

WEAK MIXING DIRECTIONS IN NON-ARITHMETIC VEECH SURFACES

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1. INTRODUCTION

1.1. Weak-mixing directions for billiards in regular polygons. Let $n \geq 3$ be an integer and consider the billiard in an n -sided regular polygon P_n . It is readily seen that the 3-dimensional phase space (the unit tangent bundle T^1P_n) decomposes into a one-parameter family of invariant surfaces, as there is a clear integral of motion. In such a setting, it is thus natural to try to understand the dynamics restricted to each of, or at least most of, the invariant surfaces.

The cases $n = 3, 4, 6$ are simple to analyze, essentially because they correspond to a lattice tiling of the plane: the dynamics is given by a linear flow on a torus, so for a countable set of surfaces all trajectories¹ are periodic, and for all others the flow is quasiperiodic and all trajectories are equidistributed with respect to Lebesgue measure on the surface.

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¹Here we restrict consideration to orbits that do not end in a singularity (i.e., a corner of the billiard table) in finite time.

For $n \neq 3, 4, 6$, the invariant surfaces have a higher genus, and quasiperiodicity is not automatic anymore. However, Veech [Ve89] showed that a dichotomy still holds: for a countable set of surfaces all infinite trajectories are periodic, and for all others all trajectories are equidistributed with respect to Lebesgue measure.

One important question that follows the work of Veech is whether weak mixing takes place (the absence of mixing is a general property in the more general class of translation flows, which is known from earlier work of Katok [Ka80]). Recall that weak mixing means that there is no remainder of periodicity or quasiperiodicity from the measurable point of view (i.e., there is no factor which is periodic or quasiperiodic), and can thus be interpreted as the complete breakdown of the nice lattice behavior seen for $n = 3, 4, 6$.

While results about the prevalence of weak mixing were obtained in the more general context of translation flows ([AF07], [AF]), the case of regular polygons proved to be much more resistant. The basic reason is that the most successful approaches so far were dependent on the presence of a suitable number of parameters which can be used in a probabilistic exclusion argument, and as a consequence they were not adapted to study the rigid situation of a specific billiard table. In this paper we address directly the problem of weak mixing for exceptionally symmetric translation flows, which include the ones arising from regular polygonal billiards.

Theorem 1. *If $n \neq 3, 4, 6$, then the restriction of the billiard flow in P_n to almost every invariant surface is weak mixing.*

Of course, in view of Veech’s remarkably precise answer to the problem of equidistribution, one could wonder whether weak mixing is not only a prevalent property, but one that might hold outside a countable set of exceptions. It turns out that this is not the case in general, and in fact we will show that the set of exceptions can be relatively large and have a positive Hausdorff dimension. However, we will prove that it can never have a full dimension.

1.2. Non-arithmetic Veech surfaces. We now turn to the general framework in which the previous result fits. A *translation surface* is a compact surface S which is equipped with an atlas defined on the complement of a finite and non-empty set of “marked points” Σ , such that the coordinate changes are translations in \mathbb{R}^2 and each marked point p has a punctured neighborhood isomorphic to a finite cover of a punctured disk in \mathbb{R}^2 . The geodesic flow in any translation surface has an obvious integral of motion, given by the angle of the direction, which decomposes it into separate dynamical systems, the *directional flows*.

An *affine homeomorphism* of a translation surface (S, Σ) is a homeomorphism of S which fixes Σ pointwise and which is affine and orientation preserving in the charts. The linear part of such homeomorphism is well defined in $\mathrm{SL}(2, \mathbb{R})$. This allows one to define a homomorphism from the group of affine homeomorphisms to $\mathrm{SL}(2, \mathbb{R})$: its image is a discrete subgroup called the *Veech group* of the translation surface.

A *Veech surface* is an “exceptionally symmetric” translation surface, in the sense that the Veech group is a (finite co-volume) lattice in $\mathrm{SL}(2, \mathbb{R})$ (it is easily seen that the Veech group is never co-compact). Simple examples of Veech surfaces are *square-tiled surfaces*, obtained by gluing finitely many copies of the unit square $[0, 1]^2$ along their sides. In this case the Veech group is commensurable with $\mathrm{SL}(2, \mathbb{Z})$. Veech surfaces that can be derived from square-tiled ones by an affine

diffeomorphism are called *arithmetic*. By construction, arithmetic Veech surfaces are branched covers of flat tori, so their directional flows are never topologically weak mixing (they admit a *continuous* quasiperiodic factor in *any* direction).

The first examples of non-arithmetic Veech surfaces were described by Veech and correspond precisely to billiard flows on regular polygons. It is easy to see that the billiard flow in P_n corresponds, up to finite cover, to the geodesic flow on a translation surface obtained by gluing the opposite sides of P_n (when n is even) or the corresponding sides of P_n and $-P_n$ (when n is odd). This construction yields indeed a Veech surface S_n which is non-arithmetic precisely when $n \neq 3, 4, 6$. The genus g of S_n is related to n by $g = \lfloor \frac{n}{4} \rfloor$ (n even) and $g = \frac{n-1}{2}$ (n odd).

We can now state the main result of this paper.

Theorem 2. *The geodesic flow in a non-arithmetic Veech surface is weakly mixing in almost every direction. Indeed the Hausdorff dimension of the set of exceptional directions is less than one.*

An important algebraic object associated to the Veech group Γ of a Veech surface (S, Σ) is the *trace field* k which is the field extension of \mathbb{Q} generated by the traces of the elements of Γ . Its degree $r = [k : \mathbb{Q}]$ satisfies $1 \leq r \leq g$ where g is the genus of S . Moreover, we have $r = 1$ if and only if S is arithmetic. As an example, the trace field of S_n is $\mathbb{Q}[\cos \frac{\pi}{n}]$ if n is odd or $\mathbb{Q}[\cos \frac{2\pi}{n}]$ if n is even.

Theorem 3. *Let S be a Veech surface with a quadratic trace field (i.e., $r = 2$). Then the set of directions for which the directional flow is not even topologically weak mixing has a positive Hausdorff dimension.*

Notice that this covers the case of certain polygonal billiards ($\mathbb{Q}[\cos \frac{\pi}{n}]$ is quadratic if and only if $n \in \{4, 5, 6\}$; hence the above result holds for S_n with $n \in \{5, 8, 10, 12\}$) and of all non-arithmetic Veech surfaces in genus 2. We point out that Theorem 3 is a particular case of a more general result, Theorem 32, which does cover some non-arithmetic Veech surfaces with non-quadratic trace fields (indeed it applies to all non-arithmetic S_n with $n \leq 16$, the degrees of their trace fields ranging from 2 to 6) and could possibly apply to all non-arithmetic Veech surfaces. Let us also note that the non-weak mixing directions obtained in Theorem 32 have multiple rationally independent eigenvalues.

One crucial aspect of our analysis is a better description of the possible eigenvalues (in any minimal direction) in a Veech surface. Using the algebraic nature of Veech surfaces, we are able to conclude several restrictions on the group of eigenvalues. For non-arithmetic Veech surfaces, the ratio of two eigenvalues always belongs to k , and the number of rationally independent eigenvalues is always at most $[k : \mathbb{Q}]$. Moreover, the group of eigenvalues is finitely generated (so the *Kronecker factor* is always a minimal translation of a finite dimensional torus), whereas for the case of arithmetic Veech surfaces, we obtain that all eigenvalues come from a ramified cover of a torus. See Theorem 23 for the precise statement.

We expect that, for a non-arithmetic Veech surface and along any minimal direction that is not weak mixing, there are always exactly $[k : \mathbb{Q}]$ independent eigenvalues, and that they are either all continuous or all discontinuous. This is the case along directions for which the corresponding forward Teichmüller geodesic is bounded in moduli space; see Remark 7.1.

1.3. Further comments. The strategy to prove weak mixing for a directional flow on a translation surface S is to show that the associated unitary flow has no non-trivial eigenvalues. It is convenient to first rotate the surface so that the directional flow goes along the vertical direction. The small scale behavior of eigenfunctions associated to a possible eigenvalue can then be studied using renormalization. Technically, one parametrizes all possible eigenvalues by the line in $H^1(S; \mathbb{R})$ through the imaginary part of the tautological one form (the Abelian differential corresponding to the translation structure) and then applies the so-called Kontsevich-Zorich cocycle over the Teichmüller flow. According to the Veech criterion any actual eigenvalue is parametrized by an element of the “weak-stable lamination” associated to an acceleration of the Kontsevich-Zorich cocycle acting modulo $H^1(S; \mathbb{Z})$. Intuitively, eigenfunctions parametrized by an eigenvalue outside the weak stable lamination would exhibit so much oscillation in small scales that measurability must be violated. The core of [AF07] is a probabilistic method to exclude non-trivial intersections of an arbitrary fixed line in $H^1(S; \mathbb{R})$ with the weak stable lamination, which uses basic information on the non-degeneracy of the cocycle.

The problem of weak mixing in the case of S_5 was asked by C. McMullen during a talk of the first author on [AF07] in 2004. It was realized during discussions with P. Hubert that the probabilistic method behind [AF07] fails for Veech surfaces, due essentially to a degeneracy (non-twisting) of the Kontsevich-Zorich cocycle. Attempts to improve the probabilistic argument using Diophantine properties of invariant subspaces turned out to lead to too weak estimates.

In this paper we prove that the locus of possible eigenvalues is much more constrained in the case of Veech surfaces: eigenvalues must be parametrized by an element in the “strong stable lamination,” consisting of all the integer translates of the stable space, which is a much simpler object than the weak stable lamination (considered modulo the strong stable space, the former is countable, while the latter is typically uncountable). Direct geometric estimates on the locus of intersection can then be obtained using much less information on the non-degeneracy of the cocycle.

