ERRATA TO
“TOTALLY POSITIVE TOEPLITZ MATRICES AND QUANTUM COHOMOLOGY OF PARTIAL FLAG VARIETIES”

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Correction to the Proof of Theorem 4.2. The text in [38] from Remark 4.3 at the bottom of p. 374 to line 8 on p. 375 should be replaced with the following. We use the same notation as in [38].

Remark 4.3. If $P$ is the parabolic subgroup, then $G^m_j$ is a well-defined (regular) function on the Bruhat cell $B^+w_P B^-/B^-$ precisely in the case $m \in I^P = \{n_1, \ldots, n_k\}$.

Proof of Theorem 4.2. (1) is proved in [33]. See also Lemma 2.3 in [35]. We will deduce (2) very explicitly from the ASK presentation. We begin by defining a particular system of coordinates on the affine space $B^+w_P B^-/B^-$. For indexing purposes introduce sets $\Omega$, $\Omega_1$, $\Omega_2$ defined by

\[ \Omega := \{(r, m) \in \mathbb{Z}^2 \mid n_1 \leq r < n, \text{ and } 1 \leq m \leq n_l \text{ if } n_l \leq r < n_{l+1}\}, \]

\[ \Omega_1 := \{(n_l, m) \in \mathbb{Z}^2 \mid \text{where } l \in \{1, \ldots, k\} \text{ and } 1 \leq m \leq n_l\}, \]

and $\Omega_2 := \Omega \setminus \Omega_1$. Consider the polynomial rings $\mathbb{C}[\Omega] := \mathbb{C}[g^r_m; (r, m) \in \Omega]$ and $\mathbb{C}[\Omega_i] := \mathbb{C}[g^r_m; (r, m) \in \Omega_i]$ for $i = 1, 2$. We have $n \times (n_{l+1} - n_l)$-matrices $U^{(l)}_{\Omega_1}$ over $\mathbb{C}[\Omega_1]$ defined by

\[
U^{(0)}_{\Omega_1} = \begin{pmatrix}
1 & g^1_{n_1} & g^2_{n_1} & \cdots & g^{n_1}_{n_1-1} \\
0 & 1 & g^1_{n_2} & \cdots & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & \cdots & \cdots & 1 & \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
\]

\[
U^{(l)}_{\Omega_1} = \begin{pmatrix}
g^{n_l}_{n_l} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
\vdots & \cdots & \ddots & \cdots \\
0 & \cdots & \cdots & g^1_{n_l} \\
0 & \cdots & \cdots & \cdots
\end{pmatrix}.
\]

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where $1 \leq l \leq k$. Furthermore we define $n \times (n_{l+1} - n_l)$-matrices $U^{(l)}_{\Omega_2}$ over $\mathbb{C}[\Omega_2]$ by

$$U^{(l)}_{\Omega_2} = \begin{pmatrix} 0 & g_{n_1}^{n_{l+1}} & g_{n_1}^{n_{l+2}} & \cdots & g_{n_1}^{n_{l+1}-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n_2}^{n_{l+1}} & g_{n_2}^{n_{l+2}} & \cdots & g_{n_2}^{n_{l+1}-1} \\ 0 & g_{n_1}^{n_{l+1}} & g_{n_1}^{n_{l+2}} & \cdots & g_{n_1}^{n_{l+1}-1} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

for $1 \leq l \leq k$ and $U^{(0)}_{\Omega_2} = 0$. The matrices defined above combine to $n \times n$ matrices $u_{\Omega_i}$ by

$$u_{\Omega_i} = \begin{pmatrix} U^{(0)}_{\Omega_i} & U^{(1)}_{\Omega_i} & \cdots & U^{(k)}_{\Omega_i} \end{pmatrix},$$

for $i = 1, 2$. Moreover we have

$$u_{\Omega} := u_{\Omega_1} + u_{\Omega_2},$$

which is an element of $U^+$ over $\mathbb{C}[\Omega]$, or equivalently a morphism $u_{\Omega} : \mathbb{C}[\Omega] \to U^+$. Composing $u_{\Omega}$ with the standard projection $U^+ \to B^+w_PB^-/B^-$ defines a map

$$\mathbb{C}[\Omega] \to B^+w_PB^-/B^-, \quad z \mapsto u_{\Omega}(z)w_PB^-.$$

It is clear that this map is an isomorphism. So we may use it to identify the coordinate ring $\mathbb{C}[B^+w_PB^-/B^-]$ with $\mathbb{C}[\Omega]$. Note that $G_{j_1}^{m_1}$ goes to $g_{j_1}^{m_1}$ under this identification. Let $\mathcal{I}_P \subset \mathbb{C}[\Omega]$ denote the defining ideal for $\mathcal{Y}_P$.

**Claim.** We have $g_{m}^{m} \in \mathcal{I}_P$ for all $(r, m) \in \Omega_2$, or equivalently $u_{\Omega_2} \equiv 0 \mod \mathcal{I}_P$.

Let $v_1, \ldots, v_n$ be the standard basis of $\mathbb{C}^n$ and $( , )$ the bilinear form given by $(v_i, v_j) = \delta_{ij}$. From the definition of the Peterson variety it follows that the ideal $\mathcal{I}_P$ is generated by the elements

$$(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_h) \in \mathbb{C}[\Omega], \quad \text{where } (r, h) \in \Omega_2 \text{ or } r = n_{l+1} \text{ and } h \in [1, n_l] \setminus \{n_{l-1} + 1\},$$

for $l = 1, \ldots, k$. Let $<$ be the lexicographical ordering on $\Omega_2$. We will prove the claim recursively. Consider $(r, m) \in \Omega_2$. Then there is an $l \in \{1, \ldots, k\}$ such that $n_l < r < n_{l+1}$ and $1 \leq m \leq n_l$, and we consider the generator

$$(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_{n_l-m+1}) \in \mathcal{I}_P.$$
Note first that \( n_l < r < n_{l+1} \) implies \( fu_{\Omega_1} \cdot v_r = u_{\Omega_{l+1}} \cdot v_{r+1} \). Therefore

\[
(u_{\Omega}^{-1} fu_{\Omega} \cdot v_r, v_{n_l-m+1}) = (u_{\Omega}^{-1} fu_{\Omega_1} \cdot v_r, v_{n_l-m+1}) + (u_{\Omega}^{-1} fu_{\Omega_2} \cdot v_r, v_{n_l-m+1})
\]

\[
= (u_{\Omega}^{-1} u_{\Omega_1} \cdot v_{r+1}, v_{n_l-m+1}) + (u_{\Omega}^{-1} fu_{\Omega_2} \cdot v_r, v_{n_l-m+1})
\]

\[
= 0 - (u_{\Omega}^{-1} u_{\Omega_2} \cdot v_{r+1}, v_{n_l-m+1}) + (u_{\Omega}^{-1} fu_{\Omega_2} \cdot v_r, v_{n_l-m+1}).
\]

Now

\[
u_{\Omega_2} \cdot v_r = \begin{cases} 
0 & \text{if } r = n_l + 1, \\
g_{n_l}^{r-1} v_1 + g_{n_l-1}^{r-1} v_2 + \ldots + g_1^{r-1} v_{n_l} & \text{otherwise}.
\end{cases}
\]

Therefore

\[
(u_{\Omega}^{-1} fu_{\Omega_2} \cdot v_r, v_{n_l-m+1}) \equiv 0 \mod (g_t^r)_{t, h < (r, m)}.
\]

Since \( u_{\Omega_2} \cdot v_{r+1} = g_{n_l}^r v_1 + g_{n_l-1}^r v_2 + \ldots + g_1^r v_{n_l} \) and \( u_{\Omega_2} \) is upper-triangular, we have all in all

\[
(u_{\Omega}^{-1} fu_{\Omega} \cdot v_r, v_{n_l-m+1}) \equiv -\sum_{j=1}^m g_j^r (u_{\Omega}^{-1} \cdot v_{n_l-1-j+1}, v_{n_l-m+1}) + 0
\]

\[
\equiv -g_m^r (u_{\Omega}^{-1} \cdot v_{n_l-m+1}, v_{n_l-m+1}) \equiv -g_m^r \mod (g_t^r)_{t, h < (r, m)}.
\]

For the minimal element \((n_l + 1, 1)\) in \( \Omega_2 \) in particular this implies

\[
(u_{\Omega}^{-1} fu_{\Omega} \cdot v_{n_l+1}, v_{n_l}) = -g_{n_l+1}^1,
\]

and so \( g_{n_l+1}^1 \) lies in \( \mathcal{I}_P \). By induction it follows that all \( g_m^r \) for \((r, m) \in \Omega_2 \) lie in \( \mathcal{I}_P \), and the claim is proved.

We have thus shown that \( \mathcal{V}_P \) lies in the subvariety \( \mathcal{V}_{\Omega_2} \) of \( B^+ w_P B^- / B^- \) defined by the ideal \( I_{\Omega_2} = (g_t^r)_{t, h < \Omega_2} \). Its coordinate ring \( \mathcal{O}(\mathcal{V}_{\Omega_2}) = \mathbb{C}[\Omega] / I_{\Omega_2} \) can be identified with the polynomial ring \( \mathbb{C}[g_t^r; (t, h) \in \Omega_1] \), or equivalently with \( \mathbb{C}[G_{n_l}^1, \ldots, G_{n_k}^n] \), where the \( G_j^r \) now denote restrictions to the affine space \( \mathcal{V}_{\Omega_2} \) of the functions defined in (4.4). We furthermore let \( u \) be the restriction of \( u_{\Omega_2} \) to \( \mathcal{V}_{\Omega_2} \), viewing \( u \) as an element of \( U^+ (\mathbb{C}[G_{n_l}^1, \ldots, G_{n_k}^n]) \). Also, we let \( \mathcal{J}_P \) denote the ideal defining \( \mathcal{V}_P \) inside \( \mathbb{C}[G_{n_l}^1, \ldots, G_{n_k}^n] \).

With this, the rest of the proof proceeds as in paragraph 2 on p. 375 in [38]. Only on line 14 of p. 375 we also need to change the morphism \( B^+ w_P B^- / B^- \to \mathfrak{gl}_n \) to a morphism \( \mathcal{V}_{\Omega_2} \to \mathfrak{gl}_n \).

**Further minor corrections.**

1. In (3.4) and (3.6) replace the exponent “\( n_1 \)” with “\( n_2 - n_1 \)” and the exponent “\( n_k - n_{k-1} \)” with “\( n - n_k \)”.

2. Similarly for the subscripts in the displayed equation after (3.4).

3. In line 2 of paragraph 5 of Section 3.4 replace “at most \((n_j - n_{j-1})\) parts” with “at most \((n_j - n_{j-1})\) parts”.

4. In line 6 of Section 3.7 replace \( \lambda_i \) with \( \lambda_{d-i+1} \).

5. Replace \( t \) with \( t^{-1} \) everywhere in displayed equation (5.2).

6. In point (6) of Section 8 replace “\( q(u(x_1, \ldots, x_n)) \)” with “\( q(u(x_1, \ldots, x_d)) \)”.

7. In the second displayed equation in the proof of Lemma 8.1, insert exponents:

\[
\sigma^P_{w_P} = (\sigma^P_{s_1} \cdots s_{n_1})^{n_2-n_1} \cdot (\sigma^P_{s_1} \cdots s_{n_2})^{n_3-n_2} \cdots (\sigma^P_{s_1} \cdots s_{n_k})^{n-n_k}.
\]
Two paragraphs down from (6) in Section 8, add the words “in type A” to read: “Peterson has announced in [33] that all the quantum cohomology rings $qH^*(G/P)$ in type A are reduced.”

The statement of Corollary 11.4(2) should be replaced with: “If $y \in X_{P,>0}$ and $\sigma_w^P(y) \geq 0$ for all $w \in W_P$, then $q_i^P(y) > 0$ for all $i = 1, \ldots, k$.”

The first line of the proof of Proposition 12.2 should read: “By (5.1) and Lemma 12.1 we have”.

Acknowledgements

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References


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