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A NEW PROOF OF GROMOV'S THEOREM ON GROUPS OF POLYNOMIAL GROWTH

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1. Introduction

1.1. **Statement of results.** Let G be a group with a finite symmetric generating set S, and let $B_G(r) \subset G$ denote the ball centered at $e \in G$ with respect to the word norm on G given by S:

$$B_G(r) = \{g \in G \mid g = g_1 \cdots g_k \text{ for some } g_1, \ldots, g_k \in S,\ k \leq r\}.$$

Definition 1.1. The group G has **polynomial growth** if for some $d \in (0, \infty)$

$$\limsup_{r \to \infty} \frac{|B_G(r)|}{r^d} < \infty, \tag{1.1}$$

and it has **weakly polynomial growth** if for some $d \in (0, \infty)$

$$\liminf_{r \to \infty} \frac{|B_G(r)|}{r^d} < \infty. \tag{1.2}$$

Our main result is a new proof of the following theorem of Gromov and Wilkie-van den Dries [Gro81, vdDW84]:

Theorem 1.2. *If a group has weakly polynomial growth, it is virtually nilpotent.*

The original proofs in [Gro81, vdDW84] are based on the Montgomery-Zippin-Yamabe structure theory of locally compact groups [MZ74]. We avoid this by following a completely different approach involving harmonic maps. The core of the argument is a new proof of (a slight generalization of) a theorem of Colding-Minicozzi:

Theorem 1.3 ([CM97])**.** *Suppose X is either a bounded geometry Riemannian manifold or a bounded degree graph and that X is quasi-isometric to a group of weakly polynomial growth. Then for all $d \in [0, \infty)$ the space of harmonic functions on X with polynomial growth at most d is finite dimensional.*

We recall that a Riemannian manifold has bounded geometry if its sectional curvature is bounded above and below and its injectivity radius is bounded away from zero. Two metric spaces are quasi-isometric if they contain bi-Lipschitz homeomorphic nets. A piecewise smooth function $u : X \to \mathbb{R}$ on a bounded degree graph is harmonic if it minimizes energy on finite subgraphs, or, equivalently, if its derivative is constant along each edge and its value at each vertex coincides with its average

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value on the adjacent vertices. A function $u : X \to \mathbb{R}$ on a metric space X has polynomial growth at most d if

$$\sup_{x\in X} \frac{|u(x)|}{(1+d_X(p,x))^d} < \infty$$

for some $p \in X$.

Note that although the main result of [CM97] is stated for groups of polynomial growth, their proof also works for groups of weakly polynomial growth, in view of [vdDW84]. The proof of Theorem 1.3 given here is independent of Gromov's theorem on groups of polynomial growth, unlike the proof in [CM97].

Remark 1.4. There are several important applications of the Wilkie-van den Dries refinement [vdDW84] of Gromov's theorem [Gro81] that do not follow from the original statement, for instance [Pap05], or the theorem of Varopoulos that a group satisfies a d-dimensional Euclidean isoperimetric inequality unless it is virtually nilpotent of growth exponent $< d$.

1.2. **Sketch of the proofs.** By a short induction argument from [Gro81, vdDW84], to prove Theorem 1.2 it suffices to show that if G is an infinite group with weakly polynomial growth, then there is a finite-dimensional representation $G \to \mathrm{GL}(n,\mathbb{R})$ with infinite image. To achieve this, we first invoke a theorem of Mok/Korevaar-Schoen[1], [Mok95, KS97], to produce a fixed-point-free isometric G-action $G \curvearrowright \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, and a G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$, where Γ is a Cayley graph of G. Theorem 1.3 then implies that f takes values in a finite-dimensional subspace of $\mathcal{H}$, and this yields the desired finite-dimensional representation of G. See Section 4 for details.

The proof of Theorem 1.3 is based on a new Poincaré inequality which holds for any Cayley graph Γ of any finitely generated group G:

$$\int_{B(R)} |f - f_R|^2 \le 8\,|S|^2\,R^2\,\frac{|B(2R)|}{|B(R)|}\int_{B(3R)} |\nabla f|^2. \tag{1.3}$$

Here f is a piecewise smooth function on $B(3R)$, f_R is the average of u over the ball $B(R)$, and S is the generating set for G.

The remainder of the proof has the same rough outline as [CM97], though the details are different. Note that [CM97] assumes a uniform doubling condition as well as a uniform Poincaré inequality. In our context, we may not appeal to such uniform bounds as their proof depends on Gromov's theorem. Instead, the idea is to use (1.3) to show that one has uniform bounds at certain scales and that this is sufficient to deduce that the space of harmonic functions in question is finite dimensional.

Contents

[1]Although the publication date of [KS97] was significantly later, the result was announced in public lectures by both Mok and Korevaar/Schoen in the spring of 1994.

2. A Poincaré inequality for finitely generated groups

Let G be a group, with a finite generating set $S \subset G$. We denote the associated word norm of $g \in G$ by $|g|$. For $R \in [0,\infty) \cap \mathbb{Z}$, let $V(R) = |B_G(R)| = |B_G(e,R)|$. We will denote the R-ball in the associated Cayley graph by $B(R) = B(e,R)$.

