Correction to the Proof of Theorem 4.2. The text in [38] from Remark 4.3 at the bottom of p. 374 to line 8 on p. 375 should be replaced with the following. We use the same notation as in [38].

Remark 4.3. If \( P \) is the parabolic subgroup, then \( G_j \) is a well-defined (regular) function on the Bruhat cell \( B^+w_PB^-/B^- \) precisely in the case \( m \in I^P = \{n_1, \ldots, n_k\} \).

Proof of Theorem 4.2. (1) is proved in [33]. See also Lemma 2.3 in [35]. We will deduce (2) very explicitly from the ASK presentation. We begin by defining a particular system of coordinates on the affine space \( B^+w_PB^-/B^- \). For indexing purposes introduce sets \( \Omega, \Omega_1, \Omega_2 \) defined by

\[
\Omega := \{(r, m) \in \mathbb{Z}^2 \mid n_1 \leq r < n, \text{ and } 1 \leq m \leq n_l \text{ if } n_l \leq r < n_{l+1}\},
\]

\[
\Omega_1 := \{(n_l, m) \in \mathbb{Z}^2 \mid \text{ where } l \in \{1, \ldots, k\} \text{ and } 1 \leq m \leq n_l\},
\]

and \( \Omega_2 := \Omega \setminus \Omega_1 \). Consider the polynomial rings \( \mathbb{C}[\Omega] := \mathbb{C}[g_m^r; (r, m) \in \Omega] \) and \( \mathbb{C}[\Omega_i] := \mathbb{C}[g_m^r; (r, m) \in \Omega_i] \) for \( i = 1, 2 \). We have \( n \times (n_{l+1} - n_l) \)-matrices \( U_{\Omega_1}^{(l)} \) over \( \mathbb{C}[\Omega_1] \) defined by

\[
U_{\Omega_1}^{(0)} = \begin{pmatrix}
1 & g_{11} & g_{12} & \cdots & g_{1n_1} \\
0 & 1 & g_{12} & \cdots & g_{1n_1} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & g_{1n_1} & \cdots \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
\]

\[
U_{\Omega_1}^{(l)} = \begin{pmatrix}
g_{n_1}^n \\
\vdots \\
g_{n_l}^1 \\
1 \\
0 \\
\vdots \\
0 \\
\cdots \\
0 \\
\cdots \\
0 \\
\cdots \\
0 \\
\cdots \\
0 \\
\cdots \\
0 \\
\cdots \\
0
\end{pmatrix}.
\]
where $1 \leq l \leq k$. Furthermore we define $n \times (n_l + 1 - n_l)$-matrices $U^{(l)}_{\Omega_2}$ over $\mathbb{C}[\Omega_2]$ by

\[
U^{(l)}_{\Omega_2} = \begin{pmatrix}
0 & g_{n_l}^{n_l+1} & g_{n_l}^{n_l+2} & \cdots & g_{n_l}^{n_l+1-1} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & g_{n_l}^{n_l+1} & g_{n_l}^{n_l+2} & \cdots & g_{n_l}^{n_l+1-1} \\
0 & g_{1}^{n_l+1} & g_{1}^{n_l+2} & \cdots & g_{1}^{n_l+1-1} \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
\]

for $1 \leq l \leq k$ and $U^{(0)}_{\Omega_2} = 0$. The matrices defined above combine to $n \times n$ matrices

\[
u_{\Omega_i} = \begin{pmatrix}
u^{(0)}_{\Omega_i} & \nu^{(1)}_{\Omega_i} & \cdots & \nu^{(k)}_{\Omega_i}
\end{pmatrix},
\]

for $i = 1, 2$. Moreover we have

\[u_{\Omega} := u_{\Omega_1} + u_{\Omega_2},\]

which is an element of $U^+$ over $\mathbb{C}[\Omega]$, or equivalently a morphism $u_{\Omega} : \mathbb{C}[\Omega] \to U^+$. Composing $u_{\Omega}$ with the standard projection $U^+ \to B^+ w_P B^- / B^-$ defines a map

\[\mathbb{C}[\Omega] \to B^+ w_P B^- / B^-,\]

\[z \mapsto u_{\Omega}(z) w_P B^- .\]

It is clear that this map is an isomorphism. So we may use it to identify the coordinate ring $\mathbb{C}[B^+ w_P B^- / B^-]$ with $\mathbb{C}[\Omega]$. Note that $G_{ji}^{ni}$ goes to $g_{ji}^{ni}$ under this identification. Let $\mathcal{I}_P \subset \mathbb{C}[\Omega]$ denote the defining ideal for $\mathcal{Y}_P$.

