

ON THE SIZE OF KAKEYA SETS IN FINITE FIELDS

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1. INTRODUCTION

Let \mathbb{F} denote a finite field of q elements. A Kakeya set (also called a Besicovitch set) in \mathbb{F}^n is a set $K \subset \mathbb{F}^n$ such that K contains a line in every direction. More formally, K is a Kakeya set if for every $x \in \mathbb{F}^n$ there exists a point $y \in \mathbb{F}^n$ such that the line

$$L_{y,x} \triangleq \{y + a \cdot x \mid a \in \mathbb{F}\}$$

is contained in K .

The motivation for studying Kakeya sets over finite fields is to try to better understand the more complicated questions regarding Kakeya sets in \mathbb{R}^n . A Kakeya set $K \subset \mathbb{R}^n$ is a compact set containing a line segment of unit length in every direction. The famous Kakeya Conjecture states that such sets must have Hausdorff (or Minkowski) dimension equal to n . The importance of this conjecture is partially due to the connections it has to many problems in harmonic analysis, number theory and PDE. This conjecture was proved for $n = 2$ [Dav71] and is open for larger values of n (we refer the reader to the survey papers [Wol99, Bou00, Tao01] for more information).

It was first suggested by Wolff [Wol99] to study finite field Kakeya sets. It was asked in [Wol99] whether there exists a lower bound of the form $C_n \cdot q^n$ on the size of such sets in \mathbb{F}^n . The lower bound appearing in [Wol99] was of the form $C_n \cdot q^{(n+2)/2}$. This bound was further improved in [Rog01, BKT04, MT04, Tao08] both for general n and for specific small values of n (e.g. for $n = 3, 4$). For general n , the most current best lower bound is the one obtained in [Rog01, MT04] (based on results from [KT99]) of $C_n \cdot q^{4n/7}$. The main technique used to show this bound is an additive number theoretic lemma relating the sizes of different sum sets of the form $A + r \cdot B$, where A and B are fixed sets in \mathbb{F}^n and r ranges over several different values in \mathbb{F} (the idea to use additive number theory in the context of Kakeya sets is due to Bourgain [Bou99]).

The next theorem, proven in Section 2, gives a near-optimal bound on the size of Kakeya sets. Roughly speaking, the proof follows by observing that any degree $q-2$ homogeneous polynomial in $\mathbb{F}[x_1, \dots, x_n]$ can be ‘reconstructed’ from its value on any Kakeya set $K \subset \mathbb{F}^n$. This implies that the size of K is at least the dimension of the space of polynomials of degree $q-2$, which is $\approx q^{n-1}$ (when q is large).

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Theorem 1.1. *Let $K \subset \mathbb{F}^n$ be a Kakeya set. Then*

$$|K| \geq C_n \cdot q^{n-1},$$

where C_n depends only on n .

The result of Theorem 1.1 can be made into an even better bound using the simple observation that a product of Kakeya sets is also a Kakeya set.

Corollary 1.2. *For every integer n and every $\epsilon > 0$ there exists a constant $C_{n,\epsilon}$, depending only on n and ϵ such that any Kakeya set $K \subset \mathbb{F}^n$ satisfies*

$$|K| \geq C_{n,\epsilon} \cdot q^{n-\epsilon}.$$

Proof. Observe that, for every integer $r > 0$, the Cartesian product $K^r \subset \mathbb{F}^{n \cdot r}$ is also a Kakeya set. Using Theorem 1.1 on this set gives

$$|K|^r \geq C_{n \cdot r} \cdot q^{n \cdot r - 1},$$

which translates into a bound of $C_{n,r} \cdot q^{n-1/r}$ on the size of K . \square

We derive Theorem 1.1 from a stronger theorem that gives a bound on the size of sets that contain only ‘many’ points on ‘many’ lines. Before stating the theorem we formally define these sets.

Definition 1.3 ((δ, γ) -Kakeya set). a set $K \subset \mathbb{F}^n$ is a (δ, γ) -Kakeya set if there exists a set $\mathcal{L} \subset \mathbb{F}^n$ of size at least $\delta \cdot q^n$ such that for every $x \in \mathcal{L}$ there is a line in direction x that intersects K in at least $\gamma \cdot q$ points.

The next theorem, proven in Section 2, gives a lower bound on the size of (δ, γ) -Kakeya sets. Theorem 1.1 will follow by setting $\delta = \gamma = 1$.

Theorem 1.4. *Let $K \subset \mathbb{F}^n$ be a (δ, γ) -Kakeya set. Then*

$$|K| \geq \binom{d+n-1}{n-1},$$

where

$$d = \lfloor q \cdot \min\{\delta, \gamma\} \rfloor - 2.$$

Notice that, in order to get a bound of $\approx q^{n(1-\epsilon)}$ on the size of K , Theorem 1.4 allows δ and γ to be as small as $q^{-\epsilon}$.

1.1. Improving the bound to $\approx q^n$. Following the initial publication of this work, Noga Alon and Terence Tao [AT08] independently observed that it is possible to turn the proof of Theorem 1.1 into a proof that gives a bound of $C_n \cdot q^n$, thus achieving an optimal bound. A proof of the following theorem appears in Section 3

Theorem 1.5. *Let $K \subset \mathbb{F}^n$ be a Kakeya set. Then*

$$|K| \geq C_n \cdot q^n,$$

where C_n depends only on n .