In order to obtain this stronger constraint on the locus of possible eigenvalues we will first carry out an analysis of the associated eigenfunctions at scales corresponding to renormalizations belonging to a large compact part of the moduli space (this refines Veech’s criterion, which handles compact sets that are small enough to be represented in spaces of interval exchange transformations). This is followed by a detailed analysis of the excursion to the non-compact part of the moduli space, which is used to forbid the occurrence of an integer “jump” in cohomology along such an excursion. In doing so, we use in an essential way the particularly simple nature of the renormalization dynamics in the non-compact part of the moduli space $SL(2, \mathbb{R})/\Gamma$ of a Veech surface (a finite union of cusps).

The fact that the flow is weakly mixing in almost every direction then follows from the absence of atoms in the harmonic measure of the Kontsevich-Zorich cocycle. To get further and obtain a bound on the Hausdorff dimension of exceptional directions, we use a Markov coding and quantitative estimates coming from a large deviation upper bound in Oseledets theorem.

We should point out that it is reasonable to expect that, in the general case of translation flows, one can construct examples of eigenvalues which do not come from the strong stable lamination. Indeed Guenais and Parreau construct suspension

flows over ergodic interval exchange transformations and with a piecewise constant roof function admitting infinitely many independent eigenvalues (see Theorem 3 of [GP]), and this provides many eigenvalues that do not come from the strong stable lamination (which can only be responsible for a subgroup of eigenvalues of finite rank). See also [BDM2], Section 6, for a different example in a related context.

2. PRELIMINARIES

2.1. Translation surfaces, moduli space, $SL(2, \mathbb{R})$ -action. A *translation surface* can also be defined as a triple (S, Σ, ω) of a closed surface S , a non-empty finite set $\Sigma \subset S$, and an Abelian differential ω on S whose zeros are contained in Σ (ω is holomorphic for a unique complex structure on S).² Writing in local coordinates $\omega = dz$, we get canonical charts to \mathbb{C} such that transition maps are translations. Such a map exists at $x \in S$ if and only if ω is non-zero at x . The zeros of ω are the *singularities* of the translation surface. Conversely, a translation surface S defined as in the Introduction (in terms of a suitable atlas on $S \setminus \Sigma$) allows one to recover the Abelian differential ω by declaring that $\omega = dz$ and extending it uniquely to the marked points. We write (S, ω) for the translation surface for which Σ is the set of zeros of ω .

Let (S, Σ, ω) be a translation surface. The form $|\omega|$ defines a flat metric except at the singularities. For each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we define the *directional flow* in the direction θ as the flow $\phi_T^{S, \theta} : S \rightarrow S$ obtained by integration of the unique vector field X_θ such that $\omega(X_\theta) = e^{i\theta}$. In local charts z such that $\omega = dz$ we have $\phi_T^{S, \theta}(z) = z + Te^{i\theta}$ for small T . The directional flows are also called *translation flows*. The (*vertical*) *flow* of (S, ω) is the flow $\phi^{S, \pi/2}$ in the vertical direction. The flow is not defined at the zeros of ω and hence not defined for all positive times on backward orbits of the singularities. The flows $\phi_T^{S, \theta}$ preserve the volume form $\frac{1}{2i}\omega \wedge \bar{\omega}$, and the ergodic properties of translation flows that we discuss below refer to this measure.

Translation surfaces were introduced to study rational billiards as the example of the regular polygons P_n in the Introduction. Each rational billiard may be seen as a translation surface by a well known construction called unfolding or Zemliakov-Katok construction (see [MT02]).

The following results hold for arbitrary translation surfaces: the directional flow is minimal except for a countable set of directions [Ke75], the translation flow is uniquely ergodic except for a set of directions of Hausdorff dimension at most $1/2$ [KMS86], [Ma92], and the translation flow is not mixing in any direction [Ka80].

The weak mixing property is more subtle. Indeed, translation flows in a genus one translation surface are never weakly mixing. The same property holds for branched coverings of genus one translation surfaces which form a *dense* subset of translation surfaces. However, for *almost every* translation surface of genus at least two, the translation flow is weakly mixing in almost every direction [AF07]. The implicit topological and measure-theoretical notions above refer to a natural structure on the space of translation structure that we introduce now.

Let $g, s \geq 1$, let S be a closed surface of genus g , let $\Sigma \subset S$ be a subset with $\#\Sigma = s$, and let $\kappa = (\kappa_x)_{x \in \Sigma}$ be a family of non-negative integers such that $\sum \kappa_i = 2g - 2$. The set of translation structures on S with prescribed conical angle $(\kappa_x + 1)2\pi$ at x ,

²A simpler definition would be to consider Σ to be the set of zeros of ω . But in that case we exclude the flat tori \mathbb{C}/Λ with the Abelian differential dz which has no zero. Except in the special case of the torus, one can safely take Σ to be the zeros of ω .

modulo isotopy relative to Σ , forms a manifold $\mathcal{T}_{S,\Sigma}(\kappa)$. The manifold structure on $\mathcal{T}_{S,\Sigma}(\kappa)$ is described by the so-called *period map* $\Theta : \mathcal{T}_{S,\Sigma}(\kappa) \rightarrow H^1(S, \Sigma; \mathbb{C})$ which associates to ω its cohomology class in $H^1(S, \Sigma; \mathbb{C})$. The period map is locally bijective and provides natural charts to $\mathcal{T}_{S,\Sigma}(\kappa)$ as well as an affine structure and a canonical Lebesgue measure. We let $\mathcal{T}_{S,\Sigma}^{(1)}(\kappa)$ denote the hypersurface of area 1 translation surfaces.

The group $\mathrm{SL}(2, \mathbb{R})$ acts (on left) on $\mathcal{T}_{S,\Sigma}(\kappa)$ by postcomposition on the charts and preserves the hypersurface $\mathcal{T}_{S,\Sigma}^{(1)}(\kappa)$. The subgroup of rotations $r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ acts by multiplication by $e^{i\theta}$ on ω . The action of the diagonal subgroup $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is called the *Teichmüller flow* and transforms $\omega = \mathrm{Re}(\omega) + i \mathrm{Im}(\omega)$ into $g_t \cdot \omega = e^t \mathrm{Re}(\omega) + ie^{-t} \mathrm{Im}(\omega)$. The *stable* and *unstable horocycle flows* are the action of the matrices $h_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and $h_s^+ = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

The *modular group* $\mathrm{MCG}(S, \Sigma)$ of (S, Σ) is the group of diffeomorphisms of S fixing Σ pointwise modulo isotopy relative to Σ . It acts discretely discontinuously (on right) on the spaces $\mathcal{T}_{S,\Sigma}(\kappa)$ and $\mathcal{T}_{S,\Sigma}^{(1)}(\kappa)$ via $(S, \omega) \mapsto (S, \omega \circ d\phi)$. The quotient, denoted $\mathcal{M}_{S,\Sigma}(\kappa)$ or $\mathcal{M}_{S,\Sigma}^{(1)}(\kappa)$, is called a *stratum of the moduli space of translation surfaces of genus g and s marked points* or shortly a *stratum*.³ The space $\mathcal{M}_{S,\Sigma}(\kappa)$ inherits from $\mathcal{T}_{S,\Sigma}(\kappa)$ is a complex affine orbifold structure. The $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{MCG}(S, \Sigma)$ actions on $\mathcal{T}_{S,\Sigma}(\kappa)$ commute. Hence the $\mathrm{SL}(2, \mathbb{R})$ action is well defined on the quotient $\mathcal{M}_{S,\Sigma}(\kappa)$. The Lebesgue measure projects on $\mathcal{M}_{S,\Sigma}(\kappa)$ (resp. $\mathcal{M}_{S,\Sigma}^{(1)}(\kappa)$) into a measure ν (resp. $\nu^{(1)}$) called the Masur-Veech measure. Masur [Ma82] and Veech [Ve82] proved independently that the measure $\nu^{(1)}$ has finite total mass, the action of $\mathrm{SL}(2, \mathbb{R})$ on each $\mathcal{M}_{S,\Sigma}(\kappa)$ is measure preserving, and moreover the Teichmüller flow g_t is ergodic on each connected component of $\mathcal{M}_{S,\Sigma}^{(1)}(\kappa)$. The moduli space, the Teichmüller flow, and the Kontsevich-Zorich cocycle are of main importance in the results we mentioned above about the ergodic properties of translation flows.

We will also sometimes use the notation $\mathcal{M}_g(\kappa)$ to denote $\mathcal{M}_{S,\Sigma}(\kappa)$, where the $(\kappa_j)_{1 \leq j \leq s}$ is obtained by reordering the $(\kappa_x)_{x \in S}$ in non-increasing order. As an example, for n even, the surface S_n built from a regular n -gon belongs to the stratum $\mathcal{M}_{\lfloor n/4 \rfloor}((n-4)/2)$ if $n \equiv 0 \pmod 4$ or $\mathcal{M}_{\lfloor n/4 \rfloor}((n-6)/4, (n-6)/4)$ if $n \equiv 2 \pmod 4$.

2.2. Veech surfaces. Our goal in this article is to study the weak-mixing property for the directional flows in a translation surface with closed $\mathrm{SL}(2, \mathbb{R})$ -orbits in $\mathcal{M}_{S,\Sigma}(\kappa)$, which are called Veech surfaces.