Claim. We have $G_{mi}^{ni} \in \mathcal{I}_P$ for all $(r, m) \in \Omega_2$, or equivalently $u_{\Omega_2} \equiv 0 \mod \mathcal{I}_P$.

Let $v_1, \ldots, v_n$ be the standard basis of $\mathbb{C}^n$ and $(~ , ~)$ the bilinear form given by $(v_i, v_j) = \delta_{ji}^i$. From the definition of the Peterson variety it follows that the ideal $\mathcal{I}_P$ is generated by the elements

\[(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_h) \in \mathbb{C}[\Omega], \text{ where } (r, h) \in \Omega_2 \text{ or } r = n_{l+1} \text{ and } h \in [1, n_l] \setminus \{n_{l-1} + 1\}, \]

for $l = 1, \ldots, k$. Let $\prec$ be the lexicographical ordering on $\Omega_2$. We will prove the claim recursively. Consider $(r, m) \in \Omega_2$. Then there is an $l \in \{1, \ldots, k\}$ such that $n_l < r < n_{l+1}$ and $1 \leq m \leq n_l$, and we consider the generator

\[(u_{\Omega}^{-1} f u_{\Omega} \cdot v_r, v_{n_l-m+1}) \in \mathcal{I}_P.\]
Note first that \( n_l < r < n_{l+1} \) implies \( fu_{\Omega_l} \cdot v_r = u_{\Omega_r} \cdot v_{r+1} \). Therefore

\[
(u_{\Omega}^{-1}fu_{\Omega_r} \cdot v_r, v_{n_l-m+1}) = (u_{\Omega}^{-1}fu_{\Omega_l} \cdot v_r, v_{n_l-m+1}) + (u_{\Omega}^{-1}fu_{\Omega_2} \cdot v_r, v_{n_l-m+1})
\]

\[
= (u_{\Omega}^{-1}u_{\Omega_l} \cdot v_{r+1}, v_{n_l-m+1}) + (u_{\Omega}^{-1}fu_{\Omega_2} \cdot v_r, v_{n_l-m+1})
\]

\[
= 0 - (u_{\Omega}^{-1}u_{\Omega_2} \cdot v_{r+1}, v_{n_l-m+1}) + (u_{\Omega}^{-1}fu_{\Omega_2} \cdot v_r, v_{n_l-m+1}).
\]

Now

\[
u_{\Omega_2} \cdot v_r = \begin{cases} 0 & \text{if } r = n_l + 1, \\ g_1^{r-1}v_1 + g_2^{r-1}v_2 + \ldots + g_{1}^{r-1}v_{n_l} & \text{otherwise}. \end{cases}
\]

Therefore

\[
(u_{\Omega}^{-1}fu_{\Omega_2} \cdot v_r, v_{n_l-m+1}) \equiv 0 \mod (g_h)^{(t,h) \prec (r,m)}.
\]

Since \( u_{\Omega_2} \cdot v_{r+1} = g_1^r v_1 + g_2^{r-1}v_2 + \ldots + g_{1}^{r-1}v_{n_l} \) and \( u_{\Omega} \) is upper-triangular, we have all in all

\[
(u_{\Omega}^{-1}fu_{\Omega} \cdot v_r, v_{n_l-m+1}) \equiv -\sum_{j=1}^{m} g_j^r \left( u_{\Omega}^{-1} \cdot v_{n_l-j+1}, v_{n_l-m+1} \right) + 0
\]

\[
\equiv -g_m^r \left( u_{\Omega}^{-1} \cdot v_{n_l-m+1}, v_{n_l-m+1} \right) \equiv -g_m^r \mod (g_h)^{(t,h) \prec (r,m)}.
\]

For the minimal element \((n_l+1,1)\) in \( \Omega_2 \) in particular this implies

\[
(u_{\Omega}^{-1}fu_{\Omega} \cdot v_{n_l+1}, v_{n_l}) = -g_1^{n_l+1},
\]

and so \( g_1^{n_l+1} \) lies in \( \mathcal{I}_P \). By induction it follows that all \( g_m^r \) for \((r,m) \in \Omega_2 \) lie in \( \mathcal{I}_P \), and the claim is proved.

We have thus shown that \( \mathcal{Y}_P \) lies in the subvariety \( \mathcal{V}_{\Omega_2} \) of \( B^+w_PB^-/B^- \) defined by the ideal \( I_{\Omega_2} = (g_h(t,h))_{t,h} \in \Omega_2 \). Its coordinate ring \( \mathcal{O}(\mathcal{V}_{\Omega_2}) = \mathbb{C}[\Omega]/I_{\Omega_2} \) can be identified with the polynomial ring \( \mathbb{C}[g_h(t,h) \in \Omega_1] \), or equivalently with \( \mathbb{C}[G_1^{n_1}, \ldots, G_n^{n_k}] \), where the \( G_j^r \) now denote restrictions to the affine space \( \mathcal{V}_{\Omega_2} \) of the functions defined in (4.4). We furthermore let \( u \) be the restriction of \( u_{\Omega_2} \) to \( \mathcal{V}_{\Omega_2} \), viewing \( u \) as an element of \( U^+(\mathbb{C}[G_1^{n_1}, \ldots, G_n^{n_k}]) \). Also, we let \( \mathcal{J}_P \) denote the ideal defining \( \mathcal{Y}_P \) inside \( \mathbb{C}[G_1^{n_1}, \ldots, G_n^{n_k}] \).

With this, the rest of the proof proceeds as in paragraph 2 on p. 375 in [38]. Only on line 14 of p. 375 we also need to change the morphism \( B^+w_PB^-/B^- \to gl_n \) to a morphism \( \mathcal{V}_{\Omega_2} \to gl_n \).

**Further minor corrections.**

1. In (3.4) and (3.6) replace the exponent “\( n_1 \)” with “\( n_2 - n_1 \)” and the exponent “\( n_k - n_{k-1} \)” with “\( n - n_k \)” Similarly for the subscripts in the displayed equation after (3.4).

2. In line 2 of paragraph 5 of Section 3.4 replace “at most \((n_j - n_{j-1})\) parts” with “at most \((n_{j+1} - n_j)\) parts”.

3. In line 6 of Section 3.7 replace \( \lambda_i \) with \( \lambda_{d-i+1} \).

4. Replace \( t \) with \( t^{-1} \) everywhere in displayed equation (5.2).

5. In point (6) of Section 8 replace “\( q(u(x_1, \ldots, x_n)) \)” with “\( q(u(x_1, \ldots, x_d)) \)”.

6. In the second displayed equation in the proof of Lemma 8.1, insert exponents:

\[
\sigma^{P}_{w^P} = (\sigma^{P}_{s_1, \ldots, s_{n_1}})^{n_2-n_1} \cdot (\sigma^{P}_{s_1, \ldots, s_{n_2}})^{n_3-n_2} \cdots (\sigma^{P}_{s_1, \ldots, s_{n_k}})^{n-n_k}.
\]
(7) Two paragraphs down from (6) in Section 8, add the words “in type A” to read: “Peterson has announced in [33] that all the quantum cohomology rings \(qH^*(G/P)\) in type A are reduced.”

(8) The statement of Corollary 11.4(2) should be replaced with: “If \(y \in X_{P,>0}\) and \(\sigma^P_w(y) \geq 0\) for all \(w \in W_P\), then \(q^P_i(y) > 0\) for all \(i = 1, \ldots, k\).”

(9) The first line of the proof of Proposition 12.2 should read: “By (5.1) and Lemma 12.1 we have”.

Acknowledgements

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References

[33] D. Peterson, Quantum cohomology of \(G/P\), Lecture Course, M.I.T., Spring Term, 1997.