Remark 2.1. We are viewing the Cayley graph as (the geometric realization of a) 1-dimensional simplicial complex, not as a discrete space. Thus $B_G(R)$ is a finite set, whereas $B(R)$ is typically 1-dimensional.

Theorem 2.2. *For every $R \in [0,\infty)\cap\mathbb{Z}$ and every smooth function $f : B(3R) \to \mathbb{R}$,*

$$\int_{B(R)} |f - f_R|^2 \leq 8\,|S|^2\,R^2\,\frac{V(2R)}{V(R)}\,\int_{B(3R)} |\nabla f|^2, \tag{2.1}$$

where f_R is the average of f over $B(R)$.

Proof. Fix $R \in [0,\infty) \cap \mathbb{Z}$.

Let $\delta f : B_G(3R-1) \to \mathbb{R}$ be given by

$$\delta f(x) = \int_{B(x,1)} |\nabla f|^2.$$

For every $y \in G$, we choose a shortest vertex path $\gamma_y : \{0,\ldots,|y|\} \to G$ from $e \in G$ to y. If $y \in B_G(2R-2)$, then

$$\sum_{x\in B(R-1)}\ \sum_{i=0}^{|y|} (\delta f)(x\,\gamma_y(i)) \leq 2R \sum_{z\in B(3R-1)} (\delta f)(z), \tag{2.2}$$

since the map $B(R-1)\times\{0,\ldots,|y|\} \to B(3R-1)$ given by $(x,i) \mapsto x\,\gamma_y(i)$ is at most $2R$-to-1.

For every ordered pair (e_1,e_2) of edges contained in $B(R)$, let $x_i \in e_i \cap G$ be elements such that $d(x_1,x_2) \leq 2R-2$, and let $y = x_1^{-1}x_2$. By the Cauchy-Schwarz inequality,

$$\int_{(p_1,p_2)\in e_1\times e_2} |f(p_1) - f(p_2)|^2\, dp_1 dp_2 \ \leq\ 2R\sum_{i=0}^{|y|} (\delta f)(x_1\,\gamma_y(i)). \tag{2.3}$$

Now

$$\begin{aligned}\int_{B(R)} |f - f_R|^2 &\leq \frac{1}{V(R)} \int_{B(R)\times B(R)} |f(p_1) - f(p_2)|^2 \, dp_1 dp_2 \\ &= \frac{1}{V(R)} \sum_{(e_1,e_2)\subset B(R)\times B(R)} \int_{(p_1,p_2)\in e_1\times e_2} |f(p_1) - f(p_2)|^2 \, dp_1 dp_2 \\ &\leq \frac{1}{V(R)} \sum_{(e_1,e_2)\subset B(R)\times B(R)} 2R \sum_{i=0}^{|y|} (\delta f)(x_1\,\gamma_y(i)),\end{aligned}$$

where x_1 and y are as defined above. The map $(e_1, e_2) \mapsto (x_1, y)$ is at most $|S|^2$-to-one, so

$$\begin{aligned}\int_{B(R)} |f - f_R|^2 \;&\leq\; 2R\,|S|^2 \frac{1}{V(R)} \sum_{x_1\in B(R-1)} \sum_{y\in B(2R-2)} \sum_{i=0}^{|y|} (\delta f)(x_1\,\gamma_y(i)) \\ &\leq\; 4\,R^2\,|S|^2 \frac{1}{V(R)} \sum_{y\in B(2R-2)} \sum_{z\in B(3R-1)} (\delta f)(z) \quad \text{by (2.2)} \\ &= 4\,R^2\,|S|^2 \frac{V(2R)}{V(R)} \sum_{z\in B(3R-1)} (\delta f)(z) \;\leq\; 8\,R^2\,|S|^2 \frac{V(2R)}{V(R)} \int_{B(3R)} |\nabla f|^2. \quad \square\end{aligned}$$

Remark 2.3. Although the theorem above is not in the literature, the proof is virtually contained in [CSC93, pp. 308–310]. When hearing of my more complicated Poincaré inequality, Laurent Saloff-Coste's immediate response was to state and prove Theorem 2.2.

3. The proof of Theorem 1.3

In this section G will be a finitely generated group with a fixed finite generating set S, and the associated Cayley graph and word norm will be denoted Γ and $\|\cdot\|$, respectively. For $R \in \mathbb{Z}_+$ we let $B(R) := B(e, R) \subset \Gamma$ and $V(R) := |B_G(R)| = |B(R) \cap G|$.

We will first give the proof in the case that $X = \Gamma$, which is the one needed for Theorem 1.2. At the end of this section we will return to the general case; see Section 3.6.

