}$ does not have a nice Legendre transform. Indeed let $y=(x,p)$ be the generic element in T^*X. Consider the obvious Cartan Lagrangian on TT^*X

$$\mathcal{L}(y,\dot{y})=\langle p,\pi_*\dot{y}\rangle-\mathcal{H}(y). \tag{3.91}$$

Let $\mathcal{I}$ be the functional on LT^*X,

$$\mathcal{I}(y)=\int_0^1\mathcal{L}(y,\dot{y})\,ds. \tag{3.92}$$

Then

$$\nabla\mathcal{I}=-\left(i_{\dot{y}}\omega+d\mathcal{H}\right). \tag{3.93}$$

Let $\overline{\mathcal{I}}$ be the restriction of $\mathcal{I}$ to the subset $\overline{L}T^*X$ of LT^*X defined by the equation

$$\dot{x}=\frac{\partial\mathcal{H}}{\partial p}(x,p). \tag{3.94}$$

Then

$$\nabla\overline{\mathcal{I}}=-\pi^*\left(\dot{p}+\frac{\partial\mathcal{H}}{\partial x}\right). \tag{3.95}$$

Note that if $\mathcal{H}$ is the Legendre transform of L, then $\overline{L}T^*X\simeq LX,\overline{\mathcal{I}}\simeq I$, so that

$$\nabla\overline{\mathcal{I}}=\nabla I. \tag{3.96}$$

By (3.95), equation (3.84) can be written in the form

$$\mathrm{Tr_s}\left[\exp\left(-2A^2_{\phi,\mathcal{H}}\right)\right] = \int_{\overline{L}T^*X} \exp\left(-\frac{1}{2}\int_0^1 \left(|\dot{x}|^2 + \left|\nabla\overline{\mathcal{I}}\right|^2\right) ds + \ldots\right). \tag{3.97}$$

We make now a final comment on the construction of the operator $A_{\phi,\mathcal{H}}$. In (2.57), given $\epsilon \geq 0$, we could as well replace ϕ by ϕ^ϵ given by

$$\phi^\epsilon = \begin{pmatrix} 1 & -1 \\ 1 & \epsilon \end{pmatrix}, \tag{3.98}$$

so that $\phi = \phi^0$. Note that

$$(\phi^\epsilon)^{-1} = \frac{1}{1+\epsilon}\begin{pmatrix} \epsilon & 1 \\ -1 & 1 \end{pmatrix}. \tag{3.99}$$

As in (2.62), we identify ϕ^ϵ and $(\phi^\epsilon)^{-1}$ with the endomorphisms of TT^*X,

$$\phi^\epsilon = \begin{pmatrix} g^{TX} & -1|_{T^*X} \\ 1|_{TX} & \epsilon g^{T^*X} \end{pmatrix}, \qquad (\phi^\epsilon)^{-1} = \frac{1}{1+\epsilon}\begin{pmatrix} \epsilon g^{T^*X} & 1|_{TX} \\ -1|_{T^*X} & g^{TX} \end{pmatrix}. \tag{3.100}$$

Let $\pi'_* : TT^*X = TX \oplus T^*X \to T^*X$ be the obvious projection. Instead of (2.63), ϕ^ϵ is now associated to the bilinear form η_ϵ on TT^*X given by

$$\eta_\epsilon(U,V) = \langle \pi_* U, \pi_* V\rangle_{g^{TX}} + \epsilon\langle \pi'_* U, \pi'_* V\rangle_{g^{T^*X}} + \omega(U,V). \tag{3.101}$$

On the same principle as before, one can construct the operator $A_{\phi^\epsilon,\mathcal{H}}$. Instead of (2.93), (2.94), we have

$$\begin{aligned} \sigma\left(A_{\phi^\epsilon,\mathcal{H}}\right) &= \frac{i}{2}\left(\xi^H + \widehat{\xi^V} + \frac{1}{1+\epsilon} i_{\xi^V - \epsilon\xi^H - \widehat{\xi^H} - \widehat{\xi^V}}\right), \\ \sigma\left(A^2_{\phi^\epsilon,\mathcal{H}}\right) &= \frac{1}{4(1+\epsilon)}\left(\epsilon\left|\xi^H\right|^2 + \left|\xi^V\right|^2\right). \end{aligned} \tag{3.102}$$

For $\epsilon > 0$, the operator $A_{\phi^\epsilon,\mathcal{H}}$ is now elliptic. We equip $TT^*X = TX \oplus TT^*X$ with the metric $g^{TT^*X}_\epsilon = g^{TX} \oplus \epsilon g^{T^*X}$. Let $g^{T^*T^*X}_\epsilon$ be the corresponding metric on $T^*T^*X = T^*X \oplus TX$.

Let $y = (x,p)$ be the generic element in T^*X. One verifies that the extension (3.97) to the present setting is given by

$$\mathrm{Tr_s}\left[\exp\left(-2A^2_{\phi^\epsilon,\mathcal{H}}\right)\right] = \int_{LT^*X} \exp\left(-\frac{1}{2}\int_0^1 \left(|\dot{y}|^2_{g^{TT^*X}_\epsilon} + |\nabla\mathcal{I}|^2_{g^{T^*T^*X}_\epsilon}\right) ds + \ldots\right). \tag{3.103}$$

The exponential factor in the right-hand side of (3.103) can be written more explicitly as

$$\begin{aligned} \exp\Bigg(-\frac{1}{2}\int_0^1 \left(|\dot{x}|^2 + \epsilon|\dot{p}|^2\right) ds - \frac{1}{2}\int_0^1 \Bigg(\frac{1}{\epsilon}\left|\dot{x} - \frac{\partial\mathcal{H}}{\partial p}(x_s,p_s)\right|^2 \\ + \left|\dot{p} + \frac{\partial\mathcal{H}}{\partial x}(x_s,p_s)\right|^2\Bigg) ds \ldots\Bigg). \end{aligned} \tag{3.104}$$

Equation (3.104) makes clear that at least formally, as $\epsilon \to 0$, the integral (3.103) localizes on the submanifold $\overline{L}T^*X$, and that we should recover (3.97). This is indeed the case.

Still the operator $A^2_{\phi^\epsilon,\mathcal{H}}$ is not self-adjoint, even when acting on functions, essentially because of the presence of $Y^{\mathcal{H}}$. Having $\epsilon > 0$ does not make the situation any

simpler analytically, even if it brings only temporary relief because the operator $A^2_{\phi^\epsilon,\mathcal{H}}$ is now elliptic.

The lack of self-adjointness can also be reformulated from another point of view. Indeed when we consider the geodesic flow on the unit cotangent bundle, when reversing the direction time, the speed $p = \dot{x}$ becomes $-p$. This reflects the fact that the operator $\nabla_{Y^{\mathcal{H}}}$ associated to $\mathcal{H} = |p|^2/2$ is only skew-adjoint. The stochastic process associated to the stochastic differential equation (3.82) is not invariant by time reversal, and its generator is not self-adjoint. The fact that as $b \to 0$, the operator $A^2_{\phi,\mathcal{H}/b^2}$ converges in a natural way to the scaled Laplacian $\Delta^X/4$ on X, which is self-adjoint, reflects also the fact that the speed p becomes infinite and, as such, is invisible by time reversal.

3.10. **The case where $X = S_1$.** Assume now that $X = S_1$, with $S_1 = \mathbf{R}/\mathbf{Z}$ equipped with its standard metric, and let F be a complex vector bundle on X, equipped with a flat metric g^F, so that $\omega\left(\nabla^F, g^F\right) = 0$. In this case, $T^*X = S_1 \times \mathbf{R}$. Let $\widehat{N}$ be the number operator of $\Lambda(\mathbf{R})$. Let $x \in \mathbf{R}$ be the canonical coordinate on S_1, and let $e = \frac{\partial}{\partial x}$ be the canonical generator of TX. Finally let $\Delta^X = \nabla^{F,2}_e$ be the Laplacian of X.

By equation (3.13) in Theorem 3.4, we get

$$\mathfrak{A}'^2_{\phi,\mathcal{H}^c} = \frac{1}{4}\left(-\Delta^V + c^2|p|^2 + c\left(2\widehat{N} - 1\right)\right) - \frac{c}{2}\nabla^F_p. \tag{3.105}$$

Proposition 3.16. *For $c \neq 0$, the following identity holds:*

$$\begin{aligned}\exp\left(\frac{1}{c}\frac{\partial^2}{\partial p \partial x}\right)\mathfrak{A}'^2_{\phi,\mathcal{H}^c}\exp\left(-\frac{1}{c}\frac{\partial^2}{\partial p\partial x}\right)&\\ = \frac{1}{4}\left(-\Delta^V + c^2|p|^2 + c\left(2\widehat{N} - 1\right)\right) - \frac{1}{4}\Delta^X.&\end{aligned} \tag{3.106}$$

Proof. We will write the operator ∇^F_p in the form $p\frac{\partial}{\partial x}$. If $c \neq 0$, we get

$$\begin{aligned}\left[\frac{1}{c}\frac{\partial^2}{\partial p\partial x}, \frac{1}{2}c^2|p|^2 - cp\frac{\partial}{\partial x}\right] &= cp\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2},\\ \frac{1}{2}\left[\frac{1}{c}\frac{\partial^2}{\partial p\partial x}, cp\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right] &= \frac{1}{2}\frac{\partial^2}{\partial x^2},\end{aligned} \tag{3.107}$$

and the next commutators obviously vanish. By (3.107), we have the equality

$$\exp\left(\frac{1}{c}\frac{\partial^2}{\partial p\partial x}\right)\left(\frac{1}{2}c^2|p|^2 - cp\frac{\partial}{\partial x}\right)\exp\left(-\frac{1}{c}\frac{\partial^2}{\partial p\partial x}\right) = \frac{1}{2}c^2|p|^2 - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{3.108}$$

Equation (3.106) is now an obvious consequence of (3.108). □

Remark 3.17. Equation (3.106) should be properly interpreted. Indeed the conjugating operator $\exp\left(\frac{1}{c}\frac{\partial^2}{\partial p\partial x}\right)$ is not even well defined as an operator.

Assume that F is the trivial vector bundle. Clearly $\mathbf{N}$ is the set of eigenvalues of $\frac{1}{2}\left(-\Delta^V + |p|^2 - 1\right)$, and the corresponding eigenfunctions are given by $Q_n(p) = \exp\left(-|p|^2/2\right)P_n(p)$, where the $P_n(p)$ are the Hermite polynomials. For $c > 0$, the spectrum of the operator in the right-hand side of (3.106) consists of the set $\frac{c}{2}n + \pi^2 m^2, m, n \in \mathbf{N}$. When acting on functions, the corresponding eigenvectors

are just $Q_n\left(\sqrt{c}p\right)\exp\left(2i\pi mx\right)$, the extension to forms of arbitrary degree being trivial. Now note that

$$\begin{aligned}\exp\left(-\frac{1}{c}\frac{\partial^2}{\partial p\partial x}\right)Q_n\left(\sqrt{c}p\right)\exp\left(2i\pi mx\right)\\ =\exp\left(-2i\pi\frac{m}{c}\frac{\partial}{\partial p}\right)Q_n\left(\sqrt{c}p\right)\exp\left(2i\pi mx\right)\\ =Q_n\left(\sqrt{c}p-2i\pi\frac{m}{\sqrt{c}}\right)\exp\left(2i\pi mx\right).\end{aligned}\tag{3.109}$$

Ultimately, one gets the full spectrum of the operator $\mathfrak{A}'^2_{\phi,\mathcal{H}^c}$ and the corresponding eigenvectors.

But even more is true. By (3.106), we find that for $t>0$, as $c\to+\infty$, the operator $\exp\left(-t\mathfrak{A}'^2_{\phi,\mathcal{H}^c}\right)$ converges in the appropriate sense to $\exp\left(t\Delta^X/4\right)$, simply because $\exp\left(-\frac{ct}{4}\left(-\Delta^V+|p|^2+2\widehat{N}-1\right)\right)$ converges to the orthogonal projection operator on its one dimensional kernel.

Finally, let us briefly explain how to give a direct proof of the Poisson formula for the heat kernel on S_1. Indeed let $\mathrm{Tr_s}$ denote the supertrace for operators acting on $\Omega^{\cdot}(\mathbf{R},\mathbf{R})$. Clearly

$$\mathrm{Tr_s}\left[\exp\left(\frac{c}{4}\left(-\Delta^V+|p|^2+2\widehat{N}-1\right)\right)\right]=1.\tag{3.110}$$

So by (3.106), for $t>0$, we get

$$\mathrm{Tr_s}\left[\exp\left(-t\mathfrak{A}'^2_{\phi,\mathcal{H}^c}\right)\right]=\mathrm{Tr}\left[\exp\left(t\Delta^X/4\right)\right].\tag{3.111}$$

By making $c\to 0$, as explained before, the left-hand side localizes on the closed geodesics in S_1. Taking the limit of (3.111) as $c\to 0$ just gives the Poisson formula.

It is a priori hopeless to extend the previous considerations in a more geometric context. Still, the above considerations make sense microlocally. Indeed the principal symbol of $\nabla_p^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}$ is $i\xi^H$. Now

$$\frac{c^2}{4}|p|^2-\frac{ic}{2}\left\langle p,\xi^H\right\rangle=\frac{c^2}{4}\left|p-\frac{i}{c}\xi^H\right|^2+\frac{1}{4}\left|\xi^H\right|^2.\tag{3.112}$$

However making the shift $p\to p+\frac{i}{c}\xi^H$ is a much too brutal transformation in the geometric setting. Still the above is indeed enough to roughly understand the situation as $c\to+\infty$.

4. Flat superconnections on the cotangent bundle of Riemannian fibres

The purpose of this section is to develop the machinery of Sections 2–3 in a relative situation, by using the formalism of Bismut and Lott [BLo95]. More precisely, if $p:M\to S$ is a fibration with compact fibre X, if F is a flat vector bundle on M, we extend the previous constructions of $d^{T^*X},\overline{d}^{T^*X}_{\phi,\mathcal{H}}$ into flat superconnections over S. Also we prove corresponding Weitzenböck formulas. Finally, by using various conjugations, we describe equivalent forms of these superconnections, in which the obtained formulas are more directly related to the formalism of Mathai and Quillen [MaQ86] on one hand and also reflect the underlying equivariant cohomology of the loop space on the other hand, this connection being obvious by looking at the formulas.

This section is organized as follows. In subsection 4.1, we describe the formalism of the creation and annihilation operators. In subsection 4.2, we recall the interpretation in [BLo95] of the de Rham operator d^M as a flat superconnection A'^M on $\Omega^{\cdot}(X, F|_X)$. In subsection 4.3, we define the adjoint of a superconnection with respect to a fibrewise sesquilinear form. In subsection 4.4, if M is equipped with a closed 2-form ω which induces a symplectic form along the fibres X, we obtain the symplectic adjoint $\overline{A}'^M$ of A'^M. In subsection 4.5, if $\mathcal{M}$ is the total space of the cotangent bundle fibration T^*X, we construct the corresponding flat superconnections $A'^{\mathcal{M}}$ and $\overline{A}'^{\mathcal{M}}$.

In subsections 4.6, 4.7 and 4.8, we recall the definition of the tools involved in the construction of the Levi-Civita superconnection [B86], which is attached to a horizontal vector bundle T^HM and to a fibrewise metric g^{TX}. In subsection 4.9, we obtain the flat superconnection $\overline{A}'^{\mathcal{M}}_{\phi}$, which specializes to $\overline{d}^{T^*X}_{\phi}$ when S is a point. In subsection 4.10, we show that $\overline{A}'^{\mathcal{M}}_{\phi}$ is also the adjoint of $A'^{\mathcal{M}}$ with respect to the Hermitian form on $\Omega^{\cdot}(X, F|_X)$ considered in subsection 2.7.

In subsections 4.11 and 4.12, we introduce the Witten twist attached to a Hamiltonian function $\mathcal{H} : \mathcal{M} \to \mathbf{R}$. In subsection 4.13, we study the behaviour of our superconnections under Poincaré duality. In subsection 4.14, we conjugate our superconnections by using the same automorphisms of TT^*X as in subsection 2.11. By adapting the formalism of [BLo95] in the present situation, we obtain, among other things, a superconnection $\mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^{\mathcal{H}}}$. In subsection 4.15, we study the effect of the scaling of the variable $p \in T^*X$. In subsection 4.16, we consider the case where $\mathcal{H} = \mathcal{H}^c$, with $\mathcal{H}^c = c\,|p|^2/2$. In subsection 4.17, we relate our superconnection to the Levi-Civita superconnection of Bismut and Lott [BLo95].

In subsection 4.18, we give a Weitzenböck formula for the curvature $\mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H}$.

In subsection 4.19, we specialize our formula to the case $\mathcal{H} = \mathcal{H}^c$. This assumption will remain in force except in subsection 4.25. In subsection 4.20, we establish a commutator identity, which will play a critical role in the corresponding local index theory in [BL05]. In subsection 4.21, we start playing with the creation and annihilation operators, first the vertical ones and later mixing them all. One purpose for this is to make the Weitzenböck formulas simpler geometrically and also accessible to methods of local index theory developed in [BL05].

In subsection 4.22, we give another construction of the flat superconnections, with less emphasis given to the symplectic geometry and more to the metric aspect. Although the superconnections look more complicated, the curvature is exactly the one obtained after many conjugations in the initial construction.

In subsection 4.23, we relate the curvature of our new superconnection to the curvature of the Levi-Civita superconnection. In subsection 4.24, we make another conjugation, to eliminate a Grassmann variable.

Finally, in subsection 4.25, we consider the case where $\mathcal{H}$ is the canonical Hamiltonian associated to a Killing vector field K along the fibres X.

4.1. **Creation and annihilation operators.** Let V be a finite dimensional real vector space of dimension n. Note that for $0 \le p \le n$, $\Lambda^p(V)$ is naturally dual to $\Lambda^p(V^*)$. Let $\rho \in \operatorname{End}(\Lambda^{\cdot}(V))$ be such that if $\alpha \in \Lambda^p(V)$,

$$\rho\alpha = (-1)^{(n-p)(n-p-1)/2}\,\alpha. \tag{4.1}$$

Here the duality between $\Lambda^p(V)$ and $\Lambda^p(V^*)$ will be given by

$$(\omega,\alpha)\in\Lambda^p(V^*)\times\Lambda^p(V)\to\langle\omega,\rho\alpha\rangle. \tag{4.2}$$

Given p with $0\le p\le n$, there is a bilinear map

$$\Lambda^p(V^*)\times\Lambda^{n-p}(V^*)\to\Lambda^n(V^*).$$

Since this bilinear map is nondegenerate, using (4.2), we get the canonical isomorphism

$$\Lambda^{\cdot}(V^*)\simeq\Lambda^{n-\cdot}(V)\otimes\Lambda^n(V^*). \tag{4.3}$$

Let $X\in V, f\in V^*$. The creation operators $f\wedge$ and the annihilation operators i_X act naturally on $\Lambda^{\cdot}(V^*)$. Under the canonical isomorphism (4.3), to the action of $f\wedge, i_X$ on $\Lambda^{\cdot}(V^*)$ there corresponds the action of $i_f, X\wedge$ on $\Lambda^{n-\cdot}(V)\otimes\Lambda^n(V^*)$.

Let $\nu\in\Lambda^{\max}(V)$ with $\nu\neq 0$, and let $\nu^{-1}\in\Lambda^n(V^*)$ be dual to ν. Then the canonical isomorphism in (4.3) is given by

$$f^1\wedge\ldots\wedge f^p\in\Lambda^p(V^*)\to i_{f^1}\ldots i_{f^p}\nu\otimes\nu^{-1}. \tag{4.4}$$

4.2. **Fibrations and flat superconnections.** Let $p:M\to S$ be a submersion of smooth manifolds, with fibre X. Let $TX\subset TM$ be the tangent bundle to the fibres. Set $n=\dim X$. Let $T^HM\subset TM$ be a horizontal subbundle of TM, so that

$$TM=T^HM\oplus TX. \tag{4.5}$$

Let $P^{TX}:TM\to TX$ be the projection operator with respect to the splitting (4.5). By (4.5),

$$T^HM\simeq p^*TS. \tag{4.6}$$

By (4.6), we get the isomorphism of the bundles of exterior algebras,

$$\Lambda^{\cdot}(T^*M)\simeq p^*\Lambda^{\cdot}(T^*S)\widehat{\otimes}\Lambda^{\cdot}(T^*X). \tag{4.7}$$

In (4.7), $\widehat{\otimes}$ is our notation for the $\mathbf{Z}$-graded tensor product.

Let F be a complex vector bundle on M, and let ∇^F be a flat connection on F. Let $\left(\Omega^{\cdot}(X,F|_X),d^X\right)$ be the fibrewise de Rham complex of smooth forms on X with values in $F|_X$, equipped with the fibrewise de Rham operator d^X.

Let $\Omega^{\cdot}(S,\Omega^{\cdot}(X,F|_X))$ be the set of smooth sections on S of $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\Omega^{\cdot}(X,F|_X)$. Then by (4.7), we have the isomorphism,

$$\Omega^{\cdot}(M,F)\simeq\Omega^{\cdot}(S,\Omega^{\cdot}(X,F|_X)). \tag{4.8}$$

Now we follow [BLo95, Section 3 (b)]. The operator d^M acts on $\Omega^{\cdot}(M,F)$ and is such that $d^{M,2}=0$. If α is a smooth section of $\Lambda^{\cdot}(T^*S)$, if $s\in\Omega^{\cdot}(M,F)$, then

$$d^M(p^*\alpha)s=\left(p^*d^S\alpha\right)s+(-1)^{\deg\alpha}(p^*\alpha)d^Ms. \tag{4.9}$$

By (4.9), d^M can be viewed as a superconnection A'^M on the infinite dimensional vector bundle $\Omega^{\cdot}(X,F|_X)$. Since $d^{M,2}=0$, A'^M is a flat superconnection, i.e.

$$A'^{M,2}=0. \tag{4.10}$$

Also A'^M is a superconnection of total degree 1 on $\Omega^{\cdot}(X,F|_X)$ in the sense of [BLo95, Section 2 (b)].

Let d^X be the de Rham operator along the fibres. If $U\in TS$, let $U^H\in T^HM$ be the horizontal lift of U. If $U\in TS$, the Lie derivative operator L_{U^H} acts naturally

on $\Omega^{\cdot}(X, F|_X)$. Let $\nabla^{\Omega^{\cdot}(X,F|_X)}$ be the connection on $\Omega^{\cdot}(X, F|_X)$, such that if $U \in TS$ and if s is a smooth section of $\Omega^{\cdot}(X, F|_X)$,

$$\nabla_U^{\Omega^{\cdot}(X,F|_X)} s = L_{U^H} s. \tag{4.11}$$

Using (4.7), (4.8), we find that $\nabla^{\Omega^{\cdot}(X,F|_X)}$ acts on $\Omega^{\cdot}(M, F)$.

Finally, if U, V are smooth sections of TS, put

$$T^H(U, V) = -P^{TX}\left[U^H, V^H\right]. \tag{4.12}$$

Using (4.7), we find that the operators d^X and i_{T^H} both act on $\Omega^{\cdot}(M, F)$.

Now we have the result in [BLo95, Proposition 3.4].

Proposition 4.1. *The following identity holds:*

$$A'^M = d^X + \nabla^{\Omega^{\cdot}(X,F|_X)} + i_{T^H}. \tag{4.13}$$

4.3. **The adjoint of a superconnection with respect to a nondegenerate bilinear form.** Let dv_X be a volume form along the fibres X. Let $\Omega^{*\cdot}\left(X, \overline{F}^*|_X\right)$ be the vector bundle over S of smooth compactly supported sections of $\Lambda(TX) \otimes F$ along the fibres X. As in subsection 2.3, using (2.22), $\Omega^{\cdot}(X, F|_X)$ and $\Omega^{*\cdot}\left(X, \overline{F}^*|_X\right)$ are antidual to each other.

Let A'^{M*} be the superconnection on $\Omega^{*\cdot}\left(X, \overline{F}^*\right)$ which is the transpose to A'^M. The definition of A'^{M*} is given in [BLo95, Section 1 (c)]. We will not recall precisely the definition of the transpose of a superconnection, except that it has all the formal properties of a transpose and moreover that if $f \in T^*S$, the transpose of $f\wedge$ is $-f\wedge$. Then A'^{M*} is a flat superconnection of total degree -1 in the sense of [BLo95, Section 2 (a)].

Let η be a smooth nondegenerate bilinear form on TX. Equivalently η defines a smooth morphism $TX \to T^*X$, which we denote by ϕ as in (2.7). Let g^F be a Hermitian metric on F. Then the couple $\left(\eta, g^F\right)$ induces an isomorphism $\Omega^{\cdot}(X, F|_X) \simeq \Omega^{*\cdot}\left(X, \overline{F}^*|_X\right)$.

Let A''^M be the pull-back of the superconnection A'^{M*} by the above isomorphism. Then A''^M is a flat superconnection on $\Omega^{\cdot}(X, F|_X)$ which has total degree -1. The superconnection A''^M will be called the adjoint of A'^M with respect to dv_{TX}, η and g^F.

4.4. **The symplectic adjoint of A'^M.** We make the same assumptions as in subsection 4.2. Also, we assume that there is a closed 2-form ω on M, whose restriction to the fibres X is a symplectic form along the fibres.

Then T^HM will be taken to be the orthogonal bundle to TX in TM with respect to ω. Using (4.7), we can split the form ω as

$$\omega = \omega^H + \omega^V, \tag{4.14}$$

where ω^H, ω^V are the restrictions of ω to T^HM, TX.

Proposition 4.2. *The form ω^V is fibrewise closed. If $U \in TS$, then*

$$L_{U^H}\omega^V = 0, \qquad \nabla^{\Omega^{\cdot}(X,F|_X)}\omega^H = 0. \tag{4.15}$$

If $U, V \in TS$, then $T^H(U, V)$ is a symplectic vector field along the fibres X and $\omega^H(U, V)$ is a Hamiltonian for this vector field.

Proof. Using (4.13) and (4.14) and writing the identity $d^M\omega = 0$ degree by degree, we get our proposition. □

Let dv_X be the symplectic fibrewise volume form on X. Let $\phi : TX \to T^*X$ be the isomorphism associated to the fibrewise symplectic form ω^V as in subsection 2.4. Let g^F be a Hermitian metric on F.

Definition 4.3. Let $\overline{A}'^M$ be the adjoint to A'^M with respect to dv_X, ω^V, g^F.

Then $\overline{A}'^M$ is a flat superconnection of total degree -1 on $\Omega^{\cdot}(X, F|_X)$. Let $\overline{d}^X$ be the fibrewise symplectic adjoint of d^X with respect to the above data. Recall that the metric g^F identifies $\Omega^{\cdot}(M, F)$ and $\Omega^{\cdot}\left(M, \overline{F}^*\right)$. Therefore $\nabla^{\Omega^{\cdot}\left(X, \overline{F}^*|_X\right)}$ can be considered as a connection on $\Omega^{\cdot}(X, F|_X)$.

Proposition 4.4. *The following identity holds:*

$$\overline{A}'^M = \overline{d}^X + \nabla^{\Omega^{\cdot}\left(X, \overline{F}^*|_X\right)} - \phi T^H \wedge . \tag{4.16}$$

Proof. By Proposition 4.2, the volume form dv_X and the morphism ϕ are parallel with respect to $\nabla^{\Omega^{\cdot}(X, F|_X)}$. It follows that the adjoint of $\nabla^{\Omega^{\cdot}(X, F|_X)}$ is just $\nabla^{\Omega^{\cdot}\left(X, \overline{F}^*|_X\right)}$. Finally, by (2.11), the symplectic adjoint of i_{T^H} is equal to $-\phi T^H \wedge$. The proof of our proposition is completed. □

4.5. **The case of the cotangent bundle fibration.** We make the same assumptions as in subsection 4.2. In particular T^HM is taken as in (4.5).

Let T^*X be the dual vector bundle to TX. Let $\mathcal{M}$ be the total space of T^*X, and let $\pi : \mathcal{M} \to M, q : \mathcal{M} \to S$ be the obvious projections. Observe that M embeds into $\mathcal{M}$ as the zero section of T^*X.

We claim that T^HM lifts to a natural horizontal bundle $T^H\mathcal{M}$ associated to the projection $q : \mathcal{M} \to S$. In fact T^HM induces a connection on the $\mathrm{Diff}(X)$ principal bundle Q associated to $p : M \to S$. Moreover $\mathrm{Diff}(X)$ acts symplectically on T^*X. Since

$$\mathcal{M} = Q \times_{\mathrm{Diff}(X)} T^*X, \tag{4.17}$$

we find that $\mathcal{M}$ is also equipped with a horizontal vector bundle $T^H\mathcal{M}$, such that $\pi_* T^H\mathcal{M} = T^HM$.

By (4.7), a form in $\Lambda^{\cdot}(T^*X)$ extends to a vertical form in $\Lambda^{\cdot}(T^*M)$. Therefore if $p \in T^*X$, the corresponding 1-form along the fibre X extends to a 1-form on M, which vanishes on T^HM. Therefore $\theta = \pi^*p$ can be considered as a globally defined 1-form on $\mathcal{M}$. Set

$$\omega = d^{\mathcal{M}}\theta. \tag{4.18}$$

Then ω is a closed 2-form on $\mathcal{M}$, whose restriction to the fibres T^*X is just the canonical symplectic form of the fibres.

Proposition 4.5. *The vector bundle $T^H\mathcal{M}$ is exactly the orthogonal bundle to TT^*X in $T\mathcal{M}$ with respect to ω.*

Proof. Let $U \in TS$, and let $U^H \in T^HM$ be its horizontal lift. By construction,

$$i_{U^H}\theta = 0. \tag{4.19}$$

Then $T^H\mathcal{M}$ consists exactly of the $V \in T\mathcal{M}$ which are such that there is $U \in TS$ with $\pi_* V = U^H$ and moreover $L_V\theta|_{TT^*X} = 0$. By (4.18) and (4.19), we get

$$L_V\theta = i_V\omega. \tag{4.20}$$

Using (4.20) and counting dimensions, our proposition follows. □

Clearly,

$$T^H\mathcal{M}|_M = T^H M. \tag{4.21}$$

If $U \in TS$, we will denote by U^H the lift of U in $T^H M$ or in $T^H\mathcal{M}$.

Let $T^H, \mathcal{T}^H$ be the tensors defined as in (4.12), which are associated, respectively, to $T^H M$ and $T^H\mathcal{M}$. By (4.21),

$$\mathcal{T}^H|_M = T^H. \tag{4.22}$$

The formalism of subsection 4.4 can be applied to the fibration $q : \mathcal{M} \to S$ with fibre T^*X. In particular, as in (4.14), we write ω in the form

$$\omega = \omega^H + \omega^V, \tag{4.23}$$

so that ω^H, ω^V are the restrictions of ω to $T^H\mathcal{M}, TT^*X$.

Proposition 4.6. *The following identity holds:*

$$\omega^H = \left\langle p, T^H \right\rangle. \tag{4.24}$$

Moreover, if $U, V \in TS$, $\mathcal{T}^H(U,V)$ is the vector field along the fibre T^*X associated to the Hamiltonian $\left\langle p, T^H(U,V)\right\rangle$. Equivalently,

$$\left[d^{T^*X}, \omega^H\right] + i_{\mathcal{T}^H}\omega^V = 0, \qquad i_{\mathcal{T}^H} = \left[\overline{d}^{T^*X}, \omega^H\right]. \tag{4.25}$$

Proof. Using Proposition 4.1, we get (4.24). The second part of our proposition follows from Proposition 4.2. The first equation in (4.25) is trivial and the second equation follows from (2.34). □

Let F be a complex flat vector bundle on M, and let ∇^F be the corresponding flat connection. Let $\left(\Omega^{\cdot}(T^*X, \pi^*F|_{T^*X}), d^{T^*X}\right)$ be the de Rham complex of smooth forms along T^*X with coefficients in $\pi^*F|_{T^*X}$, which have compact support. As in (4.11), if $U \in TS$, and if s is a smooth section of $\Omega^{\cdot}(T^*X, \pi^*F|_{T^*X})$, set

$$\nabla_U^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})} s = L_{U^H} s. \tag{4.26}$$

Then $\nabla^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}$ is a connection on $\Omega^{\cdot}(T^*X, \pi^*F|_{T^*X})$. Similarly, we will denote by $\nabla^{\Omega^{\cdot}\left(T^*X,\pi^*\overline{F}^*|_{T^*X}\right)}$ the corresponding connection on $\Omega^{\cdot}\left(T^*X, \pi^*\overline{F}^*|_{T^*X}\right)$.

We denote by $A'^{\mathcal{M}}$ the flat superconnection on $\Omega^{\cdot}(T^*X, \pi^*F|_{T^*X})$ which is associated to the de Rham operator $d^{\mathcal{M}}$.

Let g^F be a Hermitian metric on F. We still define $\omega\left(\nabla^F, g^F\right)$ as in (2.49), and the unitary connection $\nabla^{F,u}$ on F as in (2.51).

Clearly g^F maps $\Omega^{\cdot}(T^*X, \pi^*F|_{T^*X})$ into $\Omega^{\cdot}\left(T^*X, \pi^*\overline{F}^*|_{T^*X}\right)$. The connection $\nabla^{\Omega^{\cdot}\left(T^*X,\pi^*\overline{F}^*|_{T^*X}\right)}$ pulls back to a connection on $\Omega^{\cdot}(T^*X, \pi^*F|_{T^*X})$, which we still denote $\nabla^{\Omega^{\cdot}\left(T^*X,\pi^*\overline{F}^*|_{T^*X}\right)}$.

Let $f_1, \ldots, f_m$ be a basis of TS, and let $f^1, \ldots, f^m$ be the corresponding dual basis of T^*S. Clearly,

$$\nabla^{\Omega^{\cdot}\left(T^*X,\pi^*\overline{F}^*|_{T^*X}\right)} = \nabla^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})} + f^\alpha \wedge \omega\left(\nabla^F, g^F\right)\left(f_\alpha^H\right). \tag{4.27}$$

Let $\overline{A}'^{\mathcal{M}}$ be the symplectic adjoint of A'^{M}, which is associated to ω^V, g^F. Then $A'^{\mathcal{M}}$ and $\overline{A}'^{\mathcal{M}}$ are flat superconnections on $\Omega^{\cdot}\left(T^*X, \pi^*F|_{T^*X}\right)$, of total degree 1 and -1.

Theorem 4.7. *The following identities hold:*

$$\begin{aligned} A'^{\mathcal{M}} &= d^{T^*X} + \nabla^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})} + i_{\mathcal{T}^H}, \\ \overline{A}'^{\mathcal{M}} &= \overline{d}^{T^*X} + \nabla^{\Omega^{\cdot}\left(T^*X,\pi^*\overline{F}^*|_{T^*X}\right)} - d^{T^*X}\omega^H \wedge . \end{aligned} \tag{4.28}$$

If g^F is flat,

$$\left[A'^{\mathcal{M}}, \overline{A}'^{\mathcal{M}}\right] = \nabla^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X}),2}. \tag{4.29}$$

Proof. Equation (4.28) follows from Propositions 4.1, 4.4 and 4.6. Moreover, if g^F is flat,

$$\left[\nabla^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}, d^{T^*X}\right] = 0, \qquad \left[\nabla^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}, \overline{d}^{T^*X}\right] = 0. \tag{4.30}$$

By (4.25), we get

$$\left[\overline{d}^{T^*X}, i_{\mathcal{T}^H}\right] = 0. \tag{4.31}$$

Moreover by (4.15),

$$\left[\nabla^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}, \omega^H\right] = 0. \tag{4.32}$$

Now using Theorem 2.2 and (4.30)–(4.32), we get (4.29). The proof of our theorem is completed. □

4.6. **A metric and a connection on TX.** Let g^{TX} be a metric on TX. By [B86, Section 1], $\left(T^HM, g^{TX}\right)$ determine a Euclidean connection ∇^{TX} on TX. In fact let g^{TS} be a Euclidean metric on TS. We equip TM with the metric $g^{TM} = \pi^*g^{TS} \oplus g^{TX}$. Let $\nabla^{TM,L}$ be the Levi-Civita connection on $\left(TM, g^{TM}\right)$. Let ∇^{TX} be the connection on TX,

$$\nabla^{TX} = P^{TX}\nabla^{TM,L}. \tag{4.33}$$

Let ∇^{TM} be the connection on TM,

$$\nabla^{TM} = \pi^*\nabla^{TS} \oplus \nabla^{TX}. \tag{4.34}$$

Let T be the torsion of ∇^{TM}. Put

$$S = \nabla^{TM,L} - \nabla^{TM}. \tag{4.35}$$

Then S is a 1-form on M with values in antisymmetric elements of $\operatorname{End}(TX)$. Classically, if $A, B, C \in TM$,

$$\begin{aligned} &S(A)B - S(B)A + T(A,B) = 0, \\ &2\langle S(A)B, C\rangle + \langle T(A,B), C\rangle + \langle T(C,A), B\rangle - \langle T(B,C), A\rangle = 0. \end{aligned} \tag{4.36}$$

By [B86, Theorem 1.9], we know the following.

- The connection ∇^{TX} preserves the metric g^{TX}.
- The connection ∇^{TX} and the tensors T and $\langle S(\cdot), \cdot, \cdot\rangle$ do not depend on g^{TS}.

- The tensor T takes its values in TX and vanishes on $TX \times TX$. Moreover if $U, V \in TS$,

$$T\left(U^H, V^H\right) = T^H\left(U, V\right). \tag{4.37}$$

- For any $A \in TM$, $S(A)$ maps TX into $T^H M$.
- For any $A, B \in T^H M$, $S(A)B \in TX$.
- If $A \in T^H M$, $S(A)A = 0$.

From (4.36), we find that if $A \in T^H M$, $B, C \in TX$,

$$\langle T(A,B), C\rangle = \langle T(A,C), B\rangle = -\langle S(B)C, A\rangle. \tag{4.38}$$

Now, we recall a simple result stated in [B97, Theorem 1.1].

Theorem 4.8. *The connection ∇^{TX} on $\left(TX, g^{TX}\right)$ is characterized by the following two properties.*

- *On each fibre X, it restricts to the Levi-Civita connection.*
- *If $U \in TS$,*

$$\nabla^{TX}_{U^H} = L_{U^H} + \frac{1}{2}\left(g^{TX}\right)^{-1} L_{U^H} g^{TX}. \tag{4.39}$$

If $U, V \in TS$,

$$T\left(U^H, V^H\right) = T^H\left(U, V\right). \tag{4.40}$$

If $U \in TS, A \in TX$,

$$T\left(U^H, A\right) = \frac{1}{2}\left(g^{TX}\right)^{-1} L_{U^H} g^{TX} A. \tag{4.41}$$

Let dv_X be the volume along the fibre X which is associated to the metric g^{TX}. Let $e_1, \ldots, e_n$ be an orthonormal basis of TX. Set

$$e = -\sum_{i=1}^{n} S(e_i)\, e_i. \tag{4.42}$$

Then using the properties which were listed after (4.36), $e \in T^H M$.

If $U \in TS$, let $\mathrm{div}_X\left(U^H\right)$ be the smooth function along X such that

$$L_{U^H} dv_X = \mathrm{div}_X\left(U^H\right) dv_X. \tag{4.43}$$

Now we have a result stated in [BF86, Proposition 1.4].

Proposition 4.9. *If $U \in TS$,*

$$\left\langle e, U^H\right\rangle = \mathrm{div}_X\left(U^H\right). \tag{4.44}$$

4.7. **The symplectic connection.** Let $\overline{\nabla}^{TX}$ be the connection on TX which coincides with the Levi-Civita connection along the fibres of X and is such that if $U \in TS$ and if A is a smooth section of TX, then

$$\overline{\nabla}^{TX}_{U^H} A = \left[U^H, A\right]. \tag{4.45}$$

Let $\overline{R}^{TX}$ be the curvature of $\overline{\nabla}^{TX}$.

By (4.39), (4.40), we know that

$$\nabla^{TX}_{U^H} A = \overline{\nabla}^{TX}_{U^H} A + T\left(U^H, A\right). \tag{4.46}$$

Proposition 4.10. *If $A, B \in TX$, then*

$$\overline{R}^{TX}(A,B) = R^{TX}(A,B). \tag{4.47}$$

If $U, V \in TS$,

$$\overline{R}^{TX}\left(U^H, V^H\right) = \nabla^{TX}_{\cdot} T^H(U,V). \tag{4.48}$$

If $U \in TS, A \in TX$,

$$\overline{R}^{TX}\left(U^H, A\right) = L_{U^H}\left(\nabla^{TX}\right)_A. \tag{4.49}$$

In particular, if $A, B \in TX, U \in TS$,

$$\overline{R}^{TX}\left(U^H, A\right) B = \overline{R}^{TX}\left(U^H, B\right) A. \tag{4.50}$$

Proof. The identity (4.47) is trivial. Equation (4.48) is an easy consequence of (4.12) and of the fact that ∇^{TX} is fibrewise torsion free. Identity (4.49) is trivial. Since the connection $\overline{\nabla}^{TX}$ is fibrewise torsion free, (4.50) follows from (4.49). □

Remark 4.11. Needless to say, from (4.46), we deduce nontrivial identities relating R^{TX} to $\overline{R}^{TX}$.

Let $\overline{\nabla}^{T^*X}$ be the connection on T^*X induced by $\overline{\nabla}^{TX}$. Its curvature $\overline{R}^{T^*X}$ is the negative of the transpose of $\overline{R}^{TX}$.

Recall that by (2.39),

$$\Lambda^{\cdot}(T^*T^*X) \simeq \Lambda^{\cdot}(T^*X) \widehat{\otimes} \Lambda^{\cdot}(TX). \tag{4.51}$$

Let $\overline{\nabla}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F}$ be the connection on $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F$ induced by $\overline{\nabla}^{TX}$ and ∇^F.

Proposition 4.12. *If $U \in TS$, then*

$$\nabla^{\Omega^{\cdot}(T^*X,\pi^*F)}_U = \overline{\nabla}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F}_{U^H} + \left\langle \overline{R}^{T^*X}\left(U^H, e_i\right) p, e_j \right\rangle e^i i_{\widehat{e}^j}. \tag{4.52}$$

Proof. Let $\overline{\nabla}^{TT^*X}$ be the connection induced by $\overline{\nabla}^{TX}$ on $TT^*X \simeq TX \oplus T^*X$. Let A be a smooth section of TT^*X on $\mathcal{M}$. Then $\overline{R}^{T^*X}\left(U^H, \pi_* A\right) p$ is a section of T^*X and can be considered as a section of TT^*X. One verifies easily that

$$\left[U^H, A\right] = \overline{\nabla}^{TT^*X}_{U^H} A - \overline{R}^{T^*X}\left(U^H, \pi_* A\right) p. \tag{4.53}$$

By duality, we get (4.52). □

Remark 4.13. Clearly,

$$\overline{\nabla}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)} \omega^V = 0. \tag{4.54}$$

By (2.45), (4.15), (4.52) and (4.54), we should get

$$\left\langle \overline{R}^{T^*X}\left(U^H, e_i\right) p, e_j \right\rangle e^i e^j = 0. \tag{4.55}$$

In fact (4.55) is a consequence of (4.50).

We will give a proper interpretation of the first equation in (4.28), which is a formula for $A'^{\mathcal{M}}$. Recall that $A'^{\mathcal{M}}$ is just the de Rham operator $d^{\mathcal{M}}$ acting on the smooth sections of $\Lambda^{\cdot}(T^*\mathcal{M}) \otimes F$, written in an adequate trivialization. On the other hand, $\mathcal{M}$ is the total space of T^*X, which is a vector bundle on M. Finally T^*X is equipped with a connection $\overline{\nabla}^{T^*X}$. Let $h_1, \ldots, h_r$ be a basis of TM, and

let $h^1, \ldots, h^r$ be the corresponding dual basis of T^*M. Now, on $\mathcal{M}$, we have an equation which is just the obvious extension of (2.43),

$$d^{\mathcal{M}} = \widehat{e}_i \nabla_{\widehat{e}^i} + h^k \overline{\nabla}_{h_k}^{\Lambda^{\cdot}(TX)\widehat{\otimes}F} + \left\langle \overline{R}^{T^*X} p, e_i \right\rangle i_{\widehat{e}^i}. \tag{4.56}$$

Using Propositions 4.10 and 4.12, one verifies easily that the first equation in (4.28) and (4.56) are equivalent.

4.8. **Some curvature identities.** We will now denote by $\widehat{d}^X$ the fibrewise de Rham operator acting on smooth sections of $\widehat{\Lambda}^{\cdot}(T^*X)$ along the fibres X. We denote by $\widehat{\nabla}^{TX}$ the restriction of the connection ∇^{TX} to the fibres X, which acts on sections of $\widehat{\Lambda}^{\cdot}(T^*X) \otimes TX$. Observe that $\widehat{\nabla}^{TX}$ increases the degree in the exterior algebra $\widehat{\Lambda}^{\cdot}(T^*X)$ by 1. Its curvature, $\widehat{R}^{TX}$, is a smooth section of $\widehat{\Lambda}^2(T^*X) \otimes \operatorname{End}(TX)$. Also $\widehat{\nabla}^{TX}$ acts on smooth sections of $\widehat{\Lambda}^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*S)$ along the fibres.

Let $e_1, \ldots, e_n$ be an orthonormal basis of TX and let $e^1, \ldots, e^n$ be the corresponding dual basis of T^*X. Let $f_1, \ldots, f_m$ be a basis of TS and let $f^1, \ldots, f^m$ be the corresponding dual basis of T^*S. Then $\widehat{e}^1, \ldots, \widehat{e}^n$ is an orthonormal basis of T^*X, and $\widehat{e}_1, \ldots, \widehat{e}_n$ is the corresponding dual basis of TX. Using (2.38), as in Section 2, we can regard $e^1, \ldots, e^n, \widehat{e}_1, \ldots, \widehat{e}_n$ as a basis of $T^*T^*X = T^*X \oplus TX$. By (4.5) with $M = \mathcal{M}$, we can also regard these forms as being forms on $\mathcal{M}$ which vanish on $T^H\mathcal{M}$. If they are interpreted this way, they depend also on our choice of T^HM.

Definition 4.14. Set

$$\begin{aligned} \widehat{T}^H &= \frac{1}{2}\left\langle T\left(f_\alpha^H, f_\beta^H\right), e_i\right\rangle \widehat{e}^i \wedge f^\alpha \wedge f^\beta, \\ T^0 &= f^\alpha \wedge \widehat{e}^i \wedge T\left(f_\alpha^H, e_i\right). \end{aligned} \tag{4.57}$$

Note that T^0 is given by

$$T^0 = \left\langle T\left(f_\alpha^H, e_i\right), e^j\right\rangle f^\alpha \widehat{e}^i e_j. \tag{4.58}$$

Then $\widehat{T}^H$ and $\widehat{d}^X\widehat{T}^H$ are sections of $\Lambda^2(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)$. Also T^0 is a section of $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)\widehat{\otimes}TX$. Recall that we identify TX and T^*X by the metric g^{TX}. Then T^0 can be viewed as the smooth section of $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)\widehat{\otimes}T^*X$,

$$T^0 = \left\langle T\left(f_\alpha^H, e_i\right), e_j\right\rangle f^\alpha \wedge \widehat{e}^i \wedge e^j. \tag{4.59}$$

By the above, $\widehat{\nabla}_{\cdot}^{TX}T^0$ is well defined. Also the operator i_{T^0} acts on

$$\Lambda^{\cdot}(T^*S)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)$$

by interior multiplication in the variable e_j acting on $\Lambda^{\cdot}(T^*X)$, and exterior product by f^α acting on $\Lambda^{\cdot}(T^*S)$, and by exterior product by $\widehat{e}^i$ acting on $\widehat{\Lambda}^{\cdot}(T^*X)$. In particular, i_{T^0} increases the degree in $\widehat{\Lambda}^{\cdot}(T^*X)$ by 1. Set

$$\left|T^0\right|^2 = \left\langle T^0, T^0\right\rangle. \tag{4.60}$$

Equivalently,

$$\left|T^0\right|^2 = \sum_{j=1}^n \left(\sum_{\substack{1\le i\le n \\ 1\le \alpha\le m}} \left\langle T\left(f_\alpha^H, e_i\right), e_j\right\rangle f^\alpha \wedge \widehat{e}^i \right)^2, \tag{4.61}$$

where the square in the right-hand side of (4.61) is taken in $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)$. Then $\left|T^0\right|^2$ is a section of $\Lambda^2(T^*S)\widehat{\otimes}\widehat{\Lambda}^2(T^*X)$.

Theorem 4.15. *The following identity holds:*

$$\frac{1}{2}\left\langle e_i, R^{TX}e_j\right\rangle \widehat{e}^i\wedge\widehat{e}^j = \frac{1}{2}\left\langle e_i, \widehat{R}^{TX}e_j\right\rangle e^i\wedge e^j + \widehat{\nabla}^{TX}_{\cdot}T^0 + \frac{1}{2}\left|T^0\right|^2 - \frac{1}{2}\widehat{d}^X\widehat{T}^H. \tag{4.62}$$

In particular, for any $\lambda\in\mathbf{R}$, the expression in (4.62) *is invariant when replacing e^i by $e^i+\lambda\widehat{e}^i$ for $1\le i\le n$.*

Proof. Equation (4.62) was established in [BG04, Theorem 3.26]. Observe that when making the indicated replacements, in the right-hand side of (4.62), the first term is unchanged because of the circular identity for $\widehat{R}^{TX}$. By (4.38), T^0 is also unchanged. This completes the proof of our theorem. □

Assume that K is a smooth vector field along X, which is a fibrewise Killing vector field, which also preserves T^HM. Let $K'\in T^*X$ correspond to K via the metric g^{TX}. We identify K' with the corresponding vertical 1-form on M. Set

$$\widehat{K'} = \langle K, e_i\rangle\,\widehat{e}^i. \tag{4.63}$$

By definition,

$$\nabla^{\Lambda^{\cdot}(T^*X)}\widehat{K'} = \left\langle \nabla^{TX}_{e_i}K, e_j\right\rangle e^i\widehat{e}^j + \left\langle \nabla^{TX}_{f^H_\alpha}K, e_i\right\rangle f^\alpha\widehat{e}^i. \tag{4.64}$$

Now we state a simple result which complements Theorem 4.15.

Proposition 4.16. *If for $1\le i\le n$, we replace e^i by $e^i+\frac{1}{2}\widehat{e}^i$, then*

$$dK' - \nabla^{\Lambda^{\cdot}(TX)}\widehat{K'}$$

becomes

$$dK' + \frac{1}{4}\left\langle e_i, \nabla^{TX}_{e_j}K\right\rangle \widehat{e}^i\widehat{e}^j.$$

Proof. Since T is the torsion of the connection ∇^{TM} in (4.34), we get

$$dK' = \nabla^{TX}K' + f^\alpha e^i\left\langle T\left(f^H_\alpha, e_i\right), K\right\rangle + \left\langle K, T^H\right\rangle. \tag{4.65}$$

Moreover since T^HM is K-invariant, if $U\in TS$, then $\left[K, U^H\right]=0$, and so

$$\nabla^{TX}_{U^H}K = T\left(U^H, K\right). \tag{4.66}$$

So by (4.65), (4.66), we get

$$dK' = \left\langle \nabla^{TX}_{e_i}K, e_j\right\rangle e^ie^j + 2\left\langle \nabla^{TX}_{f_\alpha}K, e_i\right\rangle f^\alpha e^i + \left\langle K, T^H\right\rangle. \tag{4.67}$$

Finally, since K is fibrewise Killing, the tensor $\nabla^{TX}K$ is fibrewise antisymmetric. Our proposition follows from (4.64) and (4.67). □

4.9. **A bilinear form on T^*X and the adjoint of $A'^{\mathcal{M}}$.** Now, we use the same notation as in subsection 2.6. In particular, we still define ϕ as in (2.57). Let η be the corresponding bilinear form as in (2.23). Instead of (2.63), if $U,V\in TT^*X$,

$$\eta(U,V) = \langle \pi_*U, \pi_*V\rangle_{g^{TX}} + \omega^V(U,V). \tag{4.68}$$

Definition 4.17. Let $\overline{A}'^{\mathcal{M}}_\phi$ be the adjoint of $A'^{\mathcal{M}}$ with respect to η, g^F and dv_{T^*X}.

Then $\overline{A}_{\phi}'^{\mathcal{M}}$ is a flat superconnection of total degree -1. Let $e_1, \ldots, e_n$ be a basis of TX, and let $e^1, \ldots, e^n$ be the corresponding dual basis of T^*X. As in (2.65), (2.66), we define λ_0 by the formula

$$\lambda_0 = \left\langle g^{TX} e_i, e_j \right\rangle e^i \wedge i_{\widehat{e}^j}. \tag{4.69}$$

In the sequel, we will assume that the basis $e_1, \ldots, e_n$ is orthonormal, so that

$$\lambda_0 = e^i \wedge i_{\widehat{e}^i}. \tag{4.70}$$

Put

$$E = \left\langle L_{f_\alpha^H} g^{TX} e_i, e_j \right\rangle f^\alpha \wedge e^i \wedge i_{\widehat{e}^j}. \tag{4.71}$$

By (4.41),

$$E = 2 \left\langle T\left(f_\alpha^H, e_i\right), e_j \right\rangle f^\alpha \wedge e^i \wedge i_{\widehat{e}^j}. \tag{4.72}$$

We have the following extension of Proposition 2.10.

Proposition 4.18. *The following identity holds:*

$$\overline{A}_{\phi}'^{\mathcal{M}} = e^{\lambda_0} \overline{A}'^{\mathcal{M}} e^{-\lambda_0}. \tag{4.73}$$

Equivalently,

$$\overline{A}_{\phi}'^{\mathcal{M}} = \overline{A}'^{\mathcal{M}} - \left[\overline{A}'^{\mathcal{M}}, \lambda_0\right]. \tag{4.74}$$

More precisely,

$$\overline{A}_{\phi}'^{\mathcal{M}} = \overline{d}_{\phi}^{T^*X} + \nabla^{\Omega^\cdot\left(T^*X, \pi^*\overline{F}^*|_{T^*X}\right)} - E - \left(d^{T^*X}\omega^H + \left\langle T^H, e^i\right\rangle e^i\right). \tag{4.75}$$

Proof. By (2.59) and (2.60), (4.73) follows. Clearly,

$$\overline{\nabla}^{\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(TX)} \lambda_0 = E. \tag{4.76}$$

By (4.27), (4.52) and (4.76), we obtain,

$$\left[\nabla^{\Omega^\cdot\left(T^*X, \pi^*\overline{F}^*|_{T^*X}\right)}, \lambda_0\right] = E. \tag{4.77}$$

The remainder of the proposition is now a consequence of Proposition 4.6, Theorem 4.7, (4.74) and (4.77). □

4.10. **A fundamental symmetry.** We define the Hermitian form $\mathfrak{h}^{\Omega^\cdot(T^*X, \pi^*F|_{T^*X})}$ on $\Omega^\cdot(T^*X, \pi^*F|_{T^*X})$ as in (2.79).

Theorem 4.19. *The superconnection $\overline{A}_{\phi}'^{\mathcal{M}}$ is the $\mathfrak{h}^{\Omega^\cdot(T^*X, \pi^*F|_{T^*X})}$ adjoint of $A'^{\mathcal{M}}$.*

Proof. The proof is essentially the same as the proof of Theorem 2.13 where instead of (2.81), we use the obvious

$$r^* A'^{\mathcal{M}} r^{*-1} = A'^{\mathcal{M}}. \tag{4.78}$$

□

4.11. **A Hamiltonian function.** Let $\mathcal{H} : \mathcal{M} \to \mathbf{R}$ be a smooth function. We define the fibrewise Hamiltonian vector field $Y^{\mathcal{H}} \in TT^*X$ as in (2.33). The vector field $Y^{\mathcal{H}}$ is still given by (2.82).

One should observe here that although the fibrewise symplectic form ω^V is preserved by $Y^{\mathcal{H}}$, this is in general *not* the case for ω.

Definition 4.20. Set

$$\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}} = e^{-\mathcal{H}} A'^{\mathcal{M}} e^{\mathcal{H}}, \qquad \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}} = e^{\mathcal{H}} \overline{A}'^{\mathcal{M}}_{\phi} e^{-\mathcal{H}}. \tag{4.79}$$

Then $\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}}, \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}}$ are superconnections, and they are flat, i.e.

$$\mathfrak{C}'^{\mathcal{M},2}_{\mathcal{H}} = 0, \qquad \overline{\mathfrak{C}}'^{\mathcal{M},2}_{\phi,\mathcal{H}} = 0. \tag{4.80}$$

We still define $d^{T^*X}_{\mathcal{H}}, \overline{d}^{T^*X}_{\phi,\mathcal{H}}$ as in Definition 2.15. We can extend the previous definitions to the case where $\mathcal{H}$ is replaced by $\mathcal{H} - \omega^H$. The above properties still hold.

Proposition 4.21. *The following identities hold:*

$$\begin{aligned} \mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H} &= A'^{\mathcal{M}} + d^{\mathcal{M}}\mathcal{H} - d^{T^*X}\omega^H, \\ \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \overline{A}'^{\mathcal{M}} - i_{Y^{\mathcal{H}} - \mathcal{T}^H} - f^\alpha \nabla_{f^H_\alpha}\mathcal{H} \\ &\quad - \left[\overline{A}'^{\mathcal{M}} - i_{Y^{\mathcal{H}} - \mathcal{T}^H} - f^\alpha \nabla_{f^H_\alpha}\mathcal{H}, \lambda_0\right]. \end{aligned} \tag{4.81}$$

Moreover,

$$\begin{aligned} \mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H} &= d^{T^*X}_{\mathcal{H}} + \nabla^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})} + f^\alpha \nabla_{f^H_\alpha}\mathcal{H} + i_{\mathcal{T}^H} - d^{T^*X}\omega^H, \\ \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \overline{d}^{T^*X}_{\phi,\mathcal{H}} + \nabla^{\Omega^\cdot\left(T^*X,\pi^*\overline{F}^*|_{T^*X}\right)} - E - f^\alpha \nabla_{f^H_\alpha}\mathcal{H} + i_{\mathcal{T}^H} \\ &\quad - d^{T^*X}\omega^H - \left\langle T^H, e^i\right\rangle \left(e^i + i_{\widehat{e}^i}\right), \\ \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,2(\mathcal{H}-\omega^H)} &= \overline{d}^{T^*X}_{\phi,2\mathcal{H}} + \nabla^{\Omega^\cdot\left(T^*X,\pi^*\overline{F}^*|_{T^*X}\right)} - E - 2f^\alpha \nabla_{f^H_\alpha}\mathcal{H} + 2i_{\mathcal{T}^H} \\ &\quad - d^{T^*X}\omega^H - \left\langle T^H, e^i\right\rangle \left(e^i + i_{2\widehat{e}^i}\right). \end{aligned} \tag{4.82}$$

Proof. We use Propositions 2.3, 4.2, 4.6, Theorem 4.7 and Proposition 4.18 and we get (4.81). By Proposition 2.16, Theorem 4.7 and by (4.81), we get (4.82). □

Definition 4.22. Set

$$\begin{aligned} A^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \frac{1}{2}\left(\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,2(\mathcal{H}-\omega^H)} + A'^{\mathcal{M}}\right), & B^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \frac{1}{2}\left(\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,2(\mathcal{H}-\omega^H)} - A'^{\mathcal{M}}\right), \\ \mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \frac{1}{2}\left(\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} + \mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H}\right), & \mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \frac{1}{2}\left(\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} - \mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H}\right). \end{aligned} \tag{4.83}$$

Then $A^{\mathcal{M}}_{\phi,\mathcal{H}}, \mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}$ are superconnections, and $B^{\mathcal{M}}_{\phi,\mathcal{H}}, \mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}$ are odd sections of $\Lambda^\cdot(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\cdot\left(T^*X, \pi^*F|_{T^*X}\right)\right)$. Clearly,

$$\begin{aligned} \mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= e^{-\left(\mathcal{H}-\omega^H\right)} A^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} e^{\left(\mathcal{H}-\omega^H\right)}, \\ \mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= e^{-\left(\mathcal{H}-\omega^H\right)} B^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} e^{\left(\mathcal{H}-\omega^H\right)}. \end{aligned} \tag{4.84}$$

In the sequel we will use the couples $A^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}, B^{\mathcal{M}}_{\mathcal{H}-\omega^H}$ or $\mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}, \mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}$ indifferently. We will establish algebraic properties for one of the two couples and still use them freely for the other one. As in (2.91), we deduce from (4.80) that

(4.85)
$$A^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H} = -B^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H} = \frac{1}{4}\left[A'^{\mathcal{M}}, \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,2(\mathcal{H}-\omega^H)}\right], \qquad \left[A^{\mathcal{M}}_{\phi,\mathcal{H}}, B^{\mathcal{M}}_{\phi,\mathcal{H}}\right] = 0,$$
$$\left[A'^{\mathcal{M}}, A^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H}\right] = 0, \qquad \left[\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,2(\mathcal{H}-\omega^H)}, A^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H}\right] = 0.$$

Theorem 4.23. *The following identities hold:*

(4.86)
$$\begin{aligned}
A^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= A_{\phi,\mathcal{H}} + \nabla^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})} - \frac{1}{2}E + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) - \nabla_{f^H_\alpha}\mathcal{H}\right)\\
&\quad + \frac{3}{2}i_{\mathcal{T}^H} - \frac{1}{2}d^{T^*X}\omega^H - \frac{1}{2}\left\langle T^H, e^i\right\rangle\left(e^i + 2i_{\widehat{e}^i}\right),\\
B^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= B_{\phi,\mathcal{H}} - \frac{1}{2}E + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) - \nabla_{f^H_\alpha}\mathcal{H}\right) + \frac{1}{2}i_{\mathcal{T}^H}\\
&\quad - \frac{1}{2}d^{T^*X}\omega^H - \frac{1}{2}\left\langle T^H, e^i\right\rangle\left(e^i + 2i_{\widehat{e}^i}\right),\\
\mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \mathfrak{A}_{\phi,\mathcal{H}} + \nabla^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})} - \frac{1}{2}E + \frac{1}{2}f^\alpha\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) + i_{\mathcal{T}^H}\\
&\quad - d^{T^*X}\omega^H - \frac{1}{2}\left\langle T^H, e^i\right\rangle\left(e^i + i_{\widehat{e}^i}\right),\\
\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= \mathfrak{B}_{\phi,\mathcal{H}} - \frac{1}{2}E + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) - \nabla_{f^H_\alpha}\mathcal{H}\right)\\
&\quad - \frac{1}{2}\left\langle T^H, e^i\right\rangle\left(e^i + i_{\widehat{e}^i}\right).
\end{aligned}$$

Proof. Our theorem follows from (4.27), from Theorem 4.7 and from Proposition 4.21. □

4.12. **The symmetry in the case where $\mathcal{H}$ is r-invariant.** We modify the definition of the sesquilinear form $\mathfrak{h}^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}_{\mathcal{H}}$ in equation (2.98). If $s, s' \in \Omega^\cdot\left(T^*X, \pi^*F|_{T^*X}\right)$, put

$$\left\langle s, s'\right\rangle_{\mathfrak{h}^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}_{\mathcal{H}-\omega^H}} = \left\langle ue^{-2\left(\mathcal{H}-\omega^H\right)}s, s'\right\rangle_{\mathfrak{g}^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}}. \tag{4.87}$$

Observe that this sesquilinear form takes its values in $\Lambda^{\text{even}}\left(T^*S\right)\otimes_{\mathbf{R}}\mathbf{C}$. Still we can define the adjoint of a superconnection with respect to $\mathfrak{h}^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}_{\mathcal{H}-\omega^H}$.

Proposition 4.24. *The superconnection $\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,2(\mathcal{H}-\omega^H)}$ is the $\mathfrak{h}^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}_{\mathcal{H}-\omega^H}$ adjoint of $A'^{\mathcal{M}}$, and the superconnection $\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}_--\omega^H}$ is the $\mathfrak{h}^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}$ adjoint of $\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H}$.*

Proof. The proof of the first part of the proposition follows from Theorem 4.19. To establish the second part, we observe that ω^H is antiinvariant under r. However since ω^H is a horizontal 2-form, ω^H is indeed self-adjoint. □

Theorem 4.25. *If $\mathcal{H}$ is r-invariant, then $B^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}$ is $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}_{\mathcal{H}-\omega^H}$ skew-adjoint and $\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}$ is $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}$ skew-adjoint.*

Proof. This is an obvious consequence of Proposition 4.24. □

4.13. **Poincaré duality.** Here we adapt considerations in [BLo95]. We use the notation of subsection 2.10. In particular the operator κ^F acting on $\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})$ is defined as in (2.99). Also we use temporarily the superscript F to denote the flat vector bundle explicitly. Set

$$\kappa^F_{\mathcal{H}-\omega^H} = \kappa^F e^{-2(\mathcal{H}-\omega^H)}. \tag{4.88}$$

Using (2.100) and the fact that under r, $\mathcal{H},\omega^H$ become $\mathcal{H}_-,-\omega^H$, we get

$$\kappa^{F,-1}_{\mathcal{H}-\omega^H} = \kappa^{\overline{F}^*}_{-\mathcal{H}_--\omega^H}. \tag{4.89}$$

As a consequence of Proposition 4.24, we obtain

$$\overline{\mathfrak{C}}'^{\mathcal{M},F}_{\phi,2(\mathcal{H}-\omega^H)} = \kappa^{F,-1}_{\mathcal{H}-\omega^{\mathcal{H}}} A'^{\mathcal{M},\overline{F}^*}\kappa^F_{\mathcal{H}-\omega^H}, \quad \overline{\mathfrak{C}}'^{\mathcal{M},F}_{\phi,\mathcal{H}-\omega^H} = \kappa^{F,-1}\mathfrak{C}'^{\mathcal{M},\overline{F}^*}_{-\mathcal{H}_--\omega^H}\kappa^F. \tag{4.90}$$

By (4.90), we find that if $\mathcal{H}$ is r-invariant,

(4.91)

$$\begin{aligned} &A^{\mathcal{M},F}_{\phi,\mathcal{H}-\omega^H} = \kappa^{F,-1}_{\mathcal{H}-\omega^H} A^{\mathcal{M},\overline{F}^*}_{\phi,-\mathcal{H}-\omega^H}\kappa^F_{\mathcal{H}-\omega^H}, \quad B^{\mathcal{M},F}_{\phi,\mathcal{H}-\omega^H} = -\kappa^{F,-1}_{\mathcal{H}-\omega^H} B^{\mathcal{M},\overline{F}^*}_{\phi,-\mathcal{H}-\omega^H}\kappa^F_{\mathcal{H}-\omega^H},\\ &\mathfrak{C}^{\mathcal{M},F}_{\phi,\mathcal{H}-\omega^H} = \kappa^{F,-1}\mathfrak{C}^{\mathcal{M},\overline{F}^*}_{\phi,-\mathcal{H}-\omega^H}\kappa^F, \qquad \mathfrak{D}^{\mathcal{M},F}_{\phi,\mathcal{H}-\omega^H} = -\kappa^{F,-1}\mathfrak{D}^{\mathcal{M},\overline{F}^*}_{\phi,-\mathcal{H}-\omega^H}\kappa^F. \end{aligned}$$

As in subsection 2.10, note that because of (4.89), (4.91) is symmetric when exchanging F and $\overline{F}^*$.

4.14. **A conjugation of the superconnections.** Recall that λ_0 was defined in (2.65), (2.66) and (4.70). As in (2.110), set

$$\mu_0 = \widehat{e}_i \wedge i_{e_i}. \tag{4.92}$$

More generally, we still use the same notation as in subsection 2.11. Let $\mathfrak{C}'^{\mathcal{M}*}_{\mathcal{H}-\omega^H}$ be the superconnection adjoint to $\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H}$ with respect to $g^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}$.

Definition 4.26. Put

$$\mathfrak{E}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^{\mathcal{H}}} = \rho\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H}\rho^{-1}, \qquad \overline{\mathfrak{E}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = \sigma\mathfrak{C}'^{\mathcal{M}*}_{\mathcal{H}-\omega^H}\sigma^{-1}. \tag{4.93}$$

By (2.109), we get

$$\overline{\mathfrak{E}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = \rho\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}\rho^{-1}. \tag{4.94}$$

Theorem 4.27. *The following identities hold:*

$$\mathfrak{E}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^{\mathcal{H}}} = e^{-\mu_0}\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}-\omega^H}e^{\mu_0}, \qquad \overline{\mathfrak{E}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = e^{-\lambda_0}\mathfrak{C}'^{\mathcal{M}*}_{\mathcal{H}-\omega^H}e^{\lambda_0}. \tag{4.95}$$

Proof. This follows from (2.111) and (4.93). □

Definition 4.28. Set

(4.96)

$$\mathfrak{E}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = \frac{1}{2}\left(\overline{\mathfrak{E}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} + \mathfrak{E}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^{\mathcal{H}}}\right), \quad \mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = \frac{1}{2}\left(\overline{\mathfrak{E}}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} - \mathfrak{E}'^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^{\mathcal{H}}}\right).$$

By (4.93), (4.94), we get

$$\mathfrak{E}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = \rho\mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}\rho^{-1}, \qquad \mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = \rho\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}\rho^{-1}. \tag{4.97}$$

Theorem 4.29. *The following identity holds:*

$$\mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = \mathfrak{B}'_{\phi,\mathcal{H}} - \left\langle T\left(f^H_\alpha, e_i\right), e_j\right\rangle f^\alpha\left(e^i - \widehat{e}_i\right) i_{e_j+\widehat{e}^j} + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) - \nabla_{f^H_\alpha}\mathcal{H}\right) - \frac{1}{2}\left\langle T^H, e_i\right\rangle\left(e^i - \widehat{e}_i + i_{e_i+\widehat{e}^i}\right). \tag{4.98}$$

Moreover, if $\mathcal{H}$ is r-invariant, $\mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}$ is $h^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}$ skew-adjoint.

Proof. By (2.111) and (4.97),

$$\mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} = e^{-\mu_0}\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H}e^{\mu_0}. \tag{4.99}$$

As we saw following (2.111),

$$e^{-\mu_0}e^i e^{-\mu_0} = e^i - \widehat{e}_i, \qquad e^{-\mu_0}i_{\widehat{e}^i}e^{\mu_0} = i_{\widehat{e}^i+e_i}. \tag{4.100}$$

By (2.116), (4.72), (4.86) and (4.100), we get (4.98). The remainder of our theorem follows from (2.120) and Theorem 4.25. □

Remark 4.30. Needless to say, the second part of our theorem also follows from the explicit formula (4.98).

We denote by $\nabla^{TT^*X} = \pi^*\left(\nabla^{TX}\oplus\nabla^{T^*X}\right)$ the obvious connection on $TT^*X = \pi^*\left(TX\oplus T^*X\right)$. Let $\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F}, \nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}$ the connections induced by ∇^{TT^*X}, ∇^F or $\nabla^{TT^*X}, \nabla^{F,u}$ on $\Lambda^\cdot\left(T^*T^*X\right)\widehat{\otimes}F$.

In the sequel $i_{\widehat{R^{TX}p}}, R^{TX}p\wedge$ are still defined as in (2.42) and in (2.55). In particular they do not contain the variables f^α. More precisely

$$i_{\widehat{R^{TX}p}} = \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)p, e_k\right\rangle e^i e^j i_{\widehat{e}^k}, \quad R^{TX}p\wedge = \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)p, e_k\right\rangle i_{\widehat{e}^i}i_{\widehat{e}^j}e^k. \tag{4.101}$$

Proposition 4.31. *The following identities hold:*

$$\begin{aligned}
\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= -\frac{1}{2}\left(\widehat{e}_i + i_{\widehat{e}^i-e_i}\right)\nabla_{\widehat{e}^i} - \frac{1}{2}\left(\widehat{e}_i + i_{e_i-\widehat{e}^i}\right)\nabla_{\widehat{e}^i}\mathcal{H} \\
&- \frac{1}{2}\left(e^i + i_{\widehat{e}^i}\right)\left(\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} + \left\langle T\left(f^H_\alpha, e_i\right), e_j\right\rangle f^\alpha\left(e^j - i_{\widehat{e}^j}\right)\right. \\
&\left. + \left\langle T^H, e^i\right\rangle\right) + \frac{1}{2}\left(R^{TX}p\wedge - i_{\widehat{R^{TX}p}}\right) \\
&+ \frac{1}{2}\left(e^i - i_{\widehat{e}^i}\right)\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(e_i\right) - \nabla_{e_i}\mathcal{H}\right) + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) - \nabla_{f^H_\alpha}\mathcal{H}\right), \\
\mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} &= -\frac{1}{2}\widehat{c}\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i} - \frac{1}{2}c\left(\widehat{\nabla^V}\mathcal{H}\right) - \frac{1}{2}\left(\widehat{c}\left(e_i\right) - c\left(\widehat{e}^i\right)\right) \\
&\left(\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{e_i} + \left\langle T\left(f^H_\alpha, e_i\right), e_j\right\rangle\left(c\left(e_j\right) - \widehat{c}\left(\widehat{e}^j\right)\right) + \left\langle T^H, e_i\right\rangle\right) \\
&- \frac{1}{4}\left(\left(e^i - \widehat{e}_i\right)\left(e^j - \widehat{e}_j\right)i_{\widehat{e}^k+e_k} + i_{e_i+\widehat{e}^i}i_{e_j+\widehat{e}^j}\left(\widehat{e}_k - e^k\right)\right)\left\langle R^{TX}\left(e_i,e_j\right)p, e_k\right\rangle \\
&+ \frac{1}{2}\left(c\left(e_i\right) - \widehat{c}\left(\widehat{e}^i\right)\right)\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(e_i\right) - \nabla_{e_i}\mathcal{H}\right) \\
&\qquad + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) - \nabla_{f^H_\alpha}\mathcal{H}\right).
\end{aligned} \tag{4.102}$$

Proof. We use Propositions 2.18 and 2.28, (4.38), (4.72), Theorems 4.23 and 4.29. □

Remark 4.32. Formula (4.102) is formally closely related to corresponding formulas obtained in [BLo95, Proposition 3.9] and in [BG01, Theorem 3.14] for corresponding objects associated to the Levi-Civita superconnection of $p : M \to S$.

4.15. **The scaling of the variable p.** Instead of ϕ, for $b \in \mathbf{R}_*$, we can consider ϕ_b defined in (2.123) and the associated bilinear form η_b in (2.124). Since the vertical symplectic form ω^V is scaled by the factor b, it should be clear that in this case, ω^H should be replaced by $b\omega^H$ and λ_0 should be replaced by λ_0/b. We denote the corresponding objects with the subscript $\phi_b, \mathcal{H} - b\omega^H$.

Now we extend Proposition 2.32.

Proposition 4.33. *The following identities hold:*

(4.103)

$$\begin{aligned}
&r_b^* A'^{\mathcal{M}} r_b^{*-1} = A'^{\mathcal{M}}, && r_b^* \overline{A}'^{\mathcal{M}} r_b^{*-1} = b^{N^{T^*X}} \overline{A}'^{\mathcal{M}} b^{-N^{T^*X}},\\
&r_b^* \overline{A}'^{\mathcal{M}}_{\phi} r_b^{*-1} = \overline{A}'^{\mathcal{M}}_{\phi_b}, && \overline{A}'^{\mathcal{M}}_{\phi_b} = e^{\lambda_0/b} b^{N^{T^*X}} \overline{A}'^{\mathcal{M}} b^{-N^{T^*X}} e^{-\lambda_0/b},\\
&r_b^* \mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}_{1/b}} r_b^{*-1} = \mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}}, && r_b^* \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,\mathcal{H}_{1/b}-\omega^H} r_b^{*-1} = \overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi_b,\mathcal{H}-b\omega^H},\\
&r_b^* A^{\mathcal{M}}_{\phi,\mathcal{H}_{1/b}-\omega^H} r_b^{*-1} = A^{\mathcal{M}}_{\phi_b,\mathcal{H}-b\omega^H}, && r_b^* B^{\mathcal{M}}_{\phi,\mathcal{H}_{1/b}-\omega^H} r_b^{*-1} = B^{\mathcal{M}}_{\phi_b,\mathcal{H}-b\omega^H},\\
&r_b^* \mathfrak{C}^{\mathcal{M}}_{\phi,\mathcal{H}_{1/b}-\omega^H} r_b^{*-1} = \mathfrak{C}^{\mathcal{M}}_{\phi_b,\mathcal{H}-b\omega^H}, && r_b^* \mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}_{1/b}-\omega^H} r_b^{*-1} = \mathfrak{D}^{\mathcal{M}}_{\phi_b,\mathcal{H}-b\omega^H},\\
&K_b \mathfrak{E}^{\mathcal{M}}_{\phi,\mathcal{H}_{1/b}-\omega^H} K_b^{-1} = \mathfrak{E}^{\mathcal{M}}_{\phi_b,\mathcal{H}-b\omega^H}, && K_b \mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}_{1/b}-\omega^H} K_b^{-1} = \mathfrak{F}^{\mathcal{M}}_{\phi_b,\mathcal{H}-b\omega^H}.
\end{aligned}$$

Proof. If the base S is a point, our proposition is just Proposition 2.32. Observe that $\nabla^{\Omega^\cdot(T^*X,\pi^*F|_{T^*X})}, \nabla^{\Omega^\cdot\left(X,\pi^*\overline{F}^*|_{T^*X}\right)}, \left\langle T^H, e_i\right\rangle e^i$ are unchanged when replacing ϕ by ϕ_b and that E scales by the factor β and $d^{T^*X}\omega^H$ by the factor b. We then get (4.103) easily. □

Now we extend Proposition 2.35 to the relative situation. The radial vector field on $\mathcal{M}$ is still denoted by $\widehat{p}$. Also we will write p instead of $\pi^* p$.

Proposition 4.34. *The following identities hold:*

(4.104)

$$\left[A'^M, i_{\widehat{p}}\right] = L_{\widehat{p}}, \qquad \left[\overline{\mathfrak{C}}'^{\mathcal{M}}_{\phi,2(\mathcal{H}-\omega^H)}, p\right] = L_{\widehat{p}} - 2\nabla_{\widehat{p}}\mathcal{H} + n - N^{T^*X} + 2\omega^H + \lambda_0.$$

Proof. Since $A'^{\mathcal{M}}$ is just the total de Rham operator on $\mathcal{M}$, the first identity in (4.104) is trivial. Now we establish the second identity by first replacing $\mathcal{H}$ and ω^H by 0. Then this second identity follows from Proposition 2.35 and from (4.75). Also note that by (4.24),

$$L_{\widehat{p}}\omega^H = \omega^H. \tag{4.105}$$

The general second identity in (4.104) follows by conjugation. □

4.16. **The classical Hamiltonians.** If $c \in \mathbf{R}$, as in (2.140), let $\mathcal{H}^c : \mathcal{M} \to \mathbf{R}$ be given by

$$\mathcal{H}^c = \frac{c}{2}\left|p\right|^2. \tag{4.106}$$

In the sequel, we assume that $c : S \to \mathbf{R}_*$ is a smooth function and that the restriction of $\mathcal{H}$ to the fibre T^*X_s is just $\mathcal{H}^{c_s}$. We will write $\mathcal{H}^c$ instead of $\mathcal{H}$.

Proposition 4.35. *The following identities hold:*

$$\begin{aligned}\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} &= \mathfrak{B}_{\phi,\mathcal{H}^c} - \frac{1}{2}E + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) + c\left\langle T\left(f^H_\alpha, p\right), p\right\rangle\right)\\ &\quad - \frac{1}{2}\left\langle T^H, e^i\right\rangle\left(e^i + i_{\widehat{e}^i}\right) - \frac{dc}{2}\left|p\right|^2,\\ \mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} &= \mathfrak{B}'_{\phi,\mathcal{H}^c} - \left\langle T\left(f^H_\alpha, e_i\right), e_j\right\rangle f^\alpha\left(e^i - \widehat{e}_i\right) i_{e_j+\widehat{e}^j}\\ &\quad + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) + c\left\langle T\left(f^H_\alpha, p\right), p\right\rangle\right)\\ &\quad - \frac{1}{2}\left\langle T^H, e_i\right\rangle\left(e^i - \widehat{e}_i + i_{e_i+\widehat{e}^i}\right) - \frac{dc}{2}\left|p\right|^2.\end{aligned} \tag{4.107}$$

Proof. Clearly,

$$\left|p\right|^2 = \left\langle\left(g^{TX}\right)^{-1}p, p\right\rangle, \tag{4.108}$$

so that by (4.41),

$$f^\alpha \wedge \nabla_{f^H_\alpha}\frac{\left|p\right|^2}{2} = -f^\alpha \wedge \left\langle T\left(f^H_\alpha, p\right), p\right\rangle. \tag{4.109}$$

Combining Theorems 4.23 and 4.29 with (4.109), we get (4.107). □

4.17. **The relation to the standard Levi-Civita superconnection.** In Proposition 2.39, we gave a formula for $K_{1/\sqrt{c}}\mathfrak{A}'_{\phi,\mathcal{H}^c}K_{\sqrt{c}}$. Of course a similar formula holds for $K_{1/\sqrt{c}}\mathfrak{B}'_{\phi,\mathcal{H}^c}K_{\sqrt{c}}$. From this formula and from (4.107), we find that for $c > 0$,

$$K_{1/\sqrt{c}}\mathfrak{F}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}K_{\sqrt{c}} = \frac{\sqrt{c}}{2}\left(-\widehat{c}\left(\widehat{e}^i\right)\nabla_{\widehat{e}^i} - c\left(p\right)\right) + \mathfrak{H}_+ + \frac{1}{\sqrt{c}}\mathfrak{I}_+ - \frac{dc}{2c}\left|p\right|^2. \tag{4.110}$$

Of course, all the terms in (4.110) can be evaluated explicitly. For $c < 0$, a similar identity holds, the index $+$ being replaced by the index $-$.

Let $B_\pm$ be the odd section of

$$\Lambda^{\cdot}\left(T^*S\right)\widehat{\otimes}\mathrm{End}\left(\Omega^{\cdot}\left(X, F|_X\right)\right)$$

or of

$$\Lambda^{\cdot}\left(T^*S\right)\widehat{\otimes}\mathrm{End}\left(\Omega^{\cdot}\left(X, F\otimes o\left(TX\right)\right)\right),$$

constructed by Bismut and Lott in [BLo95], which is associated to the Levi-Civita superconnection $A_\pm$ which is attached to $\left(T^HM, g^{TX}, g^F\right)$. If S is a point, then $A_\pm = \frac{1}{2}\left(d^{X*} + d^X\right), B_\pm = \frac{1}{2}\left(d^{X*} - d^X\right)$. In general,

$$A^2_\pm = -B^2_\pm. \tag{4.111}$$

By [BLo95, Proposition 3.9], we have the formula

$$\begin{aligned}B_\pm &= \frac{1}{2}\left(d^{X*} - d^X\right) - \left\langle T\left(f^H_\alpha, e_i\right), e_j\right\rangle f^\alpha e^i i_{e_j}\\ &\quad + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) + \frac{1}{2}\left\langle T\left(f^H_\alpha, e_i\right), e_i\right\rangle\right) - \frac{1}{2}\widehat{c}\left(T^H\right).\end{aligned} \tag{4.112}$$

Recall that the orthogonal projection operators $P_\pm$ were defined in subsection 2.14. Now we establish the obvious extension of Proposition 2.41.

Proposition 4.36. *The following identity holds:*

$$P_{\pm}\mathfrak{H}_{\pm}P_{\pm}=B_{\pm}. \tag{4.113}$$

Proof. In degree 0, this is just Proposition 2.41. Since the image of P_+ is concentrated in fibrewise degree 0 and the image of P_- in fibrewise degree n, we get

$$\begin{aligned} P_+\left\langle T\left(f_\alpha^H,e_i\right),e_j\right\rangle f^\alpha\left(e^i-\widehat{e}_i\right)i_{e_j+\widehat{e}^j}P_+ &= \left\langle T\left(f_\alpha^H,e_i\right),e_j\right\rangle f^\alpha e^i i_{e_j},\\ P_-\left\langle T\left(f_\alpha^H,e_i\right),e_j\right\rangle f^\alpha\left(e^i-\widehat{e}_i\right)i_{e_j+\widehat{e}^j}P_- &= \left\langle T\left(f_\alpha^H,e_i\right),e_j\right\rangle f^\alpha e^i i_{e_j}\\ &\quad - f^\alpha\left\langle T\left(f_\alpha^H,e_i\right),e_i\right\rangle f^\alpha. \end{aligned} \tag{4.114}$$

Moreover using (3.51), we have

$$P_{\pm}f^\alpha\left\langle T\left(f_\alpha^H,p\right),p\right\rangle P_{\pm}=\frac{1}{2}f^\alpha\left\langle T\left(f_\alpha^H,e_i\right),e_i\right\rangle. \tag{4.115}$$

Finally,

$$P_{\pm}\left\langle T^H,e_i\right\rangle\left(e^i-\widehat{e}_i+i_{e_i+\widehat{e}^i}\right)P_{\pm}=\widehat{c}\left(T^H\right). \tag{4.116}$$

By (4.107), (4.112), (4.114)–(4.116), we get (4.113). □

4.18. **A formula for $\mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H}$.** In the sequel, $L_{Y^{\mathcal{H}}}$ denotes the fibrewise Lie derivative operator attached to the fibrewise vector field. In particular, $L_{Y^{\mathcal{H}}}$ acts trivially on the f_α^H. The fibrewise Lie derivative operator $L_{Y^{\mathcal{H}}}$ is to be distinguished from the full Lie derivative operator $\mathcal{L}_{Y^{\mathcal{H}}}$, which, in general, acts nontrivially on the f_α^H.

Theorem 4.37. *The following identity holds:*

$$\begin{aligned} \mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H} &= \frac{1}{4}\Bigg(-\Delta^V-\frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^ie^ji_{\widehat{e}^k}i_{\widehat{e}^l}+\left|\nabla^V\mathcal{H}\right|^2\\ &\qquad -\Delta^V\mathcal{H}+2\left(\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H}\right)\widehat{e}_ii_{\widehat{e}^j}+2\left(\nabla_{\widehat{e}^i}\nabla_{e_j}\mathcal{H}\right)e^ji_{\widehat{e}^i}\Bigg)\\ &\quad -\frac{1}{2}\Bigg(L_{Y^{\mathcal{H}}}+\frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}}\right)+\frac{1}{2}e^ii_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\\ &\qquad +\frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\Bigg)\\ &\quad -\frac{1}{2}\left\langle T\left(f_\alpha^H,e_i\right),e_j\right\rangle f^\alpha\left(\left(e^i+i_{\widehat{e}^i}\right)\nabla_{\widehat{e}^j}+\left(e^i-i_{\widehat{e}^i}\right)\nabla_{\widehat{e}^j}\mathcal{H}\right)\\ &\quad -\frac{1}{2}\left(e^k+i_{\widehat{e}^k}\right)f^\alpha e^ii_{\widehat{e}^j}\left\langle\nabla_{e_k}T\left(f_\alpha^H,e_i\right),e_j\right\rangle-\frac{1}{2}\left\langle T^H,\nabla^V\mathcal{H}\right\rangle\\ &\qquad -\frac{1}{4}\left(e^i+i_{\widehat{e}^i}\right)\left(e^j+i_{\widehat{e}^j}\right)\left\langle\nabla^{TX}_{e_i}T^H,e_j\right\rangle\\ &\qquad -\frac{1}{8}\left(e^i-i_{\widehat{e}^i}\right)f^\alpha\omega^2\left(\nabla^F,g^F\right)\left(e_i,f_\alpha^H\right)\\ &\quad +\frac{1}{2}\left(e^i+i_{\widehat{e}^i}\right)f^\alpha\left(\frac{1}{2}\nabla^{F,u}_{e_i}\omega\left(\nabla^F,g^F\right)\left(f_\alpha^H\right)-\nabla_{e_i}\nabla_{f_\alpha^H}\mathcal{H}\right)\\ &\qquad -\frac{1}{2}\left(\widehat{e}_i+i_{\widehat{e}^i-e_i}\right)f^\alpha\nabla_{\widehat{e}^i}\nabla_{f_\alpha^H}\mathcal{H}-\frac{1}{8}f^\alpha f^\beta\omega^2\left(\nabla^F,g^F\right)\left(f_\alpha^H,f_\beta^H\right). \end{aligned} \tag{4.117}$$

Proof. Set

$$M = \mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}-\omega^H} - \mathfrak{B}_{\phi,\mathcal{H}}. \tag{4.118}$$

By (4.86),

$$M = -\frac{1}{2}E + f^\alpha\left(\frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) - \nabla_{f^H_\alpha}\mathcal{H}\right) - \frac{1}{2}\left\langle T^H, e^i\right\rangle\left(e^i + i_{\widehat{e}^i}\right). \tag{4.119}$$

Then

$$\mathcal{D}^{\mathcal{M},2}_{\phi,\mathcal{H}-\omega^H} = \mathfrak{B}^2_{\phi,\mathcal{H}} + M^2 + \left[\mathfrak{B}_{\phi,\mathcal{H}}, M\right]. \tag{4.120}$$

Using (2.91) and equation (3.11) in Theorem 3.3, we get an expression for $\mathfrak{B}^2_{\phi,\mathcal{H}}$. Also by (4.72) and (4.119), we get

$$M^2 = \frac{1}{8}f^\alpha f^\beta \omega^2\left(\nabla^F, g^F\right)\left(f^H_\alpha, f^H_\beta\right). \tag{4.121}$$

Using formula (2.92) for $\mathfrak{B}_{\phi,\mathcal{H}}$, (4.38) and (4.119), one can compute $[\mathfrak{B}_{\phi,\mathcal{H}}, M]$ easily. Thus we obtain (4.117). The proof of our theorem is completed. □

4.19. **A formula for $\mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}$.** In this subsection, we assume that there is a smooth function $c : S \to \mathbf{R}_*$ such that the Hamiltonian function is now $\mathcal{H}^c$. Then dc is a smooth 1-form on S. Also we still use the notation

$$\mathcal{H} = \frac{1}{2}|p|^2, \tag{4.122}$$

so that $\mathcal{H}^c = c\mathcal{H}$.

Theorem 4.38. *The following identity holds:*

$$\begin{aligned}
\mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{4}\Bigg(&-\Delta^V + c^2|p|^2 + c\left(2\widehat{e}_i i_{\widehat{e}^i - e_i} - n\right) \\
&- \frac{1}{2}\left\langle R^{TX}\left(e_i, e_j\right)e_k, e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l}\Bigg) \\
&- \frac{c}{2}\Bigg(\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F,u}_{Y^{\mathcal{H}}} + \left\langle T\left(f^H_\alpha, p\right), e_i\right\rangle f^\alpha\left(e^i + 2\widehat{e}_i + i_{\widehat{e}^i - 2e_i}\right) \\
&+ \left\langle T^H, p\right\rangle - \left(e^i + i_{\widehat{e}^i}\right) f^\alpha\left\langle \nabla^{TX}_{e_i} T\left(f^H_\alpha, p\right), p\right\rangle - e^i i_{\widehat{e}^j}\left\langle R^{TX}\left(p, e_i\right)e_j, p\right\rangle\Bigg) \\
&- \left(\frac{1}{4}\omega\left(\nabla^F, g^F\right)\left(e_i\right) + \frac{1}{2}\left\langle T\left(f^H_\alpha, e_i\right), e_j\right\rangle f^\alpha\left(e^j + i_{\widehat{e}^j}\right)\right)\nabla_{\widehat{e}^i} \\
&- \frac{1}{2}\left(e^k + i_{\widehat{e}^k}\right) f^\alpha e^i i_{\widehat{e}^j}\left\langle \nabla^{TX}_{e_k} T\left(f^H_\alpha, e_i\right), e_j\right\rangle \\
&- \frac{1}{4}\left(e^i + i_{\widehat{e}^i}\right)\left(e^j + i_{\widehat{e}^j}\right)\left\langle \nabla^{TX}_{e_i} T^H, e_j\right\rangle \\
&- \frac{1}{4}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F, g^F\right)\left(e_j\right) - \frac{1}{8}\left(e^i - i_{\widehat{e}^i}\right) f^\alpha\omega^2\left(\nabla^F, g^F\right)\left(e_i, f^H_\alpha\right) \\
&+ \frac{1}{4}\left(e^i + i_{\widehat{e}^i}\right) f^\alpha \nabla^{F,u}_{e_i}\omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right) \\
&- \frac{1}{8}f^\alpha f^\beta \omega^2\left(\nabla^F, g^F\right)\left(f^H_\alpha, f^H_\beta\right) + \frac{1}{2}dc\left(\widehat{e}_i + i_{\widehat{e}^i - e_i}\right)\left\langle p, e_i\right\rangle.
\end{aligned} \tag{4.123}$$

Proof. Using (3.12) in Theorem 3.4, (4.109) and Theorem 4.37, we get (4.123). □

Remark 4.39. To evaluate $\mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}$, by (2.111), (4.97) and (4.100), we should just replace in (4.123) e^i by $e^i-\widehat{e}_i$ and $i_{\widehat{e}^i}$ by $i_{e_i+\widehat{e}^i}$. One can then verify directly a consequence of Theorem 4.29, i.e. that the obtained expression is $h^{\Omega^{\cdot}(T^*X,\pi^*F|_{T^*X})}$ self-adjoint.

Let $u\in\mathbf{R}$ be an extra variable.

Theorem 4.40. *For $c\in\mathbf{R}^*$, the operator $\frac{\partial}{\partial u}-\mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}$ is hypoelliptic.*

Proof. Using Theorem 4.38, the proof is the same as the proof of Theorem 3.6. □

4.20. **A commutator identity.** Recall that $\omega\left(\nabla^F,g^F\right)$ is a section of $T^*M\otimes\operatorname{End}(F)$ over M. We can write $\omega\left(\nabla^F,g^F\right)$ in the form

$$\omega\left(\nabla^F,g^F\right)=e^i\omega\left(\nabla^F,g^F\right)(e_i)+f^\alpha\omega\left(\nabla^F,g^F\right)\left(f^H_\alpha\right). \tag{4.124}$$

Definition 4.41. Set

$$\begin{aligned}\nu_c&=\frac{1}{2}\left(e^i+i_{\widehat{e}^i}\right)\left(e^i+\widehat{e}_i+i_{e_i}\right)-\nabla_{\widehat{p}}+\frac{3c}{2}\left|p\right|^2,\\ \mathfrak{G}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}&=\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}+\left[\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H},\nu_c\right].\end{aligned} \tag{4.125}$$

Theorem 4.42. *The following identity holds:*

$$\begin{aligned}\mathfrak{G}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}=\frac{1}{2}\omega\left(\nabla^F,g^F\right)&-\frac{1}{4}\left(e^i+i_{\widehat{e}^i}\right)\omega\left(\nabla^F,g^F\right)(e_i)\\ &-\frac{c}{2}\left(p+i_{\widehat{p}}+6\widehat{p}-6f^\alpha\left\langle T\left(f^H_\alpha,p\right),p\right\rangle\right)-\frac{3}{2}dc\left|p\right|^2.\end{aligned} \tag{4.126}$$

Moreover,

$$\begin{aligned}\mathfrak{G}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}+\left[\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H},\frac{3c}{2}\left|p\right|^2\right]=\frac{1}{2}\omega\left(\nabla^F,g^F\right)&-\frac{1}{4}\left(e^i+i_{\widehat{e}^i}\right)\omega\left(\nabla^F,g^F\right)(e_i)\\ &-\frac{c}{2}\left(p+i_{\widehat{p}}\right).\end{aligned} \tag{4.127}$$

Proof. Using Propositions 2.36, 4.21 and 4.35, the identities (4.38), (4.72) and (4.109), and also the circular identities on the curvature of the fibrewise Levi-Civita

connection, we get easily

$$
\begin{aligned}
(4.128)\quad & \left[\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}, \frac{1}{2}\left(e^i+i_{\widehat{e}^i}\right)\left(e^i+\widehat{e}_i+i_{e_i}\right)\right] = -\frac{c}{2}\left(p+i_p+\widehat{p}\right)+\frac{1}{2}\left(e^i+i_{\widehat{e}_i}\right) \\
& \left(\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F,u}_{e_i}+\left\langle T\left(f^H_\alpha,e_i\right),e_j\right\rangle f^\alpha\left(e^j-i_{\widehat{e}^j}\right)+\left\langle T^H,e^i\right\rangle\right) \\
& \qquad -\left(R^{TX}p-i_{\widehat{R^{TX}p}}\right), \\
& \left[\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}, -\nabla_{\widehat{p}}\right] = \frac{1}{2}\left(\widehat{e}_i+i_{\widehat{e}^i-e_i}\right)\nabla_{\widehat{e}^i}-\frac{c}{2}\left(\widehat{p}+i_{p-\widehat{p}}\right) \\
& \qquad +\frac{1}{2}\left(R^{TX}p\wedge -i_{\widehat{R^{TX}p}}\right)+2c\left\langle T\left(f^H_\alpha,p\right),p\right\rangle f^\alpha-dc\left|p\right|^2, \\
& \left[\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}, \frac{3c}{2}\left|p\right|^2\right] = -\frac{3c}{2}\left(\widehat{p}+i_{\widehat{p}-p}\right), \\
& \left[\mathfrak{C}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H}, \frac{3c}{2}\left|p\right|^2\right] = 3c\left(\widehat{p}-\left\langle T\left(f^H_\alpha,p\right),p\right\rangle f^\alpha\right)+\frac{3dc}{2}\left|p\right|^2.
\end{aligned}
$$

Using the above references again and also (4.128), we get (4.126) and (4.127). The proof of our theorem is completed. □

Remark 4.43. The main point in formula (4.126) is that the right-hand side is an operator of order 0, which has been obtained by adding to $\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}$ a commutator with $\mathfrak{D}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}$. This certainly never happens for the standard Dirac operator d^X+d^{X*} on a given fibre X.

4.21. **Interchanging the Grassmann variables.** Here, we will use the notation of subsection 4.1. By (4.3), we have the canonical isomorphism

$$
\Lambda^{\cdot}(TX)\simeq\Lambda^{n-\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX). \tag{4.129}
$$

By (4.129), we get

$$
\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\simeq\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{n-\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX). \tag{4.130}
$$

We denote by τ the canonical isomorphism in (4.130). Clearly, for $1\le i\le n$, the actions of e^i, i_{e_i} on both sides of (4.130) are the same, while the actions of $\widehat{e}_i, i_{\widehat{e}^i}$ on the left-hand side of (4.130) correspond to the action of $i_{\widehat{e}_i}, \widehat{e}^i$ on the right-hand side.

If L is an operator acting on smooth sections of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F$ along the fibres of the total space of T^*X, let $L_\dagger$ denote the corresponding operator acting on smooth sections of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F$. Of course,

$$
L_\dagger=\tau L\tau^{-1}. \tag{4.131}
$$

In particular $A'^{\mathcal{M}}_\dagger$ is the operator obtained from $A'^{\mathcal{M}}$ by using the canonical isomorphism (4.130). We have the obvious identity

$$
A'^{\mathcal{M}}_{\mathcal{H}^c\dagger}=e^{-\mathcal{H}^c}A'^{\mathcal{M}}_\dagger e^{\mathcal{H}^c}. \tag{4.132}
$$

Let $\chi \in \operatorname{End}(TX \oplus TX)$ be given in matrix form by

$$\chi = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{4.133}$$

Then χ acts naturally on the right-hand side of (4.130). This action is just

$$\chi = \exp\left(-e^i i_{\widehat{e}_i}\right). \tag{4.134}$$

By Proposition 2.5,

$$\omega^V = \widehat{e}_i \wedge e^i. \tag{4.135}$$

By (4.134), (4.135), we obtain

$$\chi = \tau e^{\omega^V} \tau^{-1}. \tag{4.136}$$

If L is an operator acting on smooth sections of $\Lambda^{\cdot}(T^*X) \widehat{\otimes} \Lambda^{\cdot}(TX) \widehat{\otimes} F$ along the fibres of the total space of T^*X, we denote by $\underline{L}$ the operator acting on smooth sections of $\Lambda^{\cdot}(T^*X) \widehat{\otimes} \Lambda^{\cdot}(T^*X) \widehat{\otimes} \Lambda^n(TX) \widehat{\otimes} F$ which is obtained from $L_\dagger$ by conjugation by χ. By (4.131), (4.134), we get

$$\underline{L} = \exp\left(-e^i i_{\widehat{e}_i}\right) \tau L \tau^{-1} \exp\left(e^i i_{\widehat{e}_i}\right). \tag{4.137}$$

By (4.134)–(4.137), we get

$$\underline{L} = \tau e^{\omega^V} L e^{-\omega^V} \tau^{-1}. \tag{4.138}$$

In the sequel, we will use the transformation $L \to \underline{L}$. Instead of this, we could as well use the transformation $L \to e^{\omega^V} L e^{-\omega^V}$.

The operator $\underline{L}$ is obtained from L by making the following changes for $1 \leq i \leq n$:

- e^i is unchanged.
- i_{e_i} is changed into $i_{e_i + \widehat{e}_i}$.
- $\widehat{e}_i$ is changed into $i_{\widehat{e}_i}$.
- $i_{\widehat{e}^i}$ is changed into $\widehat{e}^i - e^i$.

We will give formulas for the operators which were considered above using the new variables. To distinguish the new operators from the previous ones, they will be underlined. In the sequel, we will use the notation

$$\widehat{\omega}\left(\nabla^F, g^F\right) = \widehat{e}^i \omega\left(\nabla^F, g^F\right)(e_i). \tag{4.139}$$

Also $\omega\left(\nabla^F, g^F\right)$ will be considered as a 1-form on M.

Theorem 4.44. *The following identities hold:*

(4.140)

$$\underline{\mathfrak{C}}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H} = A'^{\mathcal{M}}_{\mathcal{H}^c\dagger},$$

$$\underline{\mathfrak{D}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{2}\left(e^i + i_{e_i} - \widehat{e}^i\right)\nabla_{\widehat{e}^i} - \frac{1}{2}\widehat{e}^i\Bigg(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F}_{e_i}$$

$$+\omega\left(\nabla^F, g^F\right)(e_i)\Bigg) - \frac{1}{2}\widehat{T}^H + T^0 + \frac{1}{2}\omega\left(\nabla^F, g^F\right) + \frac{1}{4}\left\langle R^{TX}\left(e_i, e_j\right)p, e_k\right\rangle \widehat{e}^i\widehat{e}^j e^k$$

$$+\frac{c}{2}\left(\widehat{e}^i - e^i - i_{e_i+2\widehat{e}_i}\right)\langle p, e_i\rangle + cf^\alpha\left\langle T\left(f^H_\alpha, p\right), p\right\rangle - \frac{dc}{2}|p|^2,$$

$$\underline{\mathfrak{C}}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{4}\left(-\left(\nabla_{\widehat{e}^i} + \left\langle T^0, e_i\right\rangle\right)^2 + c^2|p|^2 + c\left(2i_{\widehat{e}^i}\left(\widehat{e}^i - e^i - i_{e_i}\right) - n\right)\right)$$

$$+\frac{1}{4}\left\langle e_i, R^{TX}e_j\right\rangle \widehat{e}^i\widehat{e}^j - \frac{1}{4}\omega\left(\nabla^F, g^F\right)(e_i)\left(\nabla_{\widehat{e}^i} + \left\langle T^0, e_i\right\rangle\right)$$

$$-\frac{1}{4}\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F}\widehat{\omega}\left(\nabla^F, g^F\right) - \frac{1}{4}\omega\left(\nabla^F, g^F\right)^2$$

$$-\frac{c}{2}\Bigg(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F,u}_{Y^{\mathcal{H}}} + \left\langle T\left(f^H_\alpha, e_i\right)f^\alpha\left(\widehat{e}^i - 2i_{e_i}\right) + T^H, p\right\rangle$$

$$+\left\langle R^{TX}\left(\cdot, p\right)e_i, p\right\rangle \widehat{e}^i + f^\alpha\widehat{e}^i\left\langle \nabla^{TX}_{Y^{\mathcal{H}}}T^0\left(f^H_\alpha, e_i\right), p\right\rangle\Bigg) + \frac{1}{2}dc\left(\widehat{e}^i - e^i - i_{e_i}\right)\langle p, e_i\rangle,$$

$$\underline{\nu}_c = \frac{1}{2}\widehat{e}_i\left(e^i + i_{e_i+2\widehat{e}^i}\right) - \nabla_{\widehat{p}} + \frac{3c}{2}|p|^2,$$

$$\underline{\mathfrak{G}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{2}\omega\left(\nabla^F, g^F\right) - \frac{1}{4}\widehat{\omega}\left(\nabla^F, g^F\right) - \frac{c}{2}\left(\widehat{p} + 6i_{\widehat{p}} - 6f^\alpha\left\langle T\left(f^H_\alpha, p\right), p\right\rangle\right)$$

$$-\frac{3}{2}dc\left|p^2\right|,$$

$$\underline{\mathfrak{G}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} + \left[\underline{\mathfrak{C}}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H}, \frac{3c}{2}|p|^2\right] = \frac{1}{2}\omega\left(\nabla^F, g^F\right) - \frac{1}{4}\widehat{\omega}\left(\nabla^F, g^F\right) - \frac{c}{2}\widehat{p}.$$

Proof. By (4.23), (4.79) and (4.138), we get

$$\underline{\mathfrak{C}}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H} = e^{-\mathcal{H}^c}\tau e^\omega A'^{\mathcal{M}}e^{-\omega}\tau^{-1}e^{\mathcal{H}^c}. \tag{4.141}$$

Since ω is closed,

$$e^\omega A'^{\mathcal{M}}e^{-\omega} = A'^{\mathcal{M}}. \tag{4.142}$$

By (4.131), (4.141), (4.142), we get the first identity in (4.140).

To establish the second identity in (4.140), we use Propositions 4.31 and 4.35, and also the circular identity for the Levi-Civita curvature. To prove the third identity, we use equation (4.123) in Theorem 4.38, and we make the indicated replacements. Also we use the full strength of Theorem 4.15, a particular form of this theorem saying that if $U, V, W \in TX$,

$$\left\langle U, R^{TX}\left(f^H_\alpha, V\right)W\right\rangle = \left\langle -\nabla^{TX}_U T\left(f^H_\alpha, W\right) + \nabla^{TX}_W T\left(f^H_\alpha, U\right), V\right\rangle, \tag{4.143}$$

and also the fact that because of (2.53),

$$\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{e_i}\omega\left(\nabla^F,g^F\right)\left(f^H_\alpha\right)=\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{f^H_\alpha}\omega\left(\nabla^F,g^F\right)(e_i)$$
$$-\left[\omega\left(\nabla^F,g^F\right)(e_i),\omega\left(\nabla^F,g^F\right)\left(f^H_\alpha\right)\right]-\omega\left(\nabla^F,g^F\right)\left(T\left(e_i,f^H_\alpha\right)\right). \tag{4.144}$$

The last three identities in (4.140) follow from (4.125) and from Theorem 4.42. The proof of our theorem is completed. □

4.22. **A construction of hatted superconnections.** It is very interesting to rewrite the identities in (4.140) in still another form. We will use the notation of subsection 4.8.

By (4.62) or by (4.143), we get

$$\frac{1}{2}\left\langle\widehat{\nabla}^{TX}T^0,p\right\rangle=\frac{1}{4}\left\langle e_i,R^{TX}\left(f^H_\alpha,p\right)e_j\right\rangle f^\alpha\widehat{e}^i\widehat{e}^j. \tag{4.145}$$

Set

$$\underline{\widehat{\mathfrak{D}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}=\exp\left(\left\langle T^0,p\right\rangle\right)\underline{\mathfrak{D}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}\exp\left(-\left\langle T^0,p\right\rangle\right). \tag{4.146}$$

Other operators will be denoted in a similar way, by adding an extra hat when conjugating by $\exp\left(\left\langle T^0,p\right\rangle\right)$.

In the sequel i_{T^0} will denote the operator

$$i_{T^0}=\left\langle T\left(f^H_\alpha,e_i\right),e^j\right\rangle f^\alpha\wedge\widehat{e}^i\wedge i_{e_j}. \tag{4.147}$$

Consider the de Rham operator $d^{\mathcal{M}}$ on the total space $\mathcal{M}$ of T^*X over M. In the discussion which follows, we will use the Euclidean connection ∇^{T^*X} on T^*X. Let $\widehat{A}'^{\mathcal{M}}$ be the corresponding superconnection along the fibres of T^*X over M. As in (2.43) or in (4.56), we have the identity

$$\widehat{A}'^{\mathcal{M}}=\widehat{e}_i\nabla_{\widehat{e}^i}+\nabla^{T^*X}+i_{\widehat{R^{T^*X}}_p}. \tag{4.148}$$

Note that in (4.148), R^{T^*X} denotes the full curvature tensor and not only its restriction to TX as in (4.101).

Let us make here an important observation. First ∇^{T^*X} acts on smooth forms on M as the exterior differential. If we insist on equipping TX, the tangent bundle to the fibration, with the connection ∇^{TX}, the operator ∇^{T^*X} can be expressed as

$$\nabla^{T^*X}=\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F}+f^\alpha e^i i_{T(f^H_\alpha,e_i)}+i_{T^H}. \tag{4.149}$$

The interior multiplication operators in the right-hand side of (4.149) act only on the 'horizontal' exterior algebra $\Lambda^{\cdot}(T^*X)$.

Note that even though we use the same notation, as forms on $\mathcal{M}$, the $\widehat{e}_i$ are not the same in (4.28), (4.56), simply because the connections $\overline{\nabla}^{T^*X}$ and ∇^{T^*X} are distinct. Indeed using (4.39), (4.41), we find that our new $\widehat{e}_i$ in (4.148) is just $\widehat{e}_i-f^\alpha\left\langle T\left(f^H_\alpha,e_i\right),p\right\rangle$ in the notation of the previous subsections. Equivalently, if one thinks of the $\widehat{e}_i$ as being identified (as identical sections of a vector bundle), we get the obvious equation

$$\widehat{A}'^{\mathcal{M}}=\exp\left(\left\langle T\left(f^H_\alpha,e_i\right),p\right\rangle f^\alpha i_{\widehat{e}^i}\right)A'^{\mathcal{M}}\exp\left(-\left\langle T\left(f^H_\alpha,e_j\right),p\right\rangle f^\alpha i_{\widehat{e}^j}\right). \tag{4.150}$$

Equation (4.150) expresses the de Rham operator $d^{\mathcal{M}}$ in two distinct gauges, hence the conjugation.

Recall that $\widehat{A}'^{\mathcal{M}}_{\dagger}$ is obtained from $\widehat{A}'^{\mathcal{M}}$ by replacing the $\widehat{e}_i, i_{\widehat{e}^i}$ by $i_{\widehat{e}_i}, \widehat{e}^i$, so that

$$\widehat{A}'^{\mathcal{M}}_{\dagger} = \tau \widehat{A}'^{\mathcal{M}} \tau^{-1}. \tag{4.151}$$

By (4.150), we obtain

$$\widehat{A}'^{\mathcal{M}}_{\dagger} = \exp\left(\left\langle T^0, p\right\rangle\right) A'^{\mathcal{M}}_{\dagger} \exp\left(-\left\langle T^0, p\right\rangle\right). \tag{4.152}$$

Observe that the notation in (4.146) and the notation in (4.152) are compatible.

By (4.148), we get

$$\widehat{A}'^{\mathcal{M}}_{\dagger} = i_{\widehat{e}_i} \nabla_{\widehat{e}^i} + \nabla^{T^*X} + \left\langle R^{T^*X} p, e_i \right\rangle \widehat{e}^i. \tag{4.153}$$

Needless to say, equation (4.149) is still valid in (4.153).

We define $\widehat{A}'^{\mathcal{M}}_{\mathcal{H}^c\dagger}$ by the obvious conjugation.

Theorem 4.45. *The following identities hold:*

(4.154)

$$\begin{aligned}
&\widehat{\underline{\mathfrak{C}}}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H} = \widehat{A}'^{\mathcal{M}}_{\mathcal{H}^c\dagger},\\
&\widehat{\underline{\mathfrak{D}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{2}\left(e^i + i_{e_i} - \widehat{e}^i\right)\nabla_{\widehat{e}^i} - \frac{1}{2}\widehat{e}^i\Big(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^n(T^*X)\widehat{\otimes}F}_{e_i}\\
&+\omega\left(\nabla^F, g^F\right)(e_i)\Big) - \frac{1}{2}\widehat{T}^H + \frac{1}{2}\left(T^0 - i_{T^0}\right) + \frac{1}{2}\langle\widehat{\nabla}^{TX}T^0, p\rangle + \frac{1}{2}\omega\left(\nabla^F, g^F\right)\\
&+\frac{1}{4}\left\langle R^{TX}\left(e_i, e_j\right)p, e_k\right\rangle \widehat{e}^i\widehat{e}^j e^k + \frac{c}{2}\left(\widehat{e}^i - e^i - i_{e_i+2\widehat{e}_i}\right)\langle p, e_i\rangle - \frac{dc}{2}|p|^2,\\
&\widehat{\underline{\mathfrak{C}}}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{4}\left(-\Delta^V + c^2|p|^2 + c\left(2i_{\widehat{e}_i}\left(\widehat{e}^i - e^i - i_{e_i}\right) - n\right)\right)\\
&+\frac{1}{4}\left\langle e_i, R^{TX}e_j\right\rangle \widehat{e}^i\widehat{e}^j - \frac{1}{4}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i}\\
&-\frac{1}{4}\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F}\widehat{e}^j\omega\left(\nabla^F, g^F\right)(e_j) - \frac{1}{4}\omega\left(\nabla^F, g^F\right)^2\\
&-\frac{c}{2}\Bigg(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F,u}_{Y^{\mathcal{H}}} + \left\langle T\left(f^H_\alpha, e_i\right)f^\alpha\left(e^i - i_{e_i}\right) + T^H, p\right\rangle\\
&+\left\langle R^{TX}\left(\cdot, p\right)e_i, p\right\rangle \widehat{e}^i\Bigg) + \frac{1}{2}dc\left(\widehat{e}^i - e^i - i_{e_i}\right)\langle p, e_i\rangle,\\
&\widehat{\underline{\nu}}_c = \frac{1}{2}\widehat{e}^i\left(e^i + i_{e_i+2\widehat{e}_i}\right) - \nabla_{\widehat{Y^{\mathcal{H}}}} + \frac{3c}{2}|p|^2,\\
&\widehat{\underline{\mathfrak{G}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{2}\omega\left(\nabla^F, g^F\right) - \frac{1}{4}\widehat{\omega}\left(\nabla^F, g^F\right) - \frac{c}{2}\left(\widehat{p} + 6i_{\widehat{p}}\right) - \frac{3}{2}dc|p|^2,\\
&\widehat{\underline{\mathfrak{G}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} + \left[\widehat{\underline{\mathfrak{C}}}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H}, \frac{3c}{2}|p|^2\right] = \frac{1}{2}\omega\left(\nabla^F, g^F\right) - \frac{1}{4}\widehat{\omega}\left(\nabla^F, g^F\right) - \frac{c}{2}\widehat{p}.
\end{aligned}$$

Proof. The first equation in (4.154) follows from the first equation in (4.140) and from (4.152). The other formulas in (4.154) follow from Theorem 4.44 by obvious computations which are left to the reader. □

Remark 4.46. Observe that for $c = 0$, $-2\underline{\mathfrak{D}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}$ and $-2\widehat{\underline{\mathfrak{D}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}$ can be considered as superconnections over a vector bundle over a single fibre X. Indeed by (4.140), (4.154), these operators only contain creation operators $\widehat{e}^i$ and do not

contain the corresponding annihilation operators $i_{\widehat{e}_i}$. Also these two operators are odd, and moreover the leading symbol in the horizontal differentiation operators $\nabla_{e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F}$ is just $\widehat{e}^i\nabla_{e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F}$, which is a fibrewise connection. Therefore these two operators turn out to be fibrewise superconnections.

A final step in our new formalism is to explain directly the construction of $\underline{\widehat{\mathfrak{D}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}$ in terms of $\widehat{A}'^{\mathcal{M}}$.

Let $\psi \in \operatorname{End}(TX \oplus TX)$ be given in matrix form by

$$\psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.155}$$

Then ψ acts on $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}\overline{F}^*$. This action is given by

$$\psi = \exp\left(-\widehat{e}^i i_{e_i}\right). \tag{4.156}$$

Recall that by (2.46),

$$\Lambda = -i_{\widehat{e}^i} i_{e_i}. \tag{4.157}$$

By (4.156), (4.157), we get

$$\psi = \tau e^{\Lambda}\tau^{-1}. \tag{4.158}$$

If L is an operator acting on smooth sections of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F$ along the fibres of the total space of T^*X, we denote by $\overline{L}$ the operator acting on smooth sections of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F$ which is obtained from $L_\dagger$ by conjugation by ψ. By (4.156),

$$\overline{L} = \exp\left(-\widehat{e}^i i_{e_i}\right)\tau L \tau^{-1}\exp\left(\widehat{e}^i i_{e_i}\right). \tag{4.159}$$

By (4.156)–(4.159), we obtain,

$$\overline{L} = \tau e^{\Lambda} L e^{-\Lambda}\tau^{-1}. \tag{4.160}$$

The operator $\overline{L}$ is obtained from L by making the following transformations for $1 \le i \le n$:

- e^i is changed into $e^i - \widehat{e}^i$.
- i_{e_i} is unchanged.
- $\widehat{e}_i$ is replaced by $i_{e_i+\widehat{e}_i}$.
- $i_{\widehat{e}^i}$ is changed into $\widehat{e}^i$.

We identify F to $\overline{F^*}$ via the metric g^F. This is just saying that temporarily, we equip F with the connection $\nabla^{\overline{F}^*} = \nabla^F + \omega\left(\nabla^F, g^F\right)$. In particular $\overline{\widehat{A}}'^{\mathcal{M}}$ is obtained from $\widehat{A}'^{\mathcal{M}}$ by replacing F by $\overline{F}^*$ and by otherwise following the above procedure.

Proposition 4.47. *The following identity holds:*

(4.161)
$$\begin{aligned}\overline{\widehat{A}}'^{\mathcal{M}} &= i_{e_i+\widehat{e}_i}\nabla_{\widehat{e}^i} + \left(e^i - \widehat{e}^i\right)\left(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F}_{e_i} + \omega\left(\nabla^F, g^F\right)(e_i)\right)\\ &\quad + f^\alpha\left(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F}_{f^H_\alpha} + \omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right)\right)\\ &\quad + f^\alpha e^i i_{T(f^H_\alpha, e_i)} - i_{T^0} + i_{T^H} + \left\langle \widehat{\nabla}^{TX}T^0, p\right\rangle\\ &\quad + \frac{1}{2}\left\langle R^{T^*X}(e_i,e_j)p, e_k\right\rangle\left(e^ie^j\widehat{e}^k + \widehat{e}^i\widehat{e}^je^k\right) + \frac{1}{2}\left\langle R^{T^*X}\left(f^H_\alpha, f^H_\beta\right)p, e_k\right\rangle f^\alpha f^\beta\widehat{e}^k\\ &\quad + \left\langle R^{T^*X}\left(f^H_\alpha, e_i\right)p, e_k\right\rangle f^\alpha e^i\widehat{e}^k.\end{aligned}$$

Proof. We use (4.148) and (4.149). The only term in (4.148) for which the operation is nontrivial is the last term $i_{\widehat{R^{TX}p}}$. We split the curvature tensor according to its degree in the e^i, f^α. The contribution of the e^ie^j term is the first term in the fourth row in (4.161), this because of the circular identity verified by the Levi-Civita curvature of the fibre X. The $f^\alpha f^\beta$ term is unchanged and appears as the second term in the fourth row. The contribution of the $f^\alpha e^i$ term is

$$\left\langle R^{T^*X}\left(f^H_\alpha, e_i\right)p, e_k\right\rangle f^\alpha\left(e^i - \widehat{e}^i\right)\widehat{e}^k.$$

The first term above appears as the last term in (4.161). Moreover using (4.38) and (4.143), we get

$$-\left\langle R^{TX}\left(f^H_\alpha, e_i\right)p, e_k\right\rangle f^\alpha\widehat{e}^i\widehat{e}^k = \left\langle \nabla^{TX}T\left(f_\alpha, e_i\right), p\right\rangle f^\alpha\widehat{e}^i\widehat{e}^k, \tag{4.162}$$

which is equivalent to

$$-\left\langle R^{TX}\left(f^H_\alpha, e_i\right)p, e_k\right\rangle f^\alpha\widehat{e}^i\widehat{e}^k = \left\langle \widehat{\nabla}^{TX}T^0, p\right\rangle. \tag{4.163}$$

This concludes the proof of our proposition. □

Set

$$\overline{\lambda}_0 = e^i \wedge \widehat{e}^i. \tag{4.164}$$

Note that $\overline{\lambda}_0$ is obtained from λ_0 by the above procedure.

Definition 4.48. Put

$$\overline{\widehat{A}}'^{\mathcal{M}}_\phi = e^{\overline{\lambda}_0}\overline{\widehat{A}}'^{\mathcal{M}}e^{-\overline{\lambda}_0}. \tag{4.165}$$

Observe that by (4.157), (4.160), (4.164), (4.165),

$$\overline{\widehat{A}}'^{\mathcal{M}}_\phi = \tau e^{\left(e^i + i_{e_i}\right)i_{\widehat{e}^i}}\widehat{A}'^{\mathcal{M}}e^{-\left(e^i + i_{e_i}\right)i_{\widehat{e}^i}}\tau^{-1}. \tag{4.166}$$

Proposition 4.49. *The following identity holds:*

$$\begin{aligned}(4.167)\quad \overline{\widehat{A}}'^{\mathcal{M}}_{\phi} &= \left(e^i - \widehat{e}^i + i_{e_i+\widehat{e}_i}\right)\nabla_{\widehat{e}^i} \\ &+ \left(e^i - \widehat{e}^i\right)\left(\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F}_{e_i} + \omega\left(\nabla^F, g^F\right)(e_i)\right) \\ &+ f^\alpha\left(\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F}_{f^H_\alpha} + \omega\left(\nabla^F, g^F\right)\left(f^H_\alpha\right)\right) \\ &+ f^\alpha e^i i_{T(f^H_\alpha, e_i)} + T^0 - i_{T^0} + i_{T^H} - \widehat{T}^H + \left\langle \widehat{\nabla}^{TX}T^0, p\right\rangle \\ &+ \frac{1}{2}\left\langle R^{T^*X}(e_i,e_j)p, e_k\right\rangle\left(e^ie^j\widehat{e}^k + \widehat{e}^i\widehat{e}^je^k\right) + \frac{1}{2}\left\langle R^{T^*X}\left(f^H_\alpha, f^H_\beta\right)p, e_k\right\rangle f^\alpha f^\beta\widehat{e}^k \\ &+ \left\langle R^{T^*X}\left(f^H_\alpha, e_i\right)p, e_k\right\rangle f^\alpha e^i\widehat{e}^k.\end{aligned}$$

Proof. This follows easily from Proposition 4.47. □

Definition 4.50. Put

$$\overline{\widehat{A}}'^{\mathcal{M}}_{\phi,\mathcal{H}^c} = e^{\mathcal{H}^c}\overline{\widehat{A}}'^{\mathcal{M}}_{\phi}\, e^{-\mathcal{H}^c}. \tag{4.168}$$

Then we have the following important result.

Theorem 4.51. *The following identities hold:*

$$\widehat{A}'^{\mathcal{M}}_{\mathcal{H}^c\dagger} = \underline{\widehat{\mathfrak{C}}}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H}, \qquad \overline{\widehat{A}}'^{\mathcal{M}}_{\phi,\mathcal{H}^c} = \underline{\widehat{\mathfrak{C}}}'^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}. \tag{4.169}$$

In particular,

$$\frac{1}{2}\left(\overline{\widehat{A}}'^{\mathcal{M}}_{\phi,\mathcal{H}^c} - \widehat{A}'^{\mathcal{M}}_{\mathcal{H}^c\dagger}\right) = \underline{\widehat{\mathfrak{D}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H}. \tag{4.170}$$

Proof. The first identity in (4.169) was already established in Theorem 4.45. For $c = 0$, by using (4.149), (4.153), (4.154) and (4.167), we get (4.170). So the second identity in (4.169) holds. By conjugation, we get our theorem in full generality. □

Recall that $\mathcal{H} = \frac{|p|^2}{2}$. Let $\mathcal{L}_{Y^{\mathcal{H}}}$ be the global Lie derivative operator acting on the full exterior algebra of $\mathcal{M}$, which is attached to the vector field $Y^{\mathcal{H}}$. The operator $\mathcal{L}_{Y^{\mathcal{H}}}$ is to be distinguished from the fibrewise Lie derivative operator $L_{Y^{\mathcal{H}}}$, which acts trivially on the f^H_α. Clearly

$$\mathcal{L}_{Y^{\mathcal{H}}} = \left[d^{\mathcal{M}}, i_{Y^{\mathcal{H}}}\right]. \tag{4.171}$$

By (4.148), (4.149) and (4.171), we get

$$\mathcal{L}_{Y^{\mathcal{H}}} = \nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F}_{Y^{\mathcal{H}}} + \widehat{e}_i i_{e_i} - f^\alpha i_{T(f^H_\alpha,p)} + i_{\widehat{R^{T^*X}(p,\cdot)}p}. \tag{4.172}$$

Moreover,

$$L_{Y^{\mathcal{H}}} = \left[d^{T^*X}, i_{Y^{\mathcal{H}}}\right]. \tag{4.173}$$

By (3.12), (4.171), (4.173), we get

$$\begin{aligned}(4.174)\quad L_{Y^{\mathcal{H}}} &= \nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(TX)\widehat{\otimes}F}_{Y^{\mathcal{H}}} + \widehat{e}_i i_{e_i} + \left\langle R^{T^*X}(p,e_i)p, e_j\right\rangle e^i i_{\widehat{e}^j}, \\ \mathcal{L}_{Y^{\mathcal{H}}} &= L_{Y^{\mathcal{H}}} - \left\langle T\left(f^H_\alpha, p\right), e^i\right\rangle f^\alpha i_{e_i} + \left\langle R^{T^*X}\left(p, f^H_\alpha\right)p, e_i\right\rangle f^\alpha i_{\widehat{e}^i}.\end{aligned}$$

Recall that θ is the canonical 1-form on T^*X, which has been extended to a 1-form on $\mathcal{M}$ by making it vanish on T^HM and that $\omega = d^{\mathcal{M}}\theta$. Equivalently by (4.148) and (4.149), we get

$$\widehat{A}'^{\mathcal{M}}\theta = \widehat{e}_i e^i + \left\langle T\left(f_\alpha^H, e_i\right), p\right\rangle f^\alpha e^i + \left\langle T^H, p\right\rangle. \tag{4.175}$$

We still denote by $L_{Y^{\mathcal{H}}}, \mathcal{L}_{Y^{\mathcal{H}}}$ the corresponding operators acting on the sections of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F$. Equation (4.174) becomes

$$\begin{aligned} L_{Y^{\mathcal{H}}} &= \nabla_{Y^{\mathcal{H}}}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F} + i_{\widehat{e}_i} i_{e_i} + \left\langle R^{T^*X}(p, e_i)\, p, e_j\right\rangle e^i \widehat{e}^j, \\ \mathcal{L}_{Y^{\mathcal{H}}} &= L_{Y^{\mathcal{H}}} - \left\langle T\left(f_\alpha^H, p\right), e^i\right\rangle f^\alpha i_{e_i} + \left\langle R^{T^*X}\left(p, f_\alpha^H\right) p, e_i\right\rangle f^\alpha \widehat{e}^i. \end{aligned} \tag{4.176}$$

By (4.175), we get

$$\widehat{A}_\dagger'^{\mathcal{M}}\theta = i_{\widehat{e}_i} e^i + \left\langle T\left(f_\alpha^H, e_i\right), p\right\rangle f^\alpha e^i + \left\langle T^H, p\right\rangle. \tag{4.177}$$

Theorem 4.52. *The following identity holds:*

$$\begin{aligned} \widehat{\underline{\mathfrak{C}}}_{\phi,\mathcal{H}^c-\omega^H}^{\mathcal{M},2} &= \frac{1}{4}\left(-\Delta^V + c^2|p|^2 + c\left(2i_{\widehat{e}_i}\widehat{e}^i - n\right)\right) + \frac{1}{4}\left\langle e_i, R^{TX} e_j\right\rangle \widehat{e}^i \widehat{e}^j \\ &- \frac{1}{4}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i} - \frac{1}{4}\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}\widehat{\omega}\left(\nabla^F, g^F\right) - \frac{1}{4}\omega\left(\nabla^F, g^F\right)^2 \\ &- \frac{c}{2}\left(\mathcal{L}_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(Y^{\mathcal{H}}\right) + \widehat{A}_\dagger'^{\mathcal{M}}\theta\right) + \frac{1}{2}dc\left(\widehat{e}^i - e^i - i_{e_i}\right)\left\langle p, e_i\right\rangle. \end{aligned} \tag{4.178}$$

Proof. We use Theorem 4.45 and also (4.176) and (4.177). □

Remark 4.53. Expression (4.178) is by far the simplest and the most natural we have obtained for the curvature.

4.23. **Another relation to the Levi-Civita superconnection.** Here, we will establish an extension of Theorem 3.13. We fix $c \neq 0$. We replace the operator $\widehat{\underline{\mathfrak{C}}}_{\phi,\mathcal{H}^c-\omega^H}^{\mathcal{M},2}$ by its conjugate $e^{\mathcal{H}^c}\widehat{\underline{\mathfrak{C}}}_{\phi,\mathcal{H}^c-\omega^H}^{\mathcal{M},2}e^{-\mathcal{H}^c}$, and we replace the operators $\widehat{e}^i, i_{\widehat{e}_i}$ by the operators $i_{\widehat{e}^i}, \widehat{e}_i$. We will denote by $\widehat{\underline{\mathfrak{A}}}_{\phi,\mathcal{H}^c-\omega^H}^{\mathcal{M},2}$ the new operator. It is given by the formula

$$\widehat{\underline{\mathfrak{A}}}_{\phi,\mathcal{H}^c-\omega^H}^{\mathcal{M},2} = \tau^{-1}e^{\mathcal{H}^c}\widehat{\underline{\mathfrak{C}}}_{\phi,\mathcal{H}^c-\omega^H}^{\mathcal{M},2}e^{-\mathcal{H}^c}\tau. \tag{4.179}$$

Set

$$\widehat{\omega}'\left(\nabla^F, g^F\right) = \omega\left(\nabla^F, g^F\right)(e_i)\, i_{\widehat{e}^i}. \tag{4.180}$$

Then equation (4.178) can now be written in the form

$$\begin{aligned} \widehat{\underline{\mathfrak{A}}}_{\phi,\mathcal{H}^c-\omega^H}^{\mathcal{M},2} &= \frac{1}{4}\left(-\Delta^V + 2cL_{\widehat{p}} + \left\langle e_i, R^{TX} e_j\right\rangle i_{\widehat{e}^i} i_{\widehat{e}^j}\right) \\ &- \frac{1}{4}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i} - \frac{1}{4}\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}\widehat{\omega}'\left(\nabla^F, g^F\right) - \frac{1}{4}\omega\left(\nabla^F, g^F\right)^2 \\ &- \frac{c}{2}\left(\mathcal{L}_{Y^{\mathcal{H}}} + \widehat{A}'^{\mathcal{M}}\theta\right). \end{aligned} \tag{4.181}$$

Of course, in (4.181), $\mathcal{L}_{Y^{\mathcal{H}}}$ is given by (4.174), and $\widehat{A}'^{\mathcal{M}}\theta$ is given by (4.175).

Note that even when S is a point, (4.181) does not exactly coincide with the first identity in (3.13). Put

$$\begin{aligned}
\mathfrak{a}'_+ &= \frac{1}{2}\left(-\Delta^V + 2L_{\widehat{p}} + \left\langle e_i, R^{TX} e_j\right\rangle i_{\widehat{e}^i} i_{\widehat{e}^j}\right),\\
\mathfrak{b}'_+ &= -\Bigg(\mathcal{L}_{Y^{\mathcal{H}}} + \widehat{A}'^{\mathcal{M}}\theta + \frac{1}{2}\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}\widehat{\omega}'\left(\nabla^F, g^F\right)\\
&\qquad + \frac{1}{2}\omega\left(\nabla^F, g^F\right)(e_i)\,\nabla_{\widehat{e}^i}\Bigg),\\
\mathfrak{c}' &= -\frac{1}{2}\omega\left(\nabla^F, g^F\right)^2.
\end{aligned} \tag{4.182}$$

When the index is $-$ instead of $+$, we should change $L_{\widehat{p}}, \mathcal{L}_{Y^{\mathcal{H}}}, \widehat{A}'^{\mathcal{M}}\theta$ into their negative, and we get operators $\mathfrak{a}'_-, \mathfrak{b}'_-$, while $\mathfrak{c}'$ is unchanged. By (4.181), we get for $c > 0$,

$$r^*_{1/\sqrt{c}} 2\widehat{\underline{\mathfrak{A}}}^{\mathcal{M},2}_{\phi,\mathcal{H}^c - \omega^H} r^*_{\sqrt{c}} = c\mathfrak{a}'_+ + \sqrt{c}\mathfrak{b}'_+ + \mathfrak{c}'. \tag{4.183}$$

For $c < 0$, there is a corresponding obvious equality.

Definition 4.54. Let ${}^1\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F}$ be the connection on $\Lambda^{\cdot}(T^*S)\,\widehat{\otimes}\Lambda^{\cdot}(T^*X)$ along the fibres X,

$${}^1\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F}_{\cdot} = \nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F}_{\cdot} + \left\langle T\left(f^H_\alpha, e_i\right), \cdot\right\rangle f^\alpha c(e_i) + \left\langle T^H, \cdot\right\rangle. \tag{4.184}$$

Let ${}^1\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F,u}$ be the connection taken as before, replacing ∇^F by $\nabla^{F,u}$.

Put

$$\mathcal{R} = \frac{1}{4}\left\langle e_i, R^{TX} e_j\right\rangle \widehat{c}(e_i)\,\widehat{c}(e_j) - \frac{1}{4}\omega\left(\nabla^F, g^F\right)^2. \tag{4.185}$$

Recall that the formula for $B_\pm$ was given in (4.112). Let K be the scalar curvature of the fibres X. By [BLo95, Theorem 3.11], [BG01, Theorem 3.19], we have the following equation for $A^2_+ = -B^2_+$:

$$\begin{aligned}
A^2_+ &= -\frac{1}{4}{}^1\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F,u,2}_{e_i} + \frac{K}{16}\\
&\quad + \frac{1}{8}c(e_i)\,c(e_j)\,\mathcal{R}(e_i, e_j) + \frac{1}{2}f^\alpha f^\beta \mathcal{R}\left(f^H_\alpha, f^H_\beta\right) + \frac{1}{2}c(e_i)\,f^\alpha \mathcal{R}\left(e_i, f^H_\alpha\right)\\
&\quad + \frac{1}{16}\left[\omega\left(\nabla^F, g^F\right)(e_i)\right]^2 - \frac{1}{4}f^\alpha \widehat{c}(e_i)\,\nabla^{TX\otimes F,u}_{f^H_\alpha}\omega\left(\nabla^F, g^F\right)(e_i)\\
&\quad + \frac{1}{32}\widehat{c}(e_i)\,\widehat{c}(e_j)\,\omega\left(\nabla^F, g^F\right)^2(e_i, e_j) - \frac{1}{8}c(e_i)\,\widehat{c}(e_j)\,\nabla^{TX\otimes F,u}_{e_i}\omega\left(\nabla^F, g^F\right)(e_j).
\end{aligned} \tag{4.186}$$

A corresponding formula was also established in [BG01, Theorem 3.20, eq. (3.60)]. However the term $\frac{1}{4}f_\alpha \widehat{c}(e_i)\,\omega\left(\nabla^F, g^F\right)\left(T\left(f^H_\alpha, e_i\right)\right)$ is missing in the right-hand side of that equation.

First we simplify equation (4.186).

Theorem 4.55. *The following identity holds:*

$$
\begin{aligned}
(4.187)\quad A_+^2 = \frac{1}{4}\Big(-{}^1\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,2} + \big\langle e_i, R^{TX}e_j\big\rangle \widehat{c}(e_i)\widehat{c}(e_j)\Big) \\
+ \frac{1}{4}\big\langle R^{TX}(\cdot,e_i)e_i, e_j\big\rangle \widehat{c}(e_j) - \frac{1}{4}\nabla^F\omega\left(\nabla^F,g^F\right)(e_i)\widehat{c}(e_i) \\
- \frac{1}{4}\omega\left(\nabla^F,g^F\right)(e_i)\,{}^1\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F} - \frac{1}{4}\omega\left(\nabla^F,g^F\right)^2.
\end{aligned}
$$

Proof. First we will assume that $\omega\left(\nabla^F,g^F\right)=0$. Let S^{TX} be the fibrewise Ricci tensor. Then a repeated application of the circular identity and the $(2,2)$ symmetry of the Levi-Civita curvature shows that

$$
\begin{aligned}
(4.188)\quad \frac{1}{32}\big\langle e_k, R^{TX}(e_i,e_j)e_l\big\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l) + \frac{K}{16} \\
= \frac{1}{8}\big\langle e_k, R^{TX}(e_i,e_j)e_l\big\rangle e^ie^j i_{e_k}i_{e_l} + \frac{1}{4}\big\langle S^{TX}e_i,e_j\big\rangle e^i i_{e_j}.
\end{aligned}
$$

Similar arguments show that the term of degree 0 in $\Lambda^\cdot(T^*S)$ in the right-hand side of (4.187) is just (4.188). In degree 2, equation (4.187) holds. So we are left with the term of degree 1 in $\Lambda^\cdot(T^*S)$. Comparison of (4.186) and (4.188) shows that this is equivalent to the fact that for any $A\in TS$,

$$
(4.189)\quad \frac{1}{8}\big\langle R^{TX}\left(A^H,e_i\right)e_j,e_k\big\rangle \widehat{c}(e_i)\widehat{c}(e_j)\widehat{c}(e_k) = \frac{1}{4}\big\langle R^{TX}\left(A^H,e_i\right)e_i,e_j\big\rangle \widehat{c}(e_j).
$$

To establish (4.189), we use the symmetry for T in (4.38) and also (4.143). The sum in the left-hand side of (4.189) for distinct triples i,j,k vanishes identically. So we are left with the sum restricted to $i=j$ or $i=k$, which leads to (4.189). So we have established (4.187) when $\omega\left(\nabla^F,g^F\right)=0$.

Now we consider the general case. Of course, we take into account what was done before. In degree 0, what is left to prove is that the identity

$$
\begin{aligned}
(4.190)\quad \frac{1}{32}\left(-c(e_i)c(e_j)+\widehat{c}(e_i)\widehat{c}(e_j)\right)\omega\left(\nabla^F,g^F\right)^2(e_i,e_j) \\
- \frac{1}{8}c(e_i)\widehat{c}(e_j)\nabla_{e_i}^{F,u}\omega\left(\nabla^F,g^F\right)(e_j) = -\frac{1}{4}\nabla_{e_i}^F\omega\left(\nabla^F,g^F\right)(e_j)e^i\widehat{c}(e_j) \\
- \frac{1}{8}e^ie^j\omega\left(\nabla^F,g^F\right)^2(e_i,e_j),
\end{aligned}
$$

which follows from (2.53). In degree 2, identity (4.186) is trivial. In degree 1, our identity is equivalent to

$$
\begin{aligned}
(4.191)\quad -\frac{1}{8}\omega\left(\nabla^F,g^F\right)^2\left(f_\alpha^H,e_i\right)f^\alpha c(e_i) - \frac{1}{4}\nabla_{f_\alpha^H}^{TX\otimes F,u}\omega\left(\nabla^F,g^F\right)(e_i)f^\alpha\widehat{c}(e_i) \\
= -\frac{1}{4}\nabla_{f_\alpha^H}^F\omega\left(\nabla^F,g^F\right)(e_i)f^\alpha\widehat{c}(e_i) - \frac{1}{4}\omega\left(\nabla^F,g^F\right)^2\left(f_\alpha^H,e_i\right)f^\alpha e^i.
\end{aligned}
$$

Now (4.191) follows from the identity (2.51). The proof of our theorem is completed. □

When F is replaced by $F\otimes o(TX)$, there is a corresponding formula for $A_-^2 = -B_-^2$, by simply replacing F by $F\otimes o(TX)$ in the right-hand side of (4.187).

To the vector bundle T^*X on M, equipped with the metric g^{TX} and the connection ∇^{T^*X}, we can associate the projection operators $Q_\pm^{T^*X}$ which were defined in Theorem 3.11. Recall that we implicitly tensor all the objects which we will

consider by $\Lambda^{\cdot}(T^*S)$. We identify $\Omega^{\cdot}(X, F|_X)$ (resp. $\Omega^{\cdot}(X, F\otimes o(TX)|_X)$) with its image in $\Omega^{\cdot}(T^*X, \pi^*F|_{T^*X})$ by the map $\alpha\to\pi^*\alpha$ (resp. $\alpha\to\pi^*\alpha\wedge\Phi^{T^*X}$). As in subsection 3.7, $Q_+^{T^*X}$ (resp $Q_-^{T^*X}$) is a left inverse to this embedding.

Now we have the obvious extension of Theorem 3.13.

Theorem 4.56. *The following identity holds:*

$$Q_\pm^{T^*X}\left(\mathfrak{c}'-\mathfrak{b}'_\pm\mathfrak{a}'^{-1}_\pm\mathfrak{b}'_\pm\right)Q_\pm^{T^*X}=2A_\pm^2. \tag{4.192}$$

Proof. Observe that by (4.174), (4.175), (4.182) and (4.184), we have the formula,

$$\begin{aligned}\mathfrak{b}'_+=-\Bigg({}^1\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}_{Y^{\mathcal{H}}}&+\widehat{e}_i\widehat{c}(e_i)+i_{\widehat{R^{T^*X}(p,\cdot)p}}\\&+\frac{1}{2}\omega\left(\nabla^F,g^F\right)(e_i)\nabla_{\widehat{e}^i}+\frac{1}{2}\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}\widehat{\omega}'\left(\nabla^F,g^F\right)\Bigg).\end{aligned} \tag{4.193}$$

This formula should be compared with equations (3.12), (3.15) for $\mathfrak{b}_+$. In fact, the only substantial difference is that $\widehat{e}_i i_{e_i}$ has been replaced by $\widehat{e}_i\widehat{c}(e_i)$. By making the appropriate changes, we can then proceed as in the proof of Theorem 3.13. Using Theorem 4.55, we get (4.192) in the + case. The proof in the − case is similar. □

Now we will extend Theorem 3.14. Again we fix c, so that $dc=0$. By Remark 4.39, we know how to obtain $\mathfrak{E}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}$ from $\mathfrak{C}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}$ given by (4.123). Inspection of (4.123) shows that for $c>0$,

$$K_{1/\sqrt{c}}2\mathfrak{E}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}K_{\sqrt{c}}=c\alpha'_++\sqrt{c}\beta'_++\gamma'_+. \tag{4.194}$$

A similar identity holds for $c<0$, the index + being replaced by −.

Theorem 4.57. *The following identity holds:*

$$P_\pm\left(\gamma'_\pm-\beta'_\pm\alpha'^{-1}_\pm\beta'_\pm\right)P_\pm=2A_\pm^2. \tag{4.195}$$

Proof. One can give a direct proof of (4.195) similar to the proof of Theorem 3.14. A more direct procedure is to note that up to a constant, (4.194) is obtained by squaring (4.110) while making $dc=0$. Using Proposition 4.36, the proof now continues as the proof of Theorem 3.14 given in Remark 3.15. □

Remark 4.58. Theorems 4.56 and 4.57 will play the same role as Theorems 3.13 and 3.14 in showing that as $c\to\pm\infty$, $A^{\mathcal{M},2}_{\phi,2(\mathcal{H}^c-\omega^H)}$ converges in the appropriate sense to $A_\pm^2$.

4.24. **A final conjugation.** We do not assume any longer that c is fixed. For $c\neq 0$, set

$$\widehat{\underline{\mathbb{C}}}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}=\exp\left(-\frac{dc}{c}\widehat{p}\right)\widehat{\underline{\mathfrak{C}}}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H}\exp\left(\frac{dc}{c}\widehat{p}\right). \tag{4.196}$$

Other operators, which are obtained by the same conjugation as in (4.196), will be denoted in the same way.

Theorem 4.59. *For $c \neq 0$, the following identities hold:*

(4.197)

$$\begin{aligned}
&\widehat{\underline{\mathbb{C}}}^{\mathcal{M},2}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{4}\left(-\left(\nabla_{\widehat{e}^i}+\frac{dc}{c}\widehat{e}^i\right)^2 + c^2\left|p\right|^2 + c\left(2i_{\widehat{e}_i}\widehat{e}^i - n\right)\right) + \frac{1}{4}\left\langle e_i, R^{TX}e_j\right\rangle \widehat{e}^i\widehat{e}^j \\
&-\frac{1}{4}\omega\left(\nabla^F,g^F\right)(e_i)\left(\nabla_{\widehat{e}^i}+\frac{dc}{c}\widehat{e}^i\right) - \frac{1}{4}\nabla^{\Lambda^\cdot(T^*T^*X)\widehat{\otimes}F}\widehat{\omega}\left(\nabla^F,g^F\right) \\
&-\frac{1}{4}\omega\left(\nabla^F,g^F\right)^2 - \frac{c}{2}\left(\mathcal{L}_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)(p) + \underline{A}'^{\mathcal{M}}_{\dagger}\theta\right), \\
&\widehat{\underline{\mathbb{G}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} = \frac{1}{2}\omega\left(\nabla^F,g^F\right) - \frac{1}{4}\widehat{\omega}\left(\nabla^F,g^F\right) - \frac{c}{2}\left(\widehat{p}+6i_{\widehat{p}}\right) + \frac{3}{2}dc\left|p\right|^2, \\
&\widehat{\underline{\mathbb{G}}}^{\mathcal{M}}_{\phi,\mathcal{H}^c-\omega^H} + \left[\widehat{\underline{\mathbb{C}}}'^{\mathcal{M}}_{\mathcal{H}^c-\omega^H}, \frac{3c}{2}\left|p\right|^2\right] = \frac{1}{2}\omega\left(\nabla^F,g^F\right) - \frac{1}{4}\widehat{\omega}\left(\nabla^F,g^F\right) - \frac{c}{2}\widehat{p}.
\end{aligned}$$

Proof. Using (4.176)–(4.178), we get the first identity in (4.197). The other two identities follow from Theorem 4.45. □

4.25. **A Weitzenböck formula for a Killing vector field.** Now let K be a smooth section of TX. We assume that K is a fibrewise Killing vector field with respect to the metric g^{TX} and also that K preserves T^HM. Note that K preserves the connection ∇^{TX} on TX, and more generally all the objects which were considered before, including the tensor T. Moreover the tensor along the fibre $\nabla^{TX}K \in \mathrm{End}\left(TX\right)$ is antisymmetric.

Let $\mathcal{H}_K$ be the fibrewise hamiltonian canonically associated to the fibrewise action of K on T^*X. Clearly

$$\mathcal{H}_K = \left\langle p, K\right\rangle. \tag{4.198}$$

Let $Y^{\mathcal{H}_K}$ be the Hamiltonian vector field along T^*X associated to the Hamiltonian $\mathcal{H}_K$. Then $Y^{\mathcal{H}_K}$ is the natural lift of K as a vector field on T^*X. Let $L_{Y^{\mathcal{H}_K}}$ be the corresponding fibrewise Lie derivative operator. This operator acts naturally on the sections of the corresponding fibrewise exterior algebras $\Lambda^\cdot\left(T^*X\right)$. Note here that since K preserves T^HM, there is no point in distinguishing the fibrewise Lie derivative operator $L_{Y^{\mathcal{H}_K}}$ from the total Lie derivative operator $\mathcal{L}_{Y^{\mathcal{H}_K}}$.

Proposition 4.60. *The following identity holds:*

$$L_{Y^{\mathcal{H}_K}} = \nabla^{\Lambda^\cdot(T^*T^*X)}_{Y^{\mathcal{H}_K}} + \left\langle \nabla^{TX}_{e_i}K, e_j\right\rangle\left(e^i i_{e_j} + \widehat{e}_i i_{\widehat{e}^j}\right). \tag{4.199}$$

Proof. By proceeding as in the proof of (3.12) in Theorem 3.4, we get

$$\begin{aligned}
L_{Y^{\mathcal{H}_K}} = \nabla^{\Lambda^\cdot(T^*T^*X)}_{Y^{\mathcal{H}_K}} &+ \left\langle \nabla^{TX}_{e_i}K, e_j\right\rangle\left(e^i i_{e_j} + \widehat{e}_i i_{\widehat{e}^j}\right) \\
&+ \left\langle \nabla^{TX}_{e_i}\nabla^{TX}_p K + R^{TX}\left(K, e_i\right)p, e_j\right\rangle e^i i_{\widehat{e}^j}.
\end{aligned} \tag{4.200}$$

Also since K is Killing, K preserves the Levi-Civita connection along the fibres, and so classically, we have the identity of fibrewise tensors [BeGeV92, eq. (7.4)]

$$\nabla^{TX}_{\cdot}\nabla^{TX}K + R^{TX}\left(K,\cdot\right) = 0. \tag{4.201}$$

By (4.200), (4.201), we get (4.199). □

Observe that the operator $L_{Y^{\mathcal{H}_K}}$ acts naturally on smooth sections of

$$\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^n(TX)\widehat{\otimes}F.$$

Indeed by (4.200), we find that this action is given by

$$L_{Y^{\mathcal{H}_K}} = \nabla^{TX}_{Y^{\mathcal{H}_K}} + \left\langle \nabla^{TX}_{e_i}K, e_j\right\rangle \left(e^i i_{e_j} + \widehat{e}^i i_{\widehat{e}_j}\right). \tag{4.202}$$

We claim that the action $L_{Y^{\mathcal{H}_K}}$ is unchanged considering the more complicated action defined in subsection 4.21 via the conjugation by χ. Indeed this is obvious by (4.199) and by the antisymmetry of $\nabla^{TX}K$. Similarly, the operator $L_{Y^{\mathcal{H}_K}}$ in (4.202) is invariant by conjugation by $\exp\left(\left\langle T^0, p\right\rangle\right)$ as in (4.146). Indeed an explicit computation shows that this is because the tensor T^0 is itself K-invariant.

Let K' be the 1-form along the fibres X which is dual to K by the metric g^{TX}. We extend K' to a vertical 1-form on the total space of TX over M. Equivalently, K' vanishes on the horizontal vector fields. Then we write K' in the form

$$K' = \langle K, e_i\rangle e^i. \tag{4.203}$$

Similarly, set

$$\widehat{K} = \left\langle K, e^i\right\rangle \widehat{e}_i, \qquad \widehat{K'} = \langle K, e_i\rangle \widehat{e}^i. \tag{4.204}$$

Theorem 4.61. *The following identity holds:*

$$\begin{aligned}\widehat{\underline{\mathfrak{C}}}^{\mathcal{M},2}_{\phi,2\mathcal{H}_K-\omega^H} &= \frac{1}{4}\left(-\Delta^V + \left\langle e_i, R^{TX}e_j\right\rangle \widehat{e}^i\widehat{e}^j\right) - \frac{1}{4}\omega\left(\nabla^F, g^F\right)(e_i)\nabla_{\widehat{e}^i} \\ &- \frac{1}{4}\nabla^{\Lambda^{\cdot}(T^*T^*X)\widehat{\otimes}F}\widehat{\omega}\left(\nabla^F, g^F\right) - \frac{1}{4}\omega\left(\nabla^F, g^F\right)^2 - \left(dK' - |K|^2\right) + \nabla^{\Lambda^{\cdot}(T^*X)}\widehat{K'} \\ &- \left(L_{Y^{\mathcal{H}_K}} + \frac{1}{2}\omega\left(\nabla^F, g^F\right)(K)\right).\end{aligned} \tag{4.205}$$

Proof. We just give here the main steps of our computation. By Theorem 4.52 used with $c = 0$, (4.205) holds when $K = 0$. To establish (4.205) in full generality, first, we use Theorem 4.37 with $\mathcal{H} = 2\mathcal{H}_K$ and also the fact that with the notation in (4.117),

$$\nabla_{f^H_\alpha}\mathcal{H}_K = 0. \tag{4.206}$$

In fact (4.206) holds because T^HM is K-invariant so that if U is a smooth vector field on S, $\left[U^H, K\right] = 0$. We find that the contribution of $\mathcal{H}_K$ to the right-hand side of (4.117) is just

$$\begin{aligned}&-\left(L_{Y^{\mathcal{H}_K}} + \frac{1}{2}\omega\left(\nabla^F, g^F\right)(K)\right) + \left\langle \nabla_{e_i}K, e_j\right\rangle e^i i_{\widehat{e}_j} \\ &\qquad - \left\langle T\left(f^H_\alpha, K\right), e_i\right\rangle f^\alpha\left(e^i - i_{\widehat{e}^i}\right) - \left\langle T^H, K\right\rangle + |K|^2,\end{aligned} \tag{4.207}$$

where $L_{Y^{\mathcal{H}_K}}$ is given by (4.199).

As explained before, when making the transformations of subsection 4.21, $L_{Y^{\mathcal{H}_K}}$ is now given by (4.202). So we find that (4.207) becomes

$$\begin{aligned}&-\left(L_{Y^{\mathcal{H}_K}} + \frac{1}{2}\omega\left(\nabla^F, g^F\right)(K)\right) - \left\langle \nabla^{TX}_{e_i}K, e_j\right\rangle e^ie^j - 2\left\langle T\left(f^H_\alpha, K\right), e_i\right\rangle f^\alpha e^i \\ &\qquad - \left\langle T^H, K\right\rangle + |K|^2 + \left\langle \nabla^{TX}_{e_i}K, e_j\right\rangle e^i\widehat{e}^j + \left\langle T\left(f^H_\alpha, K\right), e_i\right\rangle f^\alpha\widehat{e}^i.\end{aligned} \tag{4.208}$$

Now since $\left[f_\alpha^H, K\right] = 0$, we find that

$$\nabla^{TX}_{f_\alpha^H} K = T\left(f_\alpha^H, K\right). \tag{4.209}$$

Using (4.148), (4.149) and (4.209), we get

$$\begin{aligned} dK' &= \left\langle \nabla^{TX}_{e_i} K, e_j \right\rangle e^i e^j + 2\left\langle T\left(f_\alpha^H, K\right), e_i \right\rangle f^\alpha e^i + \left\langle T^H, K \right\rangle, \\ \nabla^{TX} \widehat{K'} &= \left\langle \nabla^{TX}_{e_i} K, e_j \right\rangle e^i \widehat{e}^j + \left\langle T\left(f_\alpha^H, K\right), e_i \right\rangle f^\alpha \widehat{e}^i. \end{aligned} \tag{4.210}$$

By (4.210), we get a formula for $\underline{\mathfrak{C}}^{\mathcal{M},2}_{2\mathcal{H}_K - \omega^H}$ as in (4.140). As we saw following (4.202), conjugation by $\exp\left(\left\langle T^0, p\right\rangle\right)$ leaves $L_{Y^{\mathcal{H}_K}}$ unchanged. This is also the case for the remaining terms where K appears. The proof of our theorem is completed. □

Remark 4.62. Let us observe here that the operator $L_{Y^{\mathcal{H}_K}}$ commutes with all the terms not containing $\omega\left(\nabla^F, g^F\right)$. Also note that except for $L_{Y^{\mathcal{H}_K}}$, the terms which appear in the right-hand side of (4.205) only contain creation operators.

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