Let us recall that an *affine homeomorphism* of a translation surface (S, Σ, ω) is a homeomorphism of S which preserves Σ pointwise and is affine in the charts of S compatible with the translation structure. An affine homeomorphism ϕ has a well defined linear part, denoted by $d\phi \in \mathrm{SL}(2, \mathbb{R})$, which is the derivative of the action of ϕ in the charts. The set of linear parts of affine diffeomorphisms forms a discrete subgroup $\Gamma(S, \Sigma, \omega)$ of $\mathrm{SL}(2, \mathbb{R})$ called the *Veech group* of (S, Σ, ω) . A

³Our definition of strata is not standard as we consider a marking. It matches the standard one up to a finite cover.

translation surface is called a *Veech surface* if its Veech group is a lattice. The $SL(2, \mathbb{R})$ orbit of a Veech surface is closed in $\mathcal{M}_{S, \Sigma}(\kappa)$ and is naturally identified with the quotient $\mathcal{C} = SL(2, \mathbb{R})/\Gamma(S, \Sigma, \omega)$. The $SL(2, \mathbb{R})$ action on \mathcal{C} preserves the natural Haar measure, and the Teichmüller flow g_t is the geodesic flow on the unit tangent bundle of the hyperbolic surface $\mathbb{H}/\Gamma(S, \Sigma, \omega)$.

Veech proved that the Veech group of a Veech surface is never co-compact. Moreover, the cusp excursions may be measured in terms of systoles as in the well known case of lattices with $\Gamma = SL(2, \mathbb{Z})$. A *saddle connection* in a translation surface (S, Σ, ω) is a geodesic segment for the metric $|\omega|$ that starts and ends in Σ and whose interior is disjoint from Σ . For the square torus, $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}i)$ with the 1-form dz , the set of saddle connections identifies with primitive vectors (i.e., the vector of the form $pi + q$ with p and q relatively prime integers). For a translation surface (S, ω) the *systole* $\text{sys}(S, \omega)$ of (S, ω) is the length of the shortest saddle connection in (S, ω) . Assume that (S, ω) is a Veech surface and denote \mathcal{C} its $SL(2, \mathbb{R})$ -orbit in $\mathcal{M}_{S, \Sigma}(\kappa)$. Then the set $\mathcal{C}_\varepsilon = \{(S, \omega) \in \mathcal{C} : \text{sys}(S, \omega) \geq \varepsilon\}$ forms an exhaustion of \mathcal{C} by compact sets.

Beyond arithmetic surfaces (covers of the torus ramified over one point) the first examples of Veech surfaces are the translation surfaces S_n associated to the billiard in the regular polygon with n sides $P_n \subset \mathbb{R}^2$. The surface S_n is built from P_n (n even) or from the disjoint union of P_n and $-P_n$ (n odd) [Ve89]. In either case, S_n is defined by identifying every side of P_n with the unique other side (of either P_n or $-P_n$ according to the parity of n) parallel to it, via translations. For them, the Veech group as well as the trace field was computed by Veech.

Theorem 4 ([Ve89]). *Let S_n be the Veech surface associated to the regular polygon with n sides. Then*

- (1) *if n is odd, the Veech group of S_n is the triangle group $\Delta(2, n, \infty)$ with trace field $\mathbb{Q}[\cos(\pi/n)]$;*
- (2) *if n is even, the Veech group of S_n is the triangle group $\Delta(n/2, \infty, \infty)$ with trace field $\mathbb{Q}[\cos(2\pi/n)]$.*

Notice that for n even, $\Delta(n/2, \infty, \infty)$ is a subgroup of index 2 of $\Delta(2, n, \infty)$. Other examples with triangle groups as Veech groups were discovered by Ward [Wa98] and Bouw-Möller [BM10]. Some of them are obtained from billiards, and a general construction with polygon gluings is provided in [Ho12]. See also the article of Wright [Wr13] which explains these constructions in terms of Schwarz triangle maps.

Beyond triangle groups, Calta [Ca04] and McMullen [McM03] introduced an infinite family of Veech surfaces with quadratic trace field in the stratum $\mathcal{M}_2(2)$ (the family includes S_5 and S_8). Later on, McMullen [McM07] proved that this is the complete list of non-arithmetic surfaces in $\mathcal{M}_2(2)$ and that in $\mathcal{M}_2(1, 1)$ the surface S_{10} is the unique non-arithmetic Veech surface. Other surfaces with quadratic trace fields were discovered by McMullen [McM06] in genera 3 and 4 and further studied by Lanneau and Nguyen [LN14].

2.3. Translation flow of Veech surfaces. Let (S, Σ, ω) be a translation surface and assume that there exists an affine homeomorphism ϕ whose image under the Veech group g is parabolic. The direction determined by the eigenvector of g in \mathbb{R}^2 is a *completely periodic direction* in S : the surface (S, Σ, ω) decomposes into a finite union of cylinders whose waist curves are parallel to it. Moreover, ϕ preserves each

cylinder and acts as a power of a Dehn twist in each of them. We may assume that the eigendirection is vertical, and hence $g = h_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ for some real number s . Let h_1, h_2, \dots, h_k and w_1, w_2, \dots, w_k be the widths and the heights of the cylinders C_1, \dots, C_k in the vertical direction. For each cylinder C_i , let ϕ_i be the Dehn twist in C_i . Its linear part is $g_i = \begin{pmatrix} 1 & 0 \\ \mu(C_i)^{-1} & 1 \end{pmatrix}$ where $\mu(C_i) = w_i/h_i$ is the modulus of C_i and the real number s is such that $s/\mu(C_i)$ are integers. Reciprocally, a completely periodic direction admits a non-trivial stabilizer in $SL(2, \mathbb{R})$ if and only if the moduli $\mu(C_i)$ of the cylinders are commensurable (their ratios are rational numbers). Such a direction is called *parabolic*.

Keane and Kerckhoff-Masur-Smillie theorems about minimality and unique ergodicity of translation flows (see Section 2.1) have the following refinement.

Theorem 5 (Veech alternative [Ve89]). *Let (S, Σ, ω) be a Veech surface. Then*

- (1) *either there exists a vertical saddle connection and the vertical direction in (S, Σ, ω) is parabolic*
- (2) *or the vertical flow is uniquely ergodic.*

The $SL(2, \mathbb{R})$ orbit of a Veech surface is never compact (any fixed saddle connection can be shrunk arbitrarily by means of the $SL(2, \mathbb{R})$ action, thus escaping any compact subset of the moduli space). Nevertheless, the geometry of flat surfaces in the cusps is well understood and will be crucial in the study of eigenvalues (see Section 3). If ζ and ζ' are two saddle connections in a flat surface (S, ω) , we let $\zeta \wedge_\omega \zeta'$ denote the number in \mathbb{C} that corresponds to the (signed) area of the parallelogram determined by $\omega(\zeta)$ and $\omega(\zeta')$.

Theorem 6 (no small triangles). *Let (S, ω) be a Veech surface. Then there exists $\kappa > 0$ such that for any pair of saddle connections ζ and ζ'*

- *either $|\zeta \wedge_\omega \zeta'| > \kappa$*
- *or ζ and ζ' are parallel (i.e., $\zeta \wedge_\omega \zeta' = 0$).*

The above theorem is actually the easy part of a characterization of Veech surfaces proved in [SW10]. Note that the quantity $\zeta \wedge_\omega \zeta'$ is invariant under the Teichmüller flow (i.e., $\zeta \wedge_{g \cdot \omega} \zeta' = \zeta \wedge_\omega \zeta'$ for any $g \in SL(2, \mathbb{R})$) and corresponds to twice the area of a (virtual) triangle delimited by ζ and ζ' . We deduce in particular that if there exists a small saddle connection in a Veech surface, then any other short saddle connection would be parallel to it and that the smallness is uniform for the whole $SL(2, \mathbb{R})$ -orbit.

Corollary 7. *Let \mathcal{C} be a $SL(2, \mathbb{R})$ -orbit of a Veech surface. Then there exists $\varepsilon > 0$ such that for any $u \notin \mathcal{C}_\varepsilon$ the saddle connections in u shorter than 1 are parallel to the direction of the shortest saddle connection in u .*

2.4. Holonomy field and conjugates of Veech group. Let (S, Σ, ω) be a translation surface and let $\Lambda = \omega(H_1(S; \mathbb{Z})) \subset \mathbb{C} \simeq \mathbb{R}^2$ be the module of periods of ω . In what follows, periods will sometimes be called holonomies. Let e_1 and e_2 be two non-parallel elements in Λ . The *holonomy field* of (S, Σ, ω) is the smallest field k of \mathbb{R} such that any element in Λ may be written as a k -linear combination of e_1 and e_2 . For any subgroup Γ of $SL(2, \mathbb{R})$, the *trace field* of Γ is the field generated by the traces of the element of Γ .

Theorem 8 (Gutkin-Judge [GJ00], Kenyon-Smillie [KS00]). *Let (S, Σ, ω) be a Veech surface. Then its holonomy field k coincides with the trace field of its Veech group. The degree of k over \mathbb{Q} is at most the genus of S , and the rank of $\Lambda = \omega(H_1(S; \mathbb{Z}))$ is twice the degree of k over \mathbb{Q} .*

We will need two important facts about the holonomy field of a Veech surface.