Let $\mathcal{V}$ be a $2k$-dimensional vector space of harmonic functions on Γ. We equip $\mathcal{V}$ with the family of quadratic forms $\{Q_R\}_{R\in[0,\infty)}$, where

$$Q_R(u,u) := \int_{B(R)} u^2.$$

The remainder of this section is devoted to proving the following finite-dimensionality result:

Theorem 3.1. *For every $d \in (0,\infty)$ there is a $C = C(d) \in (0,\infty)$ such that if*

$$\liminf_{R\to\infty} \frac{V(R)\,(\det Q_R)^{\frac{1}{\dim \mathcal{V}}}}{R^d} \;<\; \infty, \tag{3.1}$$

then $\dim \mathcal{V} < C$.

Proof of Theorem 1.3 *using Theorem* 3.1. If $\mathcal{V}$ is a finite-dimensional space of harmonic functions on Γ with polynomial growth d', then

$$\limsup_{R} \frac{(\det Q_R)^{\frac{1}{\dim \mathcal{V}}}}{R^{d'}} < \infty\,.$$

This implies that (3.1) holds provided

$$\liminf_{R\to\infty} \frac{V(R)}{R^{d-d'}} < \infty\,.$$

Hence by Theorem 3.1, we obtain a uniform bound on the dimension on any space of harmonic functions with growth at most d'. □

The overall structure of the proof of Theorem 3.1 is similar to that of Colding-Minicozzi [CM97].

3.1. **Finding good scales.** We begin by using the polynomial growth assumption to select a pair of comparable scales $R_1 < R_2$ at which both the growth function V and the determinant $(\det Q_R)^{\frac{1}{\dim \mathcal{V}}}$ have doubling behavior. Later we will use this to find many functions in $\mathcal{V}$ which have doubling behavior at scale R_2. Similar scale selection arguments appear in both [Gro81] and [CM97]; the one here is a hybrid of the two.

Observe that the family of quadratic forms $\{Q_R\}_{R\in[0,\infty)}$ is nondecreasing in R, in the sense that $Q_{R'} - Q_R$ is positive semi-definite when $R' \geq R$. Also, note that Q_R is positive definite for sufficiently large R, since $Q_R(u,u) = 0$ for all R only if $u \equiv 0$. Choose $i_0 \in \mathbb{N}$ such that $Q_R > 0$ whenever $R \geq 16^{i_0}$.

We define $f : \mathbb{Z}_+ \to \mathbb{R}$ and $h : \mathbb{Z} \cap [i_0, \infty) \to \mathbb{R}$ by

$$f(R) = V(R)\,(\det Q_R)^{\frac{1}{\dim \mathcal{V}}} \quad \text{and} \quad h(i) = \log f(16^i).$$

Note that since Q_R is a nondecreasing function of R, both f and h are nondecreasing functions, and (3.1) translates to

$$\liminf_{i\to\infty}\ (h(i) - di\log 16) < \infty. \tag{3.2}$$

Put $a = 4d\log 16$, and pick $w \in \mathbb{N}$.

Lemma 3.2. *There are integers $i_1, i_2 \in [i_0, \infty)$ such that*

$$i_2 - i_1 \in (w, 3w), \tag{3.3}$$

$$h(i_2+1) - h(i_1) < wa, \tag{3.4}$$

and

$$h(i_1+1) - h(i_1) < a, \quad h(i_2+1) - h(i_2) < a. \tag{3.5}$$

Proof. There is a nonnegative integer j_0 such that

$$h(i_0 + 3w(j_0+1)) - h(i_0 + 3wj_0) < wa. \tag{3.6}$$

Otherwise, for all $l \in \mathbb{N}$ we would get

$$\begin{aligned} h(i_0 + 3wl) &= h(i_0) + \sum_{j=0}^{l-1} \left(h(i_0 + 3w(j+1)) - h(i_0 + 3wj)\right) \\ &\geq h(i_0) + wal = h(i_0) + \left(\frac{4}{3} d \log 16\right)(3wl)\,, \end{aligned}$$

which contradicts (3.2) for large l.

Let $m := i_0 + 3wj_0$.

Then there are integers $i_1 \in [m, m+w)$ and $i_2 \in [m+2w, m+3w)$ such that (3.5) holds, for otherwise we would have either $h(m+w) - h(m) \geq wa$ or $h(m+3w) - h(m+2w) \geq wa$, contradicting (3.6).

These i_1 and i_2 satisfy the conditions of the lemma, because

$$h(i_2 + 1) - h(i_1) \leq h(m + 3w) - h(m) < wa. \qquad \square$$

3.2. **A controlled cover.** Let $R_1 = 2 \cdot 16^{i_1}$ and $R_2 = 16^{i_2}$. Choose a maximal R_1-separated subset $\{x_j\}_{j \in J}$ of $B(R_2) \cap G$, and let $B_j := B(x_j, R_1)$. Then the collection $\mathcal{B} := \{B_j\}_{j\in J}$ covers $B(R_2)$, and $\frac{1}{2}\mathcal{B} := \{\frac{1}{2}B_j\}_{j \in J}$ is a disjoint collection.

Lemma 3.3.