2. PROOF OF THEOREM 1.4

We will use the following bound on the number of zeros of a degree d polynomial proven by Schwartz and Zippel [Sch80, Zip79].

Lemma 2.1 (Schwartz-Zippel). *Let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a nonzero polynomial with $\deg(f) \leq d$. Then*

$$|\{x \in \mathbb{F}^n \mid f(x) = 0\}| \leq d \cdot q^{n-1}.$$

Proof of Theorem 1.4. Suppose by contradiction that

$$|K| < \binom{d+n-1}{n-1}.$$

Then, the number of monomials in $\mathbb{F}[x_1, \dots, x_n]$ of degree d is larger than the size of K . Therefore, there exists a homogeneous degree d polynomial $g \in \mathbb{F}[x_1, \dots, x_n]$ such that g is not the zero polynomial and

$$\forall x \in K, \quad g(x) = 0$$

(this follows by solving a system of linear equations, one for each point in K , where the unknowns are the coefficients of g). Our plan is to show that g has too many zeros and therefore must be identically zero (which is a contradiction).

Consider the set

$$K' \triangleq \{c \cdot x \mid x \in K, c \in \mathbb{F}\}$$

containing all lines that pass through zero and intersect K at some point. Since g is homogeneous we have

$$g(c \cdot x) = c^d \cdot g(x),$$

and so

$$\forall x \in K', \quad g(x) = 0.$$

Since K is a (δ, γ) -Kakeya set, there exists a set $\mathcal{L} \subset \mathbb{F}^n$ of size at least $\delta \cdot q^n$ such that for every $y \in \mathcal{L}$ there exists a line with direction y that intersects K in at least $\gamma \cdot q$ points.

Claim 2.2. For every $y \in \mathcal{L}$ we have $g(y) = 0$.

Proof. Let $y \in \mathcal{L}$ be some nonzero vector (if $y = 0$, then $g(y) = 0$, since g is homogeneous). Then, there exists a point $z \in \mathbb{F}^n$ such that the line

$$L_{z,y} = \{z + a \cdot y \mid a \in \mathbb{F}\}$$

intersects K in at least $\gamma \cdot q$ points. Therefore, since $d+2 \leq \gamma \cdot q$, there exist $d+2$ distinct field elements $a_1, \dots, a_{d+2} \in \mathbb{F}$ such that

$$\forall i \in [d+2], \quad z + a_i \cdot y \in K.$$

If there exists i such that $a_i = 0$ we can remove this element from our set of $d+2$ points, and so we are left with at least $d+1$ distinct *nonzero* field elements (w.l.o.g. a_1, \dots, a_{d+1}) such that

$$\forall i \in [d+1], \quad z + a_i \cdot y \in K \quad \text{and} \quad a_i \neq 0.$$

Let $b_i = a_i^{-1}$ where $i \in [d+1]$. The $d+1$ points

$$w_i \triangleq b_i \cdot z + y, \quad i \in [d+1]$$

are all in the set K' , and so

$$g(w_i) = 0, \quad i \in [d+1].$$

If $z = 0$, then we have $w_i = y$ for all $i \in [d+1]$, and so $g(y) = 0$. We can thus assume that $z \neq 0$, which implies that w_1, \dots, w_{d+1} are $d+1$ *distinct* points belonging to the same line (the line through y with direction z). The restriction of $g(x)$ to this line is a degree $\leq d$ univariate polynomial, and so, since it has $d+1$ zeros (at the points w_i), it must be zero on the entire line. We therefore get that $g(y) = 0$, and so the claim is proven. \square

We now get a contradiction since

$$d/q < \delta$$

and, using Lemma 2.1, a polynomial of degree d can be zero on at most a d/q fraction of \mathbb{F}^n . \square

3. PROOF OF THEOREM 1.5

Suppose, by contradiction, that $K \subset \mathbb{F}^n$ is a Kakeya set such that

$$|K| < \binom{q+n-1}{n}.$$

Then, as is explained in the proof of Theorem 1.1, there exists a nonzero polynomial $g \in \mathbb{F}[x_1, \dots, x_n]$ of degree $d \leq q-1$ so that $g(x) = 0$ for all $x \in K$ (notice that g is not necessarily homogeneous). Let $\bar{g} \in \mathbb{F}[x_1, \dots, x_n]$ be the homogeneous part of degree d of g so that \bar{g} is nonzero and homogeneous. Fix some $y \in \mathbb{F}^n$. Then there exists $z \in \mathbb{F}^n$ so that the line $\{z + t \cdot y \mid t \in \mathbb{F}\}$ is contained in K . Therefore,

$$P_{y,z}(t) \triangleq g(z + t \cdot y) = 0$$

for all $t \in \mathbb{F}$. Since $P_{y,z}(t)$ is a univariate polynomial of degree $d \leq q-1$ this means that $P_{y,z}(t)$ is identically zero, and hence all its coefficients are zero. In particular, the coefficient of t^d is zero, but it is easy to see that this is exactly $\bar{g}(y)$. Since y is arbitrary it follows that the polynomial \bar{g} is identically zero – a contradiction. This concludes the proof. \square

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