ERRATA TO
“TOTALLY POSITIVE TOEPLITZ MATRICES AND QUANTUM COHOMOLOGY OF PARTIAL FLAG VARIETIES”

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Correction to the Proof of Theorem 4.2. The text in [38] from Remark 4.3 at the bottom of p. 374 to line 8 on p. 375 should be replaced with the following. We use the same notation as in [38].

Remark 4.3. If $P$ is the parabolic subgroup, then $G^m_j$ is a well-defined (regular) function on the Bruhat cell $B^+w_PB^-/B^-$ precisely in the case $m \in I^P = \{n_1, \ldots, n_k\}$.

Proof of Theorem 4.2. (1) is proved in [33]. See also Lemma 2.3 in [35]. We will deduce (2) very explicitly from the ASK presentation. We begin by defining a particular system of coordinates on the affine space $B^+w_PB^-/B^-$. For indexing purposes introduce sets $\Omega, \Omega_1, \Omega_2$ defined by

$$
\Omega := \{(r, m) \in \mathbb{Z}^2 \mid n_1 \leq r < n, \text{ and } 1 \leq m \leq n_1 \text{ if } n_1 \leq r < n_{1+1}\},
$$

$$
\Omega_1 := \{(n_l, m) \in \mathbb{Z}^2 \mid \text{ where } l \in \{1, \ldots, k\} \text{ and } 1 \leq m \leq n_l\},
$$

and $\Omega_2 := \Omega \setminus \Omega_1$. Consider the polynomial rings $\mathbb{C}[\Omega] := \mathbb{C}[g_r^m; (r, m) \in \Omega]$ and $\mathbb{C}[\Omega_i] := \mathbb{C}[g_r^m; (r, m) \in \Omega_i]$ for $i = 1, 2$. We have $n \times (n_{l+1} - n_l)$-matrices $U^{(l)}_{\Omega_1}$ over $\mathbb{C}[\Omega_1]$ defined by

$$
U^{(0)}_{\Omega_1} = \begin{pmatrix}
1 & g_1^{n_1} & g_2^{n_1} & \cdots & g_{n_1}^{n_1 - 1} \\
0 & 1 & g_1^{n_1} & \cdots & \\
& & \ddots & \ddots & \\
& & & \ddots & g_1^{n_1} \\
& & & & 1
\end{pmatrix},
$$

$$
U^{(l)}_{\Omega_1} = \begin{pmatrix}
g_{n_1}^{n_1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
& & g_1^{n_1} & \cdots \\
0 & \cdots & 1 & g_1^{n_1} \\
0 & \cdots & \cdots & 0
\end{pmatrix},
$$

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611
where $1 \leq l \leq k$. Furthermore we define $n \times (n_{l+1} - n_l)$-matrices $U_{\Omega_2}^{(l)}$ over $\mathbb{C}[\Omega_2]$ by

$$U_{\Omega_2}^{(l)} = \begin{pmatrix}
0 & g_{n_l}^{n_{l+1}} & g_{n_l}^{n_{l+2}} & \ldots & g_{n_l}^{n_{l+1}} - 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & g_{2}^{n_{l+1}} & g_{2}^{n_{l+2}} & \ldots & g_{2}^{n_{l+1}} - 1 \\
0 & g_{1}^{n_{l+1}} & g_{1}^{n_{l+2}} & \ldots & g_{1}^{n_{l+1}} - 1 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0
\end{pmatrix},$$

for $1 \leq l \leq k$ and $U_{\Omega_2}^{(0)} = 0$. The matrices defined above combine to $n \times n$ matrices

$$u_{\Omega_i} = \begin{pmatrix}
U_{\Omega_i}^{(0)} & U_{\Omega_i}^{(1)} & \cdots & U_{\Omega_i}^{(k)}
\end{pmatrix},$$

for $i = 1, 2$. Moreover we have

$$u_{\Omega} := u_{\Omega_1} + u_{\Omega_2},$$

which is an element of $U^+$ over $\mathbb{C}[\Omega]$, or equivalently a morphism $u_{\Omega} : \mathbb{C}[\Omega] \to U^+$. Composing $u_{\Omega}$ with the standard projection $U^+ \to B^+ w_P B^- / B^-$ defines a map

$$\mathbb{C}[\Omega] \to B^+ w_P B^- / B^-;$$

$$z \mapsto u_{\Omega}(z)w_P B^-.$$

It is clear that this map is an isomorphism. So we may use it to identify the coordinate ring $\mathbb{C}[B^+ w_P B^- / B^-]$ with $\mathbb{C}[\Omega]$. Note that $G_j^{m_j}$ goes to $g_j^{m_j}$ under this identification. Let $\mathcal{I}_P \subset \mathbb{C}[\Omega]$ denote the defining ideal for $\mathcal{Y}_P$.

**Claim.** We have $g_{m}^{r} \in \mathcal{I}_P$ for all $(r, m) \in \Omega_2$, or equivalently $u_{\Omega_2} \equiv 0 \mod \mathcal{I}_P$.

Let $v_1, \ldots, v_n$ be the standard basis of $\mathbb{C}^n$ and $(,)$ the bilinear form given by $(v_i, v_j) = \delta_j^i$. From the definition of the Peterson variety it follows that the ideal $\mathcal{I}_P$ is generated by the elements

$$(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_h) \in \mathbb{C}[\Omega],$$

where $(r, h) \in \Omega_2$ or $r = n_{l+1}$ and $h \in [1, n_l] \setminus \{n_{l-1} + 1\}$, for $l = 1, \ldots, k$. Let $<$ be the lexicographical ordering on $\Omega_2$. We will prove the claim recursively. Consider $(r, m) \in \Omega_2$. Then there is an $l \in \{1, \ldots, k\}$ such that $n_l < r < n_{l+1}$ and $1 \leq m \leq n_l$, and we consider the generator

$$(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_{n_l - m + 1}) \in \mathcal{I}_P.$$
Note first that \( n_l < r < n_{l+1} \) implies \( f u_{\Omega_1} \cdot v_r = u_{\Omega_1} \cdot v_{r+1} \). Therefore
\[
(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_{n_l - m + 1}) = (u_{\Omega}^{-1} f u_{\Omega_1} \cdot v_r, v_{n_l - m + 1}) + (u_{\Omega}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l - m + 1})
\]
\[
= (u_{\Omega}^{-1} u_{\Omega_1} \cdot v_{r+1}, v_{n_l - m + 1}) + (u_{\Omega}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l - m + 1})
\]
\[
= 0 - (u_{\Omega}^{-1} u_{\Omega_2} \cdot v_{r+1}, v_{n_l - m + 1}) + (u_{\Omega}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l - m + 1})
\].
Now
\[
u_{\Omega_2} \cdot v_r = \begin{cases} 
0 & \text{if } r = n_l + 1, \\
g_r^{-1} v_1 + g_r^{-1} v_2 + \ldots + g_r^{-1} v_{n_l} & \text{otherwise.}
\end{cases}
\]
Therefore
\[
(u_{\Omega}^{-1} f u_{\Omega_2} \cdot v_r, v_{n_l - m + 1}) \equiv 0 \mod (g_h^r)_{(t,h)<(r,m)}.
\]
Since \( u_{\Omega_2} \cdot v_{r+1} = g_r v_1 + g_r^{-1} v_2 + \ldots + g_r^{-1} v_{n_l} \) and \( u_{\Omega} \) is upper-triangular, we have all in all
\[
(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_{n_l - m + 1}) \equiv - \sum_{j=1}^m g_j^r (u_{\Omega}^{-1} \cdot v_{n_l - j+1}, v_{n_l - m + 1}) + 0
\]
\[
\equiv -g_m^r (u_{\Omega}^{-1} \cdot v_{n_l - m+1}, v_{n_l - m+1}) \equiv -g_m^r \mod (g_h^r)_{(t,h)<(r,m)}.
\]
For the minimal element \((n_l + 1, 1)\) in \( \Omega_2 \) in particular this implies
\[
(u_{\Omega}^{-1} f u_{\Omega} \cdot v_{n_l+1}, v_{n_l}) = -g_1^{n_l+1},
\]and so \( g_1^{n_l+1} \) lies in \( I_P \). By induction it follows that all \( g_m^r \) for \( (r,m) \in \Omega_2 \) lie in \( I_P \), and the claim is proved.

We have thus shown that \( \mathcal{Y}_P \) lies in the subvariety \( \mathcal{Y}_{\Omega_2} \) of \( B^+ w_P B^- / B^- \) defined by the ideal \( I_{\Omega_2} = (g_h^r)_{(t,h) \in \Omega_2} \). Its coordinate ring \( \mathcal{O}(\mathcal{Y}_{\Omega_2}) = \mathbb{C}[\Omega] / I_{\Omega_2} \) can be identified with the polynomial ring \( \mathbb{C}[g_i^h; (t,h) \in \Omega_1] \), or equivalently with \( \mathbb{C}[G_1^{n_1}, \ldots, G_{n_k}^{n_k}] \), where the \( G_j^m \) now denote restrictions to the affine space \( \mathcal{Y}_{\Omega_2} \) of the functions defined in (4.4). We furthermore let \( u \) be the restriction of \( u_{\Omega} \) to \( \mathcal{Y}_{\Omega_2} \), viewing \( u \) as an element of \( U^+ (\mathbb{C}[G_1^{n_1}, \ldots, G_{n_k}^{n_k}]) \). Also, we let \( J_P \) denote the ideal defining \( \mathcal{Y}_P \) inside \( \mathbb{C}[G_1^{n_1}, \ldots, G_{n_k}^{n_k}] \).

With this, the rest of the proof proceeds as in paragraph 2 on p. 375 in [38]. Only on line 14 of p. 375 we also need to change the morphism \( B^+ w_P B^- / B^- \rightarrow \mathfrak{g} \) to \( \mathfrak{gl}_n \).

**Further minor corrections.**

(1) In (3.4) and (3.6) replace the exponent “\( n_1 \)” with “\( n_2 - n_1 \)” and the exponent “\( n_k - n_{k-1} \)” with “\( n - n_k \)”. Similarly for the subscripts in the displayed equation after (3.4).

(2) In line 2 of paragraph 5 of Section 3.4 replace “at most \( (n_j - n_{j-1}) \) parts” with “at most \( (n_{j+1} - n_j) \) parts”.

(3) In line 6 of Section 3.7 replace \( \lambda_i \) with \( \lambda_{d-i+1} \).

(4) Replace \( t \) with \( t^{-1} \) everywhere in displayed equation (5.2).

(5) In point (6) of Section 8 replace “\( q(u(x_1, \ldots, x_n)) \)” with “\( q(u(x_1, \ldots, x_d)) \)”.

(6) In the second displayed equation in the proof of Lemma 8.1, insert exponents:
\[
\sigma_{w_P}^P = (\sigma_{s_1 \cdots s_{n_1}}^{P})^{n_2 - n_1} \cdot (\sigma_{s_1 \cdots s_{n_2}}^{P})^{n_3 - n_2} \cdots (\sigma_{s_1 \cdots s_{n_k}}^{P})^{n - n_k}.
\]
(7) Two paragraphs down from (6) in Section 8, add the words “in type A” to read: “Peterson has announced in [33] that all the quantum cohomology rings $qH^*(G/P)$ in type $A$ are reduced.”

(8) The statement of Corollary 11.4(2) should be replaced with: “If $y \in X_{P,>0}$ and $\sigma_w^P(y) \geq 0$ for all $w \in W_P$, then $q_i^P(y) > 0$ for all $i = 1, \ldots, k$.”

(9) The first line of the proof of Proposition 12.2 should read: “By (5.1) and Lemma 12.1 we have”.

Acknowledgements

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References