(1) *The covers $\mathcal{B}$ and $3\mathcal{B} := \{3B_j\}_{j\in J}$ have intersection multiplicity $< e^a$.*
(2) *$\mathcal{B}$ has cardinality $|J| < e^{wa}$.*
(3) *There is a $C \in (0, \infty)$ depending only on $|S|$ such that for every $j \in J$ and every smooth function $v : 3B_j \to \mathbb{R}$,*

$$\int_{B_j} |v - v_{B_j}|^2 \leq C\, e^a\, R_1^2 \int_{3B_j} |\nabla v|^2. \tag{3.7}$$

Proof. (1) If $z \in 3B_{j_1} \cap \cdots \cap 3B_{j_l}$, then $x_{j_m} \in B(x_{j_1}, 6R_1)$ for every $m \in \{1, \ldots, l\}$, so $\{B(x_{j_m}, \frac{R_1}{2})\}_{m=1}^{l}$ are disjoint balls lying in $B(x_{j_1}, 8R_1)$, and hence

$$\log l \leq \log \frac{V(3R_1)}{V(\frac{R_1}{2})} = \log V(3R_1) - \log V\left(\frac{R_1}{2}\right) \leq h(i_1 + 1) - h(i_1) < a.$$

This shows that the multiplicity of $3\mathcal{B}$ is at most e^a. This implies (1), since the multiplicity of $\mathcal{B}$ is not greater than that of $3\mathcal{B}$.

(2) The balls $\{B(x_j, \frac{R_1}{2})\}_{j\in J}$ are disjoint and are contained in $B(R_2 + \frac{R_1}{2}) \subset B(2R_2)$, so

$$|J| \leq \frac{V(2R_2)}{V(\frac{R_1}{2})} \leq \frac{V(16^{i_2+1})}{V(16^{i_1})} < e^{wa},$$

by (3.4).

(3) By Theorem 2.2 and the translation invariance of the inequality,

$$\int_{B_i} |v - v_{B_i}|^2 \;\leq\; 8\,|S|^2\, R_1^2 \frac{V(2R_1)}{V(R_1)} \int_{3B_i} |\nabla v|^2 \leq 8\,|S|^2\, R_1^2\, e^a \int_{3B_i} |\nabla v|^2.$$

$\square$

3.3. **Estimating functions relative to the cover $\mathcal{B}$.** We now estimate the size of a harmonic function in terms of its averages over the B_j's and its size on a larger ball.

We define a linear map $\Phi : \mathcal{V} \to \mathbb{R}^J$ by

$$\Phi_j(v) := \frac{1}{|B_j|} \int_{B_j} v.$$

Lemma 3.4 (Cf. [CM97, Prop. 2.5])**.** *There is a constant $C \in (0, \infty)$ depending only on the size of the generating set S, with the following properties.*

(1) *If u is a smooth function on $B(16R_2)$, then*

$$Q_{R_2}(u,u) \quad \leq \quad C\,V(R_1)\,|\Phi(u)|^2 + \; C\,e^{2a} R_1^2 \int_{B(2R_2)} |\nabla u|^2. \tag{3.8}$$

(2) *If u is harmonic on $B(16R_2)$, then*

$$Q_{R_2}(u,u) \leq C\,V(R_1)|\Phi(u)|^2 + C\,e^{2a} \left(\frac{R_1}{R_2}\right)^2 Q_{16R_2}(u,u). \tag{3.9}$$

Proof. We will use C to denote a constant which depends only on $|S|$; however, its value may vary from equation to equation.

We have

$$\begin{aligned} Q_{R_2}(u,u) &= \int_{B(R_2)} u^2 \leq \sum_{j \in J} \int_{B_j} u^2 \\ &\leq 2 \sum_{j \in J} \int_{B_j} \left(|\Phi_j(u)|^2 + |u - \Phi_j(u)|^2 \right). \end{aligned} \tag{3.10}$$

We estimate each of the terms in (3.10) in turn.

For the first term we get

$$\sum_{j \in J} \int_{B_j} |\Phi_j(u)|^2 = \sum_{j \in J} |B_j|\,|\Phi_j(u)|^2 \leq C\,V(R_1)\;|\Phi(u)|^2. \tag{3.11}$$

For the second term we have

$$\begin{aligned} \sum_{j \in J} \int_{B_j} |u - \Phi_j(u)|^2 &\leq C\,e^a\,R_1^2 \sum_{j \in J} \int_{3B_j} |\nabla u|^2 \quad \text{by Lemma 3.3(3)} \\ &\leq Ce^a R_1^2 \left(e^a \int_{B(2R_2)} |\nabla u|^2 \right) \quad \text{by Lemma 3.3(1)} \\ &= Ce^{2a}\,R_1^2 \int_{B(2R_2)} |\nabla u|^2. \end{aligned}$$

Combining this with (3.11) yields (1).

Inequality (3.9) follows from (3.8) by applying the reverse Poincaré inequality, which holds for any harmonic function v defined on $B(16R_2)$:

$$R_2^2 \int_{B(2R_2)} |\nabla v|^2 \leq C\;Q_{16R_2}(v,v).$$

(For the proof, see [SY95, Lemma 6.3], and note that for harmonic functions their condition $u \geq 0$ may be dropped.) $\square$

3.4. **Selecting functions from $\mathcal{V}$ with controlled growth.** Our next step is to select functions in $\mathcal{V}$ which have doubling behavior at scale R_2.

Lemma 3.5 (Cf. [CM97, Prop. 4.16])**.** *There is a subspace $\mathcal{U} \subset \mathcal{V}$ of dimension at least $k = \frac{\dim \mathcal{V}}{2}$ such that for every $u \in \mathcal{U}$*

$$Q_{16R_2}(u,u) \leq e^{2a}\, Q_{R_2}(u,u). \tag{3.12}$$

Proof. Since $R_2 = 16^{i_2} > 16^{i_0}$, the quadratic form Q_{R_2} is positive definite. Therefore there is a Q_{R_2}-orthonormal basis $\beta = \{v_1, \dots, v_{2k}\}$ for $\mathcal{V}$ which is orthogonal with respect to Q_{16R_2}.

Suppose there are at least l distinct elements $v \in \beta$ such that $Q_{16R_2}(v,v) \geq e^{2a}$. Then since β is Q_{R_2}-orthonormal and Q_{16R_2}-orthogonal,

$$\begin{aligned}\log\left(\frac{\det Q_{16R_2}}{\det Q_{R_2}}\right)^{\frac{1}{2k}} &= \log\left(\prod_{j=1}^{2k} \frac{Q_{16R_2}(v_j,v_j)}{Q_{R_2}(v_j,v_j)}\right)^{\frac{1}{2k}} = \log\left(\prod_{j=1}^{2k} Q_{16R_2}(v_j,v_j)\right)^{\frac{1}{2k}} \\ &\geq \log\left(e^{2al}\right)^{\frac{1}{2k}} = \frac{l}{k}a.\end{aligned}$$

On the other hand,

$$a > h(i_2+1) - h(i_2) \geq \log\left(\det Q_{16R_2}\right)^{\frac{1}{2k}} - \log\left(\det Q_{R_2}\right)^{\frac{1}{2k}}.$$

So we have a contradiction if $l \geq k$.

Therefore we may choose a k element subset $\{u_1, \dots, u_k\} \subset \{v_1, \dots, v_{2k}\}$ such that $Q_{16R_2}(u_j,u_j) < e^{2a}$ for every $j \in \{1, \dots, k\}$. Then every element of $\mathcal{U} := \operatorname{span}\{u_1, \dots, u_k\}$ satisfies (3.12). $\square$

3.5. **Bounding the dimension of $\mathcal{V}$.** We now assume that w is the smallest integer such that

$$\left(\frac{R_1}{R_2}\right)^2 = 2 \cdot 16^{i_1 - i_2} < 2 \cdot 16^{-w} < \frac{1}{2Ce^{4a}}, \tag{3.13}$$

where C is the constant in (3.9). Therefore $2 \cdot 16^{-(w-1)} \geq \frac{1}{2Ce^{4a}}$, and this implies

$$e^{wa} \leq 64\, C\, e^{64d^2 \log 16}. \tag{3.14}$$

If $u \in \mathcal{U}$ lies in the kernel of Φ, then

$$\begin{aligned} Q_{R_2}(u,u) &\leq Ce^{2a}\left(\frac{R_1}{R_2}\right)^2 Q_{16R_2}(u,u) \quad \text{by (3.9)} \\ &\leq Ce^{2a}\left(\frac{R_1}{R_2}\right)^2 \left(e^{2a}\, Q_{R_2}(u,u)\right) \quad \text{by Lemma 3.5} \\ &\leq \frac{1}{2} Q_{R_2}(u,u) \quad \text{by (3.13)}. \end{aligned}$$

Therefore $u = 0$, and we conclude that $\Phi\big|_{\mathcal{U}}$ is injective. Hence by Lemma 3.3 and (3.14),

$$\dim \mathcal{V} = 2 \dim \mathcal{U} \leq 2|J| \leq 2e^{wa} \leq 128\, C\, e^{64d^2 \log 16}.$$

3.6. **The proof in the bounded geometry case.** We now return to the general case of Theorem 1.3, where X is a bounded geometry Riemannian manifold or a bounded degree graph quasi-isometric to Γ. The main difference with the case when $X = \Gamma$ is the following result:

Lemma 3.6. *There are constants $1 \leq A_1 \leq A_2 \leq A_3 < \infty$, $C \in (0, \infty)$ such that for every $x \in X$, $R \in (0, \infty)$, and every smooth function $f : B(x, A_3 R) \to \mathbb{R}$,*

$$\int_{B(x,R)} |f - f_R|^2 \leq C\, R^2 \, \frac{V(A_2 R)}{V(A_1 R)} \int_{B(x,A_3 R)} |\nabla f|^2, \tag{3.15}$$

where f_R denotes the average of f over $B(x, R)$.

Proof. This follows from Theorem 2.2 by applying [CSC95]. □

To prove Theorem 1.3, one modifies the argument given in the $X = \Gamma$ case by using Lemma 3.6 instead of Theorem 2.2, as well as the fact that the volume functions in G and X are asymptotically equivalent, i.e., for some $A, C \in [1, \infty)$,

$$C^{-1}\, V_G(A^{-1}\, r) \leq V_X(r) \leq C\, V_G(Ar)$$

for all sufficiently large r [Šva55, Mil68]. □

Remark 3.7. Similar reasoning would apply if X were a metric measure space which is doubling and satisfies a Poincaré inequality below some scale R_0.

4. Obtaining an infinite representation using Theorem 1.3

Let G be an infinite group with weakly polynomial growth, and let Γ denote some Cayley graph of G with respect to a symmetric finite generating set S. In this section we will show that G has a finite-dimensional representation with infinite image.

Note that G is amenable, since nonamenable groups have exponential growth; as it is also infinite, it does not have Property (T) (see [dlHV89, p. 6] or the appendix where these implications are explained for the nonexpert). Therefore by a result of Mok [Mok95] and Korevaar-Schoen [KS97, Theorem 4.1.2], there is an isometric action $G \curvearrowright \mathcal{H}$ of G on a Hilbert space $\mathcal{H}$ which has no fixed points and a nonconstant G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$. In the case of Cayley graphs, the Mok/Korevaar-Schoen result is quite elementary, so we give a short proof in the appendix.

Since f is G-equivariant, it is Lipschitz.

Each bounded linear functional $\phi \in \mathcal{H}^*$ gives rise to a Lipschitz harmonic function $\phi \circ f$, and hence we have a linear map $\Phi : \mathcal{H}^* \to \mathcal{V}$, where $\mathcal{V}$ is the space of Lipschitz harmonic functions on Γ. Since the target is finite dimensional by Theorem 1.3, the kernel of Φ has finite codimension, and its annihilator $\ker(\Phi)^{\perp} \subset \mathcal{H}$ is a finite-dimensional subspace containing the image of f. It follows that the affine hull A of the image of f is finite dimensional and G-invariant. Therefore we have an induced isometric G-action $G \curvearrowright A$. This action cannot factor through a finite group, because it would then have fixed points, contradicting the fact that the original representation is fixed point free. The associated homomorphism $G \to \mathrm{Isom}(A)$ yields the desired finite-dimensional representation of G. □

5. Proof of Theorem 1.2

We now complete the proof of Gromov's theorem; this is a recapitulation of Gromov's argument, which we reproduce here for the convenience of the reader.

The proof is by induction on the degree of growth.

Definition 5.1. Let G be a finitely generated group. The **degree (of growth) of** G is the minimum $\deg(G)$ of the nonnegative integers d such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^d} < \infty.$$

A group whose degree of growth is 0 is finite, and hence Theorem 1.2 holds for such a group.

Assume inductively that for some $d \in \mathbb{N}$ every group of degree at most $d-1$ is virtually nilpotent, and suppose $\deg(G) = d$. Then G is infinite, so by Section 4 there is a finite-dimensional linear representation $G \to GL(n)$ with infinite image $H \subset GL(n)$. Since H has polynomial growth, by [Tit72] (see [Sha98] for an easier proof) it is virtually solvable and by [Wol68, Mil68] it must be virtually nilpotent.

After passing to finite index subgroups, we may assume H is nilpotent and that its abelianization is torsion-free. It follows that there is a short exact sequence

$$1 \longrightarrow K \to G \stackrel{\alpha}{\to} \mathbb{Z} \longrightarrow 1.$$

By [vdDW84, Lemma 2.1], the normal subgroup K is finitely generated, and $\deg(K) \leq \deg(G) - 1$.

By the induction hypothesis, K is virtually nilpotent. Let K' be a finite index nilpotent subgroup of K which is normal in G, and let $L \subset G$ be an infinite cyclic subgroup which is mapped isomorphically by α onto $\mathbb{Z}$. Then $K'L \subset G$ is a finite index solvable subgroup of G. As it has polynomial growth, by [Wol68, Mil68] it is virtually nilpotent. $\square$

Appendix: Property (T) and equivariant harmonic maps

In this expository section, we will give a simple proof of the special case of the Korevaar-Schoen/Mok existence result needed in the proof of Theorem 1.2. For the nonexpert, we also explain why an infinite group of subexponential growth cannot have Property (T). The material has been optimized for the specific applications needed in the paper.

In this appendix G will be a finitely generated group, $S = S^{-1} \subset G$ a symmetric finite generating set, and Γ the associated Cayley graph.

A.1. **Energy functions and Property (T).**

Given an action $G \curvearrowright X$ on a metric space X, we define the **energy function** $E : X \to \mathbb{R}$ by

$$E(x) = \sum_{s\in S} d^2(sx, x).$$

We recall that G has Property (T) iff every isometric action of G on a Hilbert space has a fixed point [dlHV89, p. 47].

The following theorem is a weak version of some results in [FM05]; see also [Gro03, pp. 115–116]:

Theorem A.1. *The following are equivalent:*

(1) *G has Property (T).*
(2) *There is a constant $D \in (0,\infty)$ such that if $G \curvearrowright \mathcal{H}$ is an isometric action on a Hilbert space and $x \in \mathcal{H}$, then G fixes a point in $B(x, D\sqrt{E(x)})$.*
(3) *There are constants $D \in (0,\infty)$, $\lambda \in (0,1)$ such that if $G \curvearrowright \mathcal{H}$ is an isometric action on a Hilbert space and $x \in \mathcal{H}$, then there is a point $x' \in B(x, D\sqrt{E(x)})$ such that $E(x') \leq \lambda E(x)$.*
(4) *There is no isometric action $G \curvearrowright \mathcal{H}$ on a Hilbert space such that the energy function $E : \mathcal{H} \to \mathbb{R}$ attains a positive minimum.*

Proof. Clearly (2) $\Longrightarrow$ (1). Also, (1) $\Longrightarrow$ (4) since the energy function E is zero at a fixed point.

(3) $\Longrightarrow$ (2). Suppose (3) holds. Let $G \curvearrowright \mathcal{H}$ be an isometric action, and pick $x_0 \in \mathcal{H}$. Define a sequence $\{x_k\} \subset \mathcal{H}$ inductively, by choosing $x_{k+1} \in B(x_k, D\sqrt{E(x_k)})$ such that $E(x_{k+1}) \leq \lambda E(x_k)$. Then $E(x_k) \leq \lambda^k E(x_0)$ and $d(x_{k+1}, x_k) \leq D\sqrt{E(x_k)} \leq D\lambda^{\frac{k}{2}}\sqrt{E(x_0)}$. Therefore $\{x_k\}$ is Cauchy, with limit x_∞ satisfying

$$d(x_\infty, x_0) \leq \frac{D\sqrt{E(x_0)}}{1-\lambda^{\frac{1}{2}}}.$$

Then $E(x_\infty) = \lim_{k\to\infty} E(x_k) = 0$, and x_∞ is fixed by G. Therefore (2) holds.

(4) $\Longrightarrow$ (3). We prove the contrapositive. Assume that (3) fails. Then for every $k \in \mathbb{N}$, we can find an isometric action $G \curvearrowright \mathcal{H}_k$ on a Hilbert space and a point $x_k \in \mathcal{H}_k$ such that

$$E(y) > \left(1 - \frac{1}{k}\right) E(x_k) \tag{A.1}$$

for every $y \in B(x_k, k\sqrt{E(x_k)})$. Note that in particular, $E(x_k) > \left(1-\frac{1}{k}\right) E(x_k)$, forcing $E(x_k) > 0$.

Let $\mathcal{H}'_k$ be the result of rescaling the metric on $\mathcal{H}_k$ by $\frac{1}{\sqrt{E(x_k)}}$. Then (A.1) implies that the induced isometric action $G \curvearrowright \mathcal{H}'_k$ satisfies $E(x_k) = 1$ and

$$E(y) \geq 1 - \frac{1}{k} \tag{A.2}$$

for all $y \in B(x_k, k)$. Then any ultralimit (see [Gro93, KL97]) of the sequence $(\mathcal{H}_k, x_k)$ of pointed Hilbert spaces is a pointed Hilbert space $(\mathcal{H}_\omega, x_\omega)$ with an isometric action $G \curvearrowright \mathcal{H}_\omega$ such that

$$E(x_\omega) = 1 = \inf_{y\in\mathcal{H}_\omega} E(y).$$

Therefore (4) fails. □

A.2. **Harmonic maps and Property (T).** Before proceeding, we recall some facts about harmonic maps on graphs. Suppose $\mathcal{G}$ is a locally finite metric graph, where all edges have length 1. If $f : \mathcal{G} \to \mathcal{H}$ is a piecewise smooth map to a Hilbert space, then the following are equivalent:

- f is harmonic.
- The Dirichlet energy of f (on any finite subgraph) is stationary with respect to compactly supported variations of f.
- The restriction of f to each edge of $\mathcal{G}$ has constant derivative, and for every vertex $v \in \mathcal{G}$,

$$\sum_{d(w,v)=1} (f(w) - f(v)) = 0.$$

Note that if $G \curvearrowright \mathcal{H}$ is an isometric action on a Hilbert space, then the energy function E is a smooth convex function, and its derivative is

$$\begin{aligned} DE(x)(v) &= 2\left(\sum_{s\in S} \langle sx - x, (Ds)(v)\rangle - \sum_{s\in S} \langle sx - x, v\rangle\right) \\ &= 2\left(\sum_{s\in S} \langle x - s^{-1}x, v\rangle + \sum_{s\in S} \langle x - sx, v\rangle\right) = 4\sum_{s\in S} \langle x - sx, v\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} & x \in \mathcal{H} \text{ is a critical point of } E \\ \iff & x \text{ is a minimum of } E \\ \iff & \sum_{s\in S} (x - sx) = 0. \end{aligned} \tag{A.3}$$

It follows that the G-equivariant map $f_0 : G \to \mathcal{H}$ given by $f_0(g) := gx$ extends to a G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$ if and only if

$$\sum_{s\in S} (f_0(se) - f_0(e)) = \sum_{s\in S} (sx - x) = 0 \iff x \text{ is a minimum of } E.$$

The next result is a special case of a theorem from [Mok95, KS97].

Lemma A.2. *The following are equivalent:*

1. *G does not have Property (T).*
2. *There is an isometric action $G \curvearrowright \mathcal{H}$ on a Hilbert space $\mathcal{H}$ and a nonconstant G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$.*

Proof. (1) $\implies$ (2). If G does not have Property (T), then by Theorem A.1 there is an isometric action $G \curvearrowright \mathcal{H}$ on a Hilbert space and a point $x \in \mathcal{H}$ with $E(x) = \inf_{y\in\mathcal{H}} E(y) > 0$. Let $f : \Gamma \to \mathcal{H}$ be the G-equivariant map with $f(g) = gx$ for every $g \in G \subset \Gamma$ and whose restriction to each edge e of Γ has constant derivative. Then f is harmonic and obviously nonconstant.

(2) $\implies$ (1). Suppose (2) holds and $f : \Gamma \to \mathcal{H}$ is the G-equivariant harmonic map. Then $f(e)$ is a positive minimum of $E : \mathcal{H} \to \mathbb{R}$; in particular the action $G \curvearrowright \mathcal{H}$ has no fixed points. Therefore G does not have Property (T). $\square$

A.3. **Amenability and Property (T).** We now recall, using the definitions most closely tied to the situation of this paper, why an infinite amenable group—for instance a group of weakly polynomial growth—does not satisfy Property (T).

Definition A.3. If $F \subset G$, then the *boundary of* F is the set ∂F of elements $g \in F$ at distance 1 from the complement $G \setminus F$.

We may define a map $\partial F \to N_1(F) \setminus F$ by sending $g \in \partial F$ to some adjacent element of $G \setminus F$; since every element of G is adjacent to $|S|$ elements, it follows that this map is at most $|S|$-to-1, and so

$$|N_1(F) \setminus F| \geq \frac{1}{|S|} |\partial F| . \tag{A.4}$$

Definition A.4. The group G is *amenable* if it contains a *Folner sequence*, i.e., there is a sequence $\{F_k\}$ of finite subsets of G, such that $\frac{|\partial F_k|}{|F_k|} \to 0$ as $k \to \infty$, and it is *nonamenable* otherwise. Thus G is nonamenable iff there is a constant $C \in (0, \infty)$ such that

$$|\partial F| \geq C\, |F| \tag{A.5}$$

for all finite subsets $F \subset G$.

Lemma A.5. *Nonamenable groups have exponential growth. In particular, a group of weakly polynomial growth is amenable.*

Proof. Pick $C \in (0, \infty)$ so that (A.5) holds. Let $B(r)$ denote the r-ball centered at $e \in G$, for r a nonnegative integer. Then

$$|B(r+1)| = |B(r)| + |B(r+1) \setminus B(r)| \geq |B(r)| + \frac{1}{|S|}|\partial B(r)| \geq |B(r)| + \frac{C}{|S|}|B(r)| .$$

Thus $|B(r)| \geq (1 + \frac{C}{|S|})^r$. □

Lemma A.6. *Consider the left regular representation $G \curvearrowright \ell^2(G)$ of G, where $(g \cdot u)(h) = u(g^{-1}h)$ for all $g, h \in G$ and $u \in \ell^2(G)$.*

(1) *If G is amenable, there is a sequence $\{u_k\}$ of unit vectors in $\ell^2(G)$ such that*
$$\limsup_{k \to \infty} \max_{s \in S} \|s \cdot u_k - u_k\| = 0 .$$
(2) *If G is amenable and has Property (T), it is finite.*

Proof. (1) Let $\{F_k\}$ be a Folner sequence in G. Define $v_k \in \ell^2(G)$ by $v_k(g) = \chi_{F_k}(g^{-1})$, where χ_F is the characteristic function of F; put $u_k = \frac{1}{|F_k|^{\frac{1}{2}}} v_k$. Then for all $g \in G$, $s \in S$,

$$((s \cdot v_k) - v_k)(g) = \chi_{F_k}(g^{-1}s) - \chi_{F_k}(g^{-1})$$

which is nonzero only if either g^{-1} or $g^{-1}s$ is in ∂F_k; therefore

$$\|s \cdot v_k - v_k\|^2 \leq 2|\partial F_k| ,$$

and

$$\|s \cdot u_k - u_k\| = \frac{1}{|F_k|^{\frac{1}{2}}}\|s \cdot v_k - v_k\| \leq \frac{\sqrt{2}|\partial F_k|^{\frac{1}{2}}}{|F_k|^{\frac{1}{2}}},$$

which tends to zero as $k \to \infty$.

(2) Let $\{u_k\}$ be as in (1). For the isometric action given by the left regular representation, the energy satisfies $E(u_k) \to 0$ as $k \to \infty$. By Theorem A.1, it follows that the action $G \curvearrowright \ell^2(G)$ has a fixed point $v \in \ell^2(G)$ which is nonzero. Then v is a nonzero G-invariant ℓ^2 function, which forces G to be finite. □

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