Theorem 9 (Gutkin-Judge [GJ00]). *A Veech surface (S, Σ, ω) is arithmetic (i.e., a ramified cover of a torus over one point) if and only if its holonomy field is \mathbb{Q} .*

Theorem 10 ([LH06]). *The holonomy field of a Veech surface is totally real.⁴*

Now, we define the Galois conjugates of the Veech group. Let (S, Σ, ω) be a Veech surface, let Γ be its Veech group and let k be its holonomy field. Let e_1 and e_2 be two non-parallel elements in the set of holonomies $\Lambda = \omega(H_1(S; \mathbb{Z}))$. For each element $v \in H_1(S; \mathbb{Z})$ there exist unique elements α and β of k such that $\omega(v) = \alpha e_1 + \beta e_2$. The maps α and β are linear with values in k ; in other words, they belong to $H^1(S; k)$, and moreover, the tautological space $V = \mathbb{R} \operatorname{Re}(\omega) \oplus \mathbb{R} \operatorname{Im}(\omega)$ can be rewritten as $V = \mathbb{R}\alpha \oplus \mathbb{R}\beta$ in $H^1(S; \mathbb{R})$. Note that an alternative definition of the trace field would be the field of definition of the plane $\mathbb{R} \operatorname{Re}(\omega) + \mathbb{R} \operatorname{Im}(\omega)$ in $H^1(S; \mathbb{R})$. For any embedding $\sigma : k \rightarrow \mathbb{R}$, we may define new linear forms $\sigma \circ \alpha$ and $\sigma \circ \beta$. Those linear forms generate a 2-dimensional subspace in $H^1(S; \mathbb{R})$. The subspace does not depend on the choice of α and β , and we call it the *conjugate by σ* of the tautological subspace V . This subspace (which is indeed defined in $H^1(S; k)$) will be denoted by V^σ . Because the action of the affine group on homology is defined over \mathbb{Z} and preserves V , it preserves as well the conjugates V^σ . The following result appeared in [Mo06] in terms of decomposition of variation of Hodge structures.

Theorem 11 ([Mo06]). *Let (S, Σ, ω) be a Veech surface, k be its holonomy field, and $V = \mathbb{R} \operatorname{Re} \omega \oplus \mathbb{R} \operatorname{Im} \omega$ be the tautological subspace. Then for any embedding $\sigma : k \rightarrow \mathbb{R}$ the subspace V^σ is invariant under the action of the affine group of (S, Σ, ω) . Moreover, the space generated by the $[k : \mathbb{Q}]$ subspaces $V^\sigma \subset H^1(S; \mathbb{R})$ is the smallest rational subspace of $H^1(S; \mathbb{R})$ containing V and is the direct sum of the V^σ .*

The fact that the sum is direct follows from the presence of hyperbolic elements in the Veech group (see Theorem 28 of [KS00]).

For an affine homeomorphism ϕ of a translation surface (S, Σ, ω) we have

$$\begin{pmatrix} \phi^* \operatorname{Re}(\omega) \\ \phi^* \operatorname{Im}(\omega) \end{pmatrix} = D\phi \cdot \begin{pmatrix} \operatorname{Re}(\omega) \\ \operatorname{Im}(\omega) \end{pmatrix}.$$

Hence, the Veech group Γ is canonically identified to the action of the affine group on the tautological subspace $V = \mathbb{R} \operatorname{Re} \omega \oplus \mathbb{R} \operatorname{Im} \omega$. The choice of two elements of $H_1(S; \mathbb{Z})$ with non-parallel holonomy provides an identification of Γ as a subgroup of $\operatorname{SL}(2, k)$. Given an embedding of k in \mathbb{R} we may conjugate the coefficients of the matrices in $\operatorname{SL}(2, k)$ and get a new embedding of Γ into $\operatorname{SL}(2, \mathbb{R})$. This embedding is canonically identified to the action of the affine group on the conjugate of the tautological bundle V^σ . We denote by Γ^σ this group and call it the *conjugate of the Veech group by σ* .

⁴Recall that a subfield $k \subset \mathbb{R}$ is called totally real if its image under any embedding $k \rightarrow \mathbb{C}$ is contained in \mathbb{R} .

2.5. Kontsevich-Zorich cocycle. Over a Teichmüller space $\mathcal{T}_{S,\Sigma}(\kappa)$, let us consider the trivial cocycle $g_t \times id$ on $\mathcal{T}_{S,\Sigma}(\kappa) \times H^1(S; \mathbb{R})$. The modular group $MCG(S, \Sigma)$ acts on $\mathcal{T}_{S,\Sigma}(\kappa) \times H^1(S; \mathbb{R})$, and the quotient bundle is a flat orbifold vector bundle over $\mathcal{M}_{S,\Sigma}(\kappa)$ called the *Hodge bundle*. The *Kontsevich-Zorich cocycle* is the projection of $g_t \times id$ to the Hodge bundle. We will also need a slightly different form of the Kontsevich-Zorich cocycle, namely the projection of $g_t \times id$ on $\mathcal{T}_{S,\Sigma}(\kappa) \times H^1(S \setminus \Sigma; \mathbb{R})$ that we call the *extended Kontsevich-Zorich cocycle* (on the *extended Hodge bundle*).

We will often use a discrete version of the Teichmüller flow and the Kontsevich-Zorich cocycle built as follows. Let Q be a transversal to the Teichmüller flow which is compact and simply connected. There is a natural trivialization of the Hodge bundle and extended Hodge bundle over Q given by the Gauss-Manin connection (there is a unique way to identify two nearby fibers preserving the integer lattice). Once we fixed a reference based point $(S, \Sigma, \omega) \in Q$, the Poincaré section $F : Q \rightarrow Q$ of the Teichmüller flow hence provides two cocycles $A : Q \rightarrow Sp(H^1(S; \mathbb{R}))$ and $A' : Q \rightarrow Sp(H^1(S \setminus \Sigma; \mathbb{R}))$ that preserve the integer lattices.

In the case of Veech surfaces, the cocycles $A : Q \rightarrow Sp(H^1(S; \mathbb{R}))$ and $A' : Q \rightarrow Sp(H^1(S \setminus \Sigma; \mathbb{R}))$ preserve the tautological bundle V and its conjugates V^σ . As each homotopy class of closed curve on a hyperbolic surface has either a geodesic or a horocyclic representative, one can see that the matrices $A(x)$ which appear in this discrete version are induced by affine homeomorphisms of the surface (S, Σ, ω) .

Because the modular group acts by symplectic transformation on $H^1(S; \mathbb{R})$ (with respect to the intersection form) the $2g$ Lyapunov exponents $\lambda_1^\mu \geq \lambda_2^\mu \geq \dots \geq \lambda_{2g}^\mu$ of the Kontsevich-Zorich cocycle satisfy

$$\forall 1 \leq k \leq g, \quad \lambda_k^\mu = -\lambda_{2g-k-1}^\mu \geq 0.$$

Because of the natural injection $H^1(S; \mathbb{R}) \rightarrow H^1(S \setminus \Sigma; \mathbb{R})$, the Lyapunov spectrum of the extended Kontsevich-Zorich cocycle contains the one of the Kontsevich-Zorich cocycle. It may be proved that the remaining exponents are $s - 1$ zeros where s is the cardinality of Σ .

The Kontsevich-Zorich cocycle preserves the tautological bundle, and one sees directly from the definition that the Lyapunov exponents on the tautological bundle are 1 and -1 . For the remaining exponents we have the following inequality.

Theorem 12 (Forni [Fo02]). *Let μ be an ergodic invariant probability measure of the Teichmüller flow on some stratum $\mathcal{M}_{S,\Sigma}(\kappa)$. Then the second Lyapunov exponent λ_2^μ of the Kontsevich-Zorich cocycle satisfies $1 > \lambda_2^\mu$.*

Remark 2.1. Forni's proof indeed shows that there exists a natural Hodge norm on the Hodge bundle such that for any $x \in \mathcal{M}_{S,\Sigma}(\kappa)$, the Kontsevich-Zorich cocycle starting from x has a norm strictly less than e^t at any time $t > 0$, when restricted to the symplectic orthogonal to the tautological space.

Given a Veech surface (S, Σ, ω) , the conjugates of the tautological subspace V^σ are well defined over the whole Teichmüller curve: they form subbundles of the Hodge bundle invariant for the Kontsevich-Zorich cocycle. In other words, we may restrict the Kontsevich-Zorich cocycle to any of the V^σ and consider the associated pair $(\lambda^\sigma, -\lambda^\sigma)$ of Lyapunov exponents. The following result shows that these exponents are non-zero.

Lemma 13 ([BM10]). *Let (S, Σ, ω) be a Veech surface, k be its holonomy field, and V be the tautological subbundle of the Hodge bundle of its $SL(2, \mathbb{R})$ orbit. Then for any non-identity embedding $\sigma : k \rightarrow \mathbb{R}$, the non-negative Lyapunov exponent λ^σ of the Kontsevich-Zorich cocycle restricted to V^σ satisfies $0 < \lambda^\sigma < 1$.*

Note that the upper bound can be deduced from Theorem 12. The lower bound can also be obtained from the fact that the group Γ^σ is non-elementary and the hyperbolicity properties of the geodesic flow on \mathcal{C} .

One consequence of the upper bound for λ^σ is the following result that will be used to prove Theorem 3.

Proposition 14 ([Mo06]). *Let (S, Σ, ω) be a Veech surface, k its holonomy field and Γ its Veech group. Then, for any non-identity embedding $\sigma : k \rightarrow \mathbb{R}$ the group Γ^σ is non-discrete.*

3. EIGENFUNCTIONS IN VEECH SURFACES

Let (S, Σ, ω) be a translation surface and $\phi_t : S \rightarrow S$ the vertical flow. We say that $\nu \in \mathbb{R}$ is an *eigenvalue* of ω if there exists a non-zero measurable function $f : S \rightarrow \mathbb{C}$ such that for almost every $x \in S$, we have $f(\phi_T(x)) = \exp(2\pi i\nu T)f(x)$ for all $T \in \mathbb{R}$. The eigenvalue is *continuous* if the map f may be chosen continuous. The flow is *weak-mixing* if it admits no eigenvalue except 0 with multiplicity one.

The Veech criterion that appeared in [Ve84] was of main importance in [AF07] to prove the genericity of weak-mixing among translation flows. This criterion depends on the consideration of appropriate compact transversal to the Teichmüller flow which is “small enough” to fit inside “zippered rectangles” charts and also satisfy some additional boundedness properties.

Theorem 15 (Veech criterion). *Let $\mathcal{M}_g(\kappa)$ be a stratum of translation surfaces. For all (S, Σ, ω) in $\mathcal{M}_g(\kappa)$ there exists a small compact transversal Δ of the Teichmüller flow containing (S, Σ, ω) such that the following holds. Let $A_n : \Delta \rightarrow Sp(2g, \mathbb{Z})$ be the Kontsevich-Zorich cocycle on that transversal. Then for every $x \in \Delta$ with an eigenvalue ν and such that its forward Teichmüller geodesic comes back infinitely often to Δ we have*

$$\lim_{n \rightarrow \infty} \text{dist}(A_n(x) \cdot (\nu \text{Im}(\tau_x)), H^1(S \setminus \Sigma, \mathbb{Z})) = 0,$$

where τ_x is the differential associated to the surface $x \in \Delta$.

The above theorem holds in great generality as soon as the dynamical system is described by Rokhlin towers (see [BDM1]).

Veech’s criterion tells us that we can prove that the vertical flow is weak mixing if we can show that the line $\mathbb{R} \text{Im}(\omega)$ intersects the “weak stable lamination,” the set of all $w \in H^1(S \setminus \Sigma; \mathbb{R})$ such that

$$\lim_{n \rightarrow \infty} \text{dist}(A_n \cdot w, H^1(S \setminus \Sigma, \mathbb{Z})) = 0,$$

only at the origin. Unfortunately, the nature of the weak stable lamination is rather complicated. It is, of course, a union of translates of the stable space, the set of all

$w \in H^1(S \setminus \Sigma; \mathbb{R})$ such that

$$\lim_{n \rightarrow \infty} \text{dist}(A_n \cdot w, 0) = 0,$$

and it contains the “strong stable space” consisting of the integer translates of the stable space. However, in general it is much larger, being transversely uncountable.

The main objective of this section is to show that for Veech surfaces, any eigenvalue ν must be such that $\nu \text{Im}(\omega)$ must belong to the smaller (and much simpler) strong stable space.

Theorem 16. *Let (S, Σ, ω) be a Veech surface with no vertical saddle connection and whose linear flow admits an eigenvalue ν . Consider a compact transversal Δ large enough so that the forward Teichmüller geodesic of (S, Σ, ω) comes back infinitely often to Δ .*

Let $A : \Delta \rightarrow \text{Sp}(H^1(S \setminus \Sigma, \mathbb{Z}))$ the associated Kontsevich-Zorich cocycle. Then there exists $v \in H^1(S \setminus \Sigma; \mathbb{Z})$ such that

$$\lim_{n \rightarrow \infty} A_n \cdot (\nu \text{Im}(\omega) - v) = 0.$$

Note that the existence of such transversals is due to the fact that geodesics in finite volume hyperbolic surfaces always come back to a fixed compact set.

In order to prove Theorem 16, we will need to control all the renormalizations of eigenfunctions, and not only those corresponding to returns to a small compact transversal. It will be crucial for our strategy that for a Veech surface we can choose ε such that the following holds. Let $\mathcal{C}_\varepsilon = \{\omega : \text{sys}(S, \Sigma, \omega) \geq \varepsilon\}$. Then the set of “moments of compactness” $\{t > 0; g_t \cdot \omega \in \mathcal{C}_\varepsilon\}$ for the forward Teichmüller geodesic is unbounded if and only if there are no vertical saddle connections in ω . And, most important, any orbit segment away from the moments of compactness (the cusp excursions) can easily be described geometrically.

3.1. Tunneling curves and a dual Veech criterion. In this section (which is not restricted to Veech surfaces) we show that the existence of eigenfunctions yields information about all times of the Teichmüller flow and not only return times to a (small or large) compact transversal. This is based on a refinement of the Veech criterion (Theorem 18) which is formulated in terms of homology cycles called *tunneling curves* (which are designed to follow closely the vertical flow in a suitable sense). We show that tunneling curves always see the expected property of approximation to integers. In a second step (Lemma 19), we prove that in any compact part of the moduli space, the tunneling curves generate $H_1(S \setminus \Sigma; \mathbb{Z})$. Those two results together allow us to remove the smallness condition on the transversal in the formulation of the Veech criterion, hence allowing us to consider the large compact set \mathcal{C}_ε when analyzing Veech surfaces later.

Before defining tunneling curves we need the notion of a cycle of rectangles. A *rectangle* for (S, Σ, ω) is an isometric immersion of an euclidean rectangle with horizontal and vertical sides. In other words, a rectangle is a map $R : [0, w] \times [0, h] \rightarrow S \setminus \Sigma$ such that $R^*(\text{Re}(\omega)) = \pm dx$ and $R^*(\text{Im}(\omega)) = \pm dy$. The number w is called the *width* of the rectangle and the number h its *height*. Note that with our convention we may not identify a rectangle with its image in S , but we care about the direction: a rectangle is determined by its image in S and an element of $\{+1, -1\} \times \{+1, -1\}$.

Definition 17. A (k, δ, h) -cycle of rectangles for ω is a set of $2k$ rectangles denoted H_j and V_j for $j \in \mathbb{Z}/k\mathbb{Z}$ such that

- the height of H_j is δ and its width is $w_j \geq \delta$;
- the width of V_j is δ and its height is $\delta \leq h_j \leq h$;
- $H_j^*(\operatorname{Re}(\omega)) = \pm dx$ and $H_j^*(\operatorname{Im}(\omega)) = dy$;
- $V_j^*(\operatorname{Re}(\omega)) = dx$ and $V_j^*(\operatorname{Im}(\omega)) = \pm dy$;
- each rectangle H_j is embedded in the surface;
- for each j ; $H_j([0, \delta] \times [0, \delta]) = V_{j-1}([0, \delta] \times [h_j - \delta, h_j])$ and $H_j([w_j - \delta, w_j] \times [0, \delta]) = V_j([0, \delta] \times [0, \delta])$.

In other words, a (k, δ, h) -cycle of rectangles is a thin tube of width δ in (S, Σ, ω) made of k horizontal and k vertical pieces and that forms a cycle in the surface. We will sometimes drop the condition on the heights and write (k, δ) for (k, δ, ∞) .

A tunneling curve is a curve which belongs in a cycle of rectangles. More precisely, let $R = (H_j, V_j)_{j \in \mathbb{Z}/k\mathbb{Z}}$ be a (k, δ, h) -cycle of rectangles for ω . We may build a curve ζ as follows: for $j \in \mathbb{Z}/k\mathbb{Z}$, we define vertical segments $\zeta_j^v : [0, h_j - \delta] \rightarrow S \setminus \Sigma$ by $\zeta_j^v(t) = V_j(\frac{\delta}{2}, t + \frac{\delta}{2})$ and horizontal segments $\zeta_j^h : [0, w_j - \delta] \rightarrow S \setminus \Sigma$ by $\zeta_j^h(t) = H_j(t + \frac{\delta}{2}, \frac{\delta}{2})$. The curve ζ is the concatenation of $\zeta_1^h, \zeta_1^v, \dots, \zeta_k^h, \zeta_k^v$ and forms a loop in the surface $S \setminus \Sigma$. The homology class of ζ in $H_1(S \setminus \Sigma; \mathbb{Z})$ is the *homology class* of the cycle of rectangles R .

A homology class $\zeta \in H_1(S \setminus \Sigma; \mathbb{Z})$ is said to be (k, δ, h) -tunneling if there exists a set of (k_i, δ, h) -cycles of rectangles for $i = 1, \dots, n$ such that $k_1 + \dots + k_n \leq k$ and whose homology classes ζ_i satisfy $\sum \zeta_i = \zeta$.

Note that if ζ is a tunneling curve in a (k, δ, h) -cycle of rectangles, then

$$|\operatorname{Re}(\omega)(\zeta)| \leq \int_{\zeta} |\operatorname{Re}(\omega)| \leq \frac{k}{\delta} \operatorname{Area}(\omega) \quad \text{and} \quad |\operatorname{Im}(\omega)(\zeta)| \leq \int_{\zeta} |\operatorname{Im}(\omega)| \leq kh.$$

In particular, in a fixed translation surface (S, Σ, ω) there is only a finite number of (k, δ, h) -tunneling curves. The set of (k, δ, h) -tunneling homology classes in $H_1(S \setminus \Sigma; \mathbb{Z})$ for ω is denoted $TC_{k, \delta, h}(\omega)$. The set $TC_{k, \delta}(\omega) = TC_{k, \delta, \infty}(\omega)$ denotes the set of (k, δ) -tunneling homology classes. Note that, if $k' \leq k$, $\delta' \geq \delta$ and $h' \leq h$, then a (k', δ', h') -tunneling curve is also (k, δ, h) -tunneling.

We will now adapt Veech's original proof of his criterion in Veech [Ve84] to obtain a dual version with respect to the tunneling curves in $TC_{k, \delta}$.

Theorem 18 (dual Veech criterion). *Let (S, Σ, ω) be a translation surface without vertical saddle connections and that admits an eigenvalue ν . Then, for any positive integer k and positive real number δ we have*

$$\lim_{t \rightarrow \infty} \sup_{\zeta \in TC_{k, \delta}(g_t \cdot \omega)} \operatorname{dist}(\nu \operatorname{Im}(\omega)(\zeta), \mathbb{Z}) = 0.$$

Proof. We assume that $\operatorname{Area}(\omega) = 1$.

Fix k and δ . We fix a small number α , and we prove that for t big enough, all (k, δ) -tunneling curves for $\omega_t = g_t \cdot \omega = e^t \operatorname{Re}(\omega) + Ie^{-t} \operatorname{Im}(\omega)$ are such that $\operatorname{dist}(\nu \operatorname{Im}(\omega)(\zeta), \mathbb{Z}) < k\alpha$. It is enough to prove the theorem for a curve that

belongs to a cycle of rectangles (recall that a tunneling curve may be a sum of curves associated to a cycle of rectangles).

Let $(H_j, V_j)_{j \in \mathbb{Z}/k\mathbb{Z}}$ be a (k, δ) -cycle of rectangles for ω_t , and let ζ be its homology class. We define the *signed height* of the vertical rectangle V_j by $\widetilde{h}_j = h_j - \delta$ if $(V_j)^*(\text{Im}(\omega)) = dy$ and $\widetilde{h}_j = \delta - h_j$ otherwise (it is precisely the value of the integral of $\text{Im}(\omega)$ along the component ζ_j^v of the curve ζ as above). In particular, we have $\text{Im}(\omega)(\zeta) = \widetilde{h}_1 + \dots + \widetilde{h}_k$. For each j , we define $I_j = H_j([0, w_j] \times \{\delta/2\})$ the middle interval of the rectangle H_j . The segment I_j is a horizontal interval of length $e^{-t}w_j$ for ω .

We assume that the surface S admits a non-trivial eigenvalue $\nu \neq 0$ and denote f an associated eigenfunction $f : S \rightarrow \mathbb{C}$ with $|f| = 1$. The set $\Omega \subset S$ of points x such that for all $T \in \mathbb{R}$ the vertical flow is defined and $f(\phi_T(x)) = e^{2\pi i\nu T} f(x)$ has full measure. Moreover, Ω intersects any horizontal segment in a subset of full linear measure. Define measurable functions $f_j : [0, w_j] \rightarrow \mathbb{C}$ by $f_j(t) = f(H_j(t, \delta/2))$. For each j and almost every $x \in [0, \delta]$ we have

$$\begin{aligned} f_j(w_j - \delta + x) &= e^{-2\pi i\nu \widetilde{h}_j} f_{j+1}(x) && \text{if } H_j^*(\text{Re}(\omega)) = dx \text{ and } H_{j+1}^*(\text{Re}(\omega)) = dx, \\ f_j(w_j - x) &= e^{-2\pi i\nu \widetilde{h}_j} f_{j+1}(x) && \text{if } H_j^*(\text{Re}(\omega)) = -dx \text{ and } H_{j+1}^*(\text{Re}(\omega)) = dx, \\ f_j(w - \delta + x) &= e^{-2\pi i\nu \widetilde{h}_j} f_{j+1}(\delta - x) && \text{if } H_j^*(\text{Re}(\omega)) = dx \text{ and } H_{j+1}^*(\text{Re}(\omega)) = -dx, \\ f_j(w_j - x) &= e^{-2\pi i\nu \widetilde{h}_j} f_{j+1}(\delta - x) && \text{if } H_j^*(\text{Re}(\omega)) = -dx \text{ and } H_{j+1}^*(\text{Re}(\omega)) = -dx. \end{aligned}$$

These formulas can be rewritten in a more compact form as follows. Let $s_j : [0, \delta] \rightarrow [0, \delta]$ be given by $s_j(x) = \delta - x$ if $H_j^*(\text{Re}(\omega)) = dx$ and by $s_j(x) = x$ if $H_j^*(\text{Re}(\omega)) = -dx$. Then $f_j(w_j - s_j(x)) = e^{-2\pi i\nu \widetilde{h}_j} f_{j+1}(\delta - s_{j+1}(x))$ in all cases.

Recall that we fix α an arbitrarily small positive real number. The strategy of the proof consists in proving that if t is big enough, independently of the choice of the (k, δ) -cycle of rectangles, we may find points $x_j \in [0, \delta]$, $j \in \mathbb{Z}/k\mathbb{Z}$ such that for each j we have $|f_j(\delta - s_j(x_{j-1})) - f_j(w_j - s_j(x_j))| < \alpha$. In particular, we can write $\frac{f_j(\delta - s_j(x_{j-1}))}{f_j(w_j - s_j(x_j))} = e^{2\pi i\lambda_j}$ where $\lambda_j \in (-\alpha, \alpha)$.

Assuming that such points do exist, we prove how to derive our theorem. Using the points x_j we may write

$$\begin{aligned} 1 &= \prod_{j \in \mathbb{Z}/k\mathbb{Z}} \frac{f_j(w_j - s_j(x_j))}{f_{j-1}(w_{j-1} - s_{j-1}(x_{j-1}))} = \prod_{j \in \mathbb{Z}/k\mathbb{Z}} e^{2\pi i\nu \widetilde{h}_{j-1}} \frac{f_j(w_j - s_j(x_j))}{f_j(\delta - s_j(x_{j-1}))} \\ &= \prod_{j \in \mathbb{Z}/k\mathbb{Z}} e^{2\pi i\nu \widetilde{h}_{j-1}} e^{-2\pi i\lambda_j} \end{aligned}$$

so that $\text{Im}(\omega)(\zeta) = \sum \lambda_j \pmod{\mathbb{Z}}$, implying the result.

Now, we show how to find points x_j using a measure theoretic argument. More precisely, we prove that the measure of the set of points $(x, y) \in [0, \delta] \times [w_j - \delta, w_j]$ such that $|f_j(x) - f_j(y)| < \alpha$ becomes arbitrarily close to δ^2 as t goes to infinity independently of the choice of the cycle of rectangles. Since $\delta \leq w_j \leq \delta^{-1}$, it is enough to show that there exists a compact subset $K_j \subset I_j$ with measure close to 1 such that $|f(x) - f(y)| < \alpha$ for every $x, y \in K_j$.

Fix some small constant $\chi > 0$. By Lusin's theorem, there exists a compact subset $K \subset S$ of measure $1 - \chi$ such that $f|_K$ is continuous (up to a measure zero set). In particular, there exists $\varepsilon > 0$ such that if $x, y \in K$ are ε -close, then $|f(x) - f(y)| < \alpha$.

Notice that the rectangle H_j has width $e^{-t}w_j$ and height $e^t\delta$ in (S, Σ, ω) . Recall that we also have $\delta \leq w_j \leq \frac{1}{\delta}$. In particular, the area of H_j is at least δ^2 , so K must intersect it in a subset of measure at least $1 - \delta^{-2}\kappa$. It follows that some (full) horizontal segment $I'_j = H_j([0, w_j] \times \{T\})$ in this rectangle intersects K into a subset K'_j of measure at least $1 - \delta^{-2}\chi$ as well. Take t large enough so that $e^{-t}\delta^{-1} < \varepsilon$. Then $|f(x) - f(y)| < \alpha$ for every $x, y \in K'_j$. Note that $I'_j = \phi_{\pm e^t(T-\delta/2)}(I_j)$ (the same sign as when writing $H_j^*(\text{Im } \omega) = \pm dy$) so by the functional equation we have $|f(x) - f(y)| < \alpha$ for every $x, y \in K_j = \phi_{\mp e^t(T-\delta/2)}(K'_j)$, as desired. \square

We now prove that any translation surface admits the tunneling basis and the constant may be taken uniform in compact sets.

Lemma 19. *Let $\mathcal{M}_{S, \Sigma}^{(1)}(\kappa)$ be a stratum in moduli space, and let $\varepsilon > 0$. Let $K_\varepsilon \subset \mathcal{M}_{S, \Sigma}(\kappa)$ be the set of translation surfaces whose systole is at least ε . Then there exists (k, δ, h) such that for any translation surface $\omega \in K_\varepsilon$, the (k, δ, h) -tunneling curves for ω generate $H_1(S \setminus \Sigma; \mathbb{Z})$.*

Proof. The set $A_{k, \delta, h}$ of surfaces in $\mathcal{M}_{S, \Sigma}(\kappa)$ that admits a (k, δ', h') -tunneling basis with $\delta' > \delta$ and $h' < h$ is an open set. From compactness of K_ε it is hence enough to prove that for any translation surface in $\mathcal{M}_{S, \Sigma}(\kappa)$, every closed curve in $S \setminus \Sigma$ is homotopic to a (k, δ, h) -tunneling curve, for some k, δ , and h . Indeed, up to homotopy we may assume that a closed curve is built by concatenating small (and hence embedded) horizontal and vertical segments in alternation. Those segments can then be slightly thickened to build the desired cycle of rectangles. \square

3.2. Excursions in cusps.

In this section we prove Theorem 16.

We first give another formulation of Theorem 16 in terms of tunneling basis (in order to use Theorem 18). Let $\mathcal{C} \subset \mathcal{M}_{S, \Sigma}(\kappa)$ be the $\text{SL}(2, \mathbb{R})$ orbit of a Veech surface, let $\varepsilon > 0$ be small, and let \mathcal{C}_ε be the set of surfaces in \mathcal{C} whose systole is at least ε . From Lemma 19, we get k, δ , and h such that each translation structure $\omega \in \mathcal{C}_\varepsilon$ has a (k, δ, h) -tunneling basis in $H_1(S \setminus \Sigma; \mathbb{Z})$. Using compactness and the finiteness of (k, δ, h) -tunneling curves, there exists a constant $M > 1$ such that for any translation structure ω in \mathcal{C}_ε any (k, δ, h) -tunneling basis $\{\zeta_j\}$ and any (k, δ, h) -tunneling curve ζ for ω , the coefficients of $\zeta = \sum c_j \zeta_j$ with respect to the basis satisfy $\sum |c_j| < M$.

Let $\omega \in \mathcal{C}$ be a translation surface that admits an eigenvalue ν . From Theorem 18, there exists t_0 such that for any t larger than t_0 , to each (k, δ, h) -tunneling curve ζ for ω_t , we may associate a unique $n_t(\zeta) \in \mathbb{Z}$ such that $|\nu \text{Im}(\omega)(\zeta) - n_t(\zeta)| < 1/(2M)$. Let $t \geq t_0$ be such that $\omega_t \in \mathcal{C}_\varepsilon$. If $\{\zeta_j\}$ is a (k, δ, h) -tunneling basis and $\zeta = \sum c_j \zeta_j$ is a (k, δ, h) -tunneling curve, then

$$|\nu \text{Im}(\omega)(\zeta) - \sum c_j n_t(\zeta_j)| = |\sum c_j (\nu \text{Im}(\omega)(\zeta_j) - n_t(\zeta_j))| < \frac{1}{2M} \sum |c_j| < \frac{1}{2},$$

so that $n_t(\zeta) = \sum c_j n_t(\zeta_j)$. As the (k, δ, h) -tunneling curves form a basis in \mathcal{C}_ε , the mapping $n_t : TC_{k, \delta, h}(\omega_t) \rightarrow \mathbb{Z}$ extends in a unique way to a linear map $n_t : H_1(S \setminus \Sigma; \mathbb{Z}) \rightarrow \mathbb{Z}$. The convergence to an integer element in Theorem 16 is then equivalent to the following statement.

Lemma 20. *Let ω be a Veech surface in \mathcal{C} without vertical saddle connections for which the translation flow admits an eigenvalue ν , and let $\omega_t = g_t \cdot \omega$. Let $\varepsilon > 0$, and let t_0 and $n_t \in H^1(S \setminus \Sigma; \mathbb{Z})$ be as above. Then the family $(n_t)_{t \geq t_0, \omega_t \in \mathcal{C}_\varepsilon}$ is eventually constant.*

The proof of Lemma 20 follows by analyzing parts of Teichmüller geodesics that go off \mathcal{C}_ε because, by construction, n_t is locally constant.

Until the end of this section, fix $\varepsilon > 0$ such that the cusps of \mathcal{C} are isolated in the complement of \mathcal{C}_ε and that the conclusion of Corollary 7 holds (the former is actually a consequence of the latter). A *cuspid excursion* of length $t > 0$ is a segment of a Teichmüller orbit $\omega_s = g_s \cdot \omega_0$, $s \in [0, t]$, such that $\omega_s \in \mathcal{C}_\varepsilon$ only for $s = 0, t$. Note that the shortest saddle connection γ at the beginning of a cuspid excursion is never horizontal or vertical, and indeed we have $|\text{Im}(\omega_0)(\gamma)| > |\text{Re}(\omega_0)(\gamma)| > 0$, with the length of the cuspid excursion given by $t = \log \frac{|\text{Im}(\omega_0)(\gamma)|}{|\text{Re}(\omega_0)(\gamma)|}$.

By our choice of ε (see Corollary 7), if ω belong to $\partial\mathcal{C}_\varepsilon = \{(S, \Sigma, \omega) \in \mathcal{C} : \text{sys}(S, \Sigma, \omega) = \varepsilon\}$, then S admits a canonical decomposition as a finite union of maximal cylinders C_i , $1 \leq i \leq c$, with waist curve γ_i parallel to the saddle connection γ .

Lemma 21. *For any (k, δ, h) with $\delta > 0$ small enough and $h > 0$ big enough, there exists an integer $k' \geq k$ with the following property. Let us consider a cuspid excursion $(\omega_s)_{0 \leq s \leq t}$ of length t . Let γ be a saddle connection of the shortest length on ω_0 . Let κ be the sign of $\frac{\text{Im}(\omega_0)(\gamma)}{\text{Re}(\omega_0)(\gamma)}$ and let $m_i = \left\lfloor \frac{e^t}{\mu(C_i)} \right\rfloor$, where $\mu(C_i)$ is the modulus of the cylinder C_i in the canonical decomposition of (S, Σ, ω_0) . For each*

(k, δ, h) -tunneling curve ζ for ω_0 the class $\zeta - \kappa \sum_{i=1}^c m_i \langle \zeta, \gamma_i \rangle \gamma_i$ is (k', δ, h) -tunneling for ω_t , and for any integers ℓ_i such that $0 \leq \ell_i \leq m_i$ the classes $\zeta - \kappa \sum_{i=1}^c \ell_i \langle \zeta, \gamma_i \rangle \gamma_i$ are (k', δ) -tunneling for ω_0 .

Proof. We first reduce our study to (not necessarily closed) paths inside a single cylinder. Let $X = \{x_1, x_2, \dots, x_p\}$ be the middle points of saddle connections in the direction of the shortest saddle connection γ for ω_0 . We call *transversal* a flat geodesic segment γ' that joins two points of X and is disjoint from saddle connections parallel to γ . Any curve in (S, Σ, ω_0) is freely homotopic in $S \setminus \Sigma$ to a concatenation of transversals γ'_i such that $|\text{Im}(\omega_0)(\gamma'_i)| < 2R$ and $|\text{Re}(\omega_0)(\gamma'_i)| < 2R$ where the constant R may be chosen independently of (S, Σ, ω) in $\partial\mathcal{C}_\varepsilon$. Moreover, if ζ is a (k, δ, h) -tunneling curve for ω_0 , the minimal number of pieces is uniformly bounded in terms of ε, k, δ , and h .

Let us fix x and y on the boundary of some cylinder C_i and denote by $T(x, y)$ the set of transversals that join x to y . We build rectangles around the curves in $T(x, y)$ in order to be able to reconstruct a cycle of rectangles. A (k', δ, h) -path of rectangles for x and y is a set of rectangles $H_1, V_1, \dots, H_{k'}, V_{k'}, H_{k'+1}$ such that

- the height of H_j is δ and its width is $w_j \geq \delta$;
- the width of V_j is δ and its height h_j satisfies $\delta \leq h_j \leq h$;
- $H_j^*(\text{Re}(\omega)) = \pm dx$ and $H_j^*(\text{Im}(\omega)) = dy$;
- $V_j^*(\text{Re}(\omega)) = dx$ and $V_j^*(\text{Im}(\omega)) = \pm dy$;
- each rectangle H_j is embedded in the surface;
- for each j , $H_j([0, \delta] \times [0, \delta]) = V_{j-1}([0, \delta] \times [h_j - \delta, h_j])$ and $H_j([w_j - \delta, w_j] \times [0, \delta]) = V_j([0, \delta] \times [0, \delta])$;
- $H_1(\delta/2, \delta/2) = x$ and $H_{k'+1}(w_{k'+1} - \delta/2, \delta/2) = y$.

As we did for cycles of rectangles, to a (k', δ, h) -path of rectangles for x and y we may associate its homology class in $H_1(S \setminus \Sigma, \{x, y\}; \mathbb{Z})$.

Let us fix a transversal γ' joining x and y inside some cylinder C_i and such that $|\text{Im}(\omega_0)(\gamma')| < 2R$ and $|\text{Re}(\omega_0)(\gamma')| < 2R$. Since any (k, δ, h) -tunneling curve can be decomposed into a uniformly bounded number of such transversals, it will be enough for us to prove that there exists $k' > 0$ (only depending on $\varepsilon, \delta, h, R$) with the following properties:

- (1) The transversal $\gamma'' \in T(x, y)$ in the class of $\gamma' - \kappa m_i \langle \gamma', \gamma_i \rangle \gamma_i$ is a (k', δ, h) -tunneling path for ω_t ;
- (2) For every $0 \leq \ell \leq m_i$, the class of $\ell \gamma_i$ is (k', δ) -tunneling for ω_0 .

Note that the width $w(C_i)$, the height $h(C_i)$, and the modulus $\mu(C_i)$ are all bounded away from zero and infinity, independently of i , through $\partial \mathcal{C}_\varepsilon$. In particular, we may assume that $w(C_i)$ and $h(C_i)$ are bigger than 10δ . Note also that $\langle \gamma', \gamma_i \rangle = \pm 1$.

Write $\frac{\text{Im}(\omega_0)(\gamma')}{\text{Re}(\omega_0)(\gamma')}$ as $\kappa \tan \theta$ with $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. We may assume that the cusp excursion of the surface (S, Σ, ω_0) in \mathcal{C}_ε is long enough, so that in particular $m_i \geq 10$ and $\tan \theta > \frac{2R}{h(C_i)}$.

Let us first show the second property. It is enough to show that for $0 \leq \ell \leq \frac{3m_i}{4}$, $\ell \gamma_i$ can be represented by a $(2, \delta)$ -tunneling curve. Let \hat{C}_i denote the ‘‘core of C_i ,’’ obtained by removing a $\frac{h(C_i)}{8}$ -neighborhood of its boundary. If $l \leq \frac{3m_i}{4}$, then we can represent $\ell \gamma_i$ by a concatenation γ'_i of a vertical path of length $\ell w(C_i) \sin \theta$ and a horizontal path of length $\ell w(C_i) \cos \theta$ inside \hat{C}_i .⁵ It easily follows that the δ -enlargement of the horizontal part of γ'_i is embedded in C_i , while the δ -enlargement of the vertical part of γ'_i is contained in C_i , so that γ is $(2, \delta)$ -tunneling.

Let us now show the first property. By compactness considerations, it will be enough to show that the transversal $\gamma'' = \gamma' - \kappa m_i \langle \gamma', \gamma_i \rangle \gamma_i$ has a bounded length with respect to ω_t . Up to changing the orientation of γ' , we may assume that $\langle \gamma', \gamma_i \rangle = 1$. Then, for the imaginary part we have

$$\begin{aligned} |\text{Im}(\omega_t)(\gamma' - \kappa m_i \gamma_i)| &= e^{-t} |\text{Im}(\omega_0)(\gamma' - \kappa m_i \gamma_i)| \\ &< e^{-t} 2M + e^{-t} \left\lfloor \frac{e^t}{\mu(C_i)} \right\rfloor \leq 2M + \frac{1}{\mu(C_i)}. \end{aligned}$$

For the real part, note first that

$$\begin{aligned} \text{Re}(\omega_t)(\gamma' - \kappa m_i \gamma_i) &= e^t \text{Re}(\omega_0)(\gamma' - \kappa m_i \gamma_i) \\ &= \frac{|\text{Im}(\omega_0)(\gamma_i)|}{|\text{Re}(\omega_0)(\gamma_i)|} (\text{Re}(\omega_0)(\gamma') - \kappa m_i \text{Re}(\omega_0)(\gamma_i)) \\ &= \pm |\text{Im}(\omega_0)(\gamma_i)| \left(\frac{\text{Re}(\omega_0)(\gamma')}{\text{Re}(\omega_0)(\gamma_i)} - \kappa m_i \right). \end{aligned}$$

⁵To see that such a concatenation lies inside \hat{C}_i , note that the maximal length of a vertical path in \hat{C}_i is exactly $\frac{3}{4} \frac{h(C_i)}{\cos \theta}$ and $\frac{3}{4} m_i w(C_i) \sin \theta \leq \frac{3}{4} h(C_i) \frac{\sin^2 \theta}{\cos \theta} \leq \frac{3}{4} \frac{h(C_i)}{\cos \theta}$.

Recall that $m_i = \lfloor \frac{h(C_i)}{u(C_i)} \tan \theta \rfloor$ while $\text{Im}(\omega_0)(\gamma_i)$ is uniformly bounded. In order to conclude, let us show that $\frac{\text{Re}(\omega_0)(\gamma')}{\text{Re}(\omega_0)(\gamma_i)}$ is at a uniformly bounded distance from $\kappa \frac{h(C_i)}{w(C_i)} \tan \theta$. Using that $\langle \gamma', \gamma_i \rangle = 1$ and that $\tan \theta > \frac{2M}{h(C_i)}$, we see that $\text{Re}(\omega_0)(\gamma')$ and $\text{Im}(\omega_0)(\gamma_i)$ have the same sign,⁶ which implies that $\frac{\text{Re}(\omega_0)(\gamma')}{\text{Re}(\omega_0)(\gamma_i)}$ and $\kappa \frac{h(C_i)}{w(C_i)} \tan \theta$ have the same sign as well. We have $|\text{Re}(\omega_0)(\gamma_i)| = w(C_i) \cos \theta$ and

$$|\text{Re}(\omega_0)(\gamma')| \pm |\text{Im}(\omega_0)(\gamma')| \cot \theta = \frac{h(C_i)}{\sin \theta},$$

so that

$$\frac{|\text{Re}(\omega_0)(\gamma')|}{|\text{Re}(\omega_0)(\gamma_i)|} = \frac{h(C_i)}{w(C_i)} \frac{1}{\sin \theta \cos \theta} \mp \frac{|\text{Im}(\omega_0)(\gamma')|}{w(C_i) \sin \theta}.$$

It follows that

$$\frac{|\text{Re}(\omega_0)(\gamma')|}{|\text{Re}(\omega_0)(\gamma_i)|} - \frac{h(C_i)}{w(C_i)} \tan \theta = \frac{h(C_i)}{w(C_i)} \cot \theta \mp \frac{|\text{Im}(\omega_0)(\gamma')|}{w(C_i) \sin \theta}.$$

Since $h(C_i)$ is uniformly bounded away from infinity, $w(C_i)$ is uniformly bounded away from 0, $\cot \theta < 1$, $\sin \theta > 2^{-1/2}$, and $|\text{Im}(\omega_0)(\gamma')| < 2M$, the result follows. \square

We now prove how Lemma 21 may be used to conclude the proof of Lemma 20.

Proof of Lemma 20. Let (k, δ, h) with δ small enough and h large enough so that every surface in \mathcal{C}_ε admits a (k, δ, h) -tunneling basis, and such that for every surface in $\partial\mathcal{C}_\varepsilon$, the waist curves γ_i of the canonical cylinder decomposition are (k, δ, h) -tunneling. Let $k' \geq k$ be such that the conclusion of Lemma 21 holds. Let M' be, as in the beginning of the section, an upper bound for $\sum |c_\alpha|$ over all expressions $\sum c_\alpha \zeta_\alpha$ of a (k', δ, h) -tunneling curve in a (k', δ, h) -tunneling basis. Let (S, Σ, ω) be a surface that admits an eigenvalue ν . We know from Theorem 18 that there exists a time t_0 such that for every $t \geq t_0$ and every (k', δ) -tunneling curve α for $\omega_t = g_t \cdot \omega$ we have $\text{dist}(\nu \text{Im}(\omega)(\alpha), \mathbb{Z}) < 1/(4M')$.

Recall that for $t \geq t_0$ such that $\omega_t \in \mathcal{C}_\varepsilon$, we may define $n_t \in H^1(S \setminus \Sigma; \mathbb{Z})$ from the nearest integer vectors of elements of the (k', δ, h) -tunneling basis. By construction, n_t remains constant in an interval of times for which $\omega_t \in \mathcal{C}_\varepsilon$. Let $t \geq t_0$ be such that ω_t is the beginning of a cusp excursion of length τ . We prove that $n_{t+\tau} = n_t$.

From Lemma 21, we know that there exist bases $\{\zeta_j^0\}$ and $\{\zeta_j^1\}$ of $H_1(S \setminus \Sigma; \mathbb{Z})$ such that the ζ_j^0 are (k, δ, h) -tunneling for ω_t , the ζ_j^1 are (k', δ, h) -tunneling for $\omega_{t+\tau}$,

and for each j , $\zeta_j^1 - \zeta_j^0 = -\kappa \sum_{i=1}^c m_i \langle \zeta_j^0, \gamma_i \rangle \gamma_i$ is a sum of multiples of waist curves of cylinders in the canonical decomposition of (S, Σ, ω_t) . Moreover, each partial sum

$$\zeta_j^\ell = \zeta_j^0 - \kappa \sum_{i=1}^c \ell_i \langle \zeta_j^0, \gamma_i \rangle \gamma_i, \text{ with } 0 \leq \ell_i \leq m_i, \text{ is } (k', \delta)\text{-tunneling for } \omega_t.$$

⁶Indeed, let us consider a horizontal path γ'' in C_i joining the boundaries of C_i , which is homotopic to γ' relative to ∂C_i . Since $\text{Im}(\omega_0)(\gamma') < 2M$, the condition $\tan \theta > \frac{2M}{h(C_i)}$ implies that the sign of $\text{Re}(\omega_0)(\gamma')$ is the same as the sign of $\text{Re}(\omega_0)(\gamma'')$, and since $\langle \gamma'', \gamma_i \rangle = \langle \gamma', \gamma_i \rangle = 1$, this must have the same sign as $\text{Im}(\omega_0)(\gamma_i)$.

Let us fix j . We know that ζ_j^0 and ζ_j^1 are (k', δ, h) - ω_t tunneling. Hence both $\nu \operatorname{Im}(\omega)(\zeta_j^0)$ and $\nu \operatorname{Im}(\omega)(\zeta_j^1)$ are $1/(4M')$ close from two integers $v_j^0 = n_t(\zeta_j^0)$ and $v_j^1 = n_{t+\tau}(\zeta_j^1)$. The curves ζ_j^ℓ are (k', δ, ∞) -tunneling for ω_t , and hence $\nu \operatorname{Im}(\omega)(\zeta_j^\ell)$ is also $1/(4M')$ -close to an integer w_j^ℓ . Two consecutive interpolating curves ζ_j^ℓ and $\zeta_j^{\ell'} = \zeta_j^\ell \pm \gamma_i$ differ by $\pm \gamma_i$. The waist curves γ_i are (k', δ, h) -tunneling for both ω_t and $\omega_{t+\tau}$, and hence $1/(4M')$ is close to the integer $n_t(\gamma_i) = n_{t+1}(\gamma_i)$. Hence we have $w_j^{\ell'} = w_j^\ell \pm n_t(\gamma_i)$ from which follows that $n_t(\zeta_j^1) = n_{t+\tau}(\zeta_j^1)$.

We have shown that n_t and $n_{t+\tau}$ coincide on the basis $\{\zeta_j^1\}_j$. They are hence equal in $H^1(S \setminus \Sigma; \mathbb{Z})$. □

4. GENERIC WEAK-MIXING

4.1. Weak-mixing in almost every direction. Let (S, Σ, ω) be a non-arithmetic Veech surface, and let \mathcal{C} be its $\operatorname{SL}(2, \mathbb{R})$ orbit.

Let k be the holonomy field of (S, Σ, ω) . We recall from Section 2.4 that the vector space $H^1(S; \mathbb{R})$ admits $[k : \mathbb{Q}]$ invariant planes V^σ associated to the embeddings $\sigma : k \rightarrow \mathbb{R}$. Let $W = \bigoplus_{\sigma} V^\sigma$ and $\pi_\sigma : W \rightarrow V^\sigma$ be the natural projection. The real

vector space W is actually defined over \mathbb{Q} , and we denote $W_{\mathbb{Z}} = W \cap H^1(S; \mathbb{Z})$ the integer lattice in W . By definition, π_σ restricted to $W_{\mathbb{Z}}$ is injective (because W is the smallest subspace defined over \mathbb{Q} that contains $V = V^{\operatorname{id}}$).

By Theorem 16, if a surface $u \in \mathcal{C}$ has no vertical saddle connection and admits an eigenvalue, then there exists a vector $v \in H^1(S \setminus \Sigma; \mathbb{Z})$ such that $v - \nu \operatorname{Im}(\omega)$ belongs to the stable space of the Kontsevich-Zorich cocycle. Notice that in this case, we necessarily have $v \in W_{\mathbb{Z}}$. Indeed, $\operatorname{Im}(\omega) \in W$, so the projection of $v - \nu \operatorname{Im}(\omega)$ on the quotient $H^1(S \setminus \Sigma; \mathbb{R})/W$ is contained in $H^1(S \setminus \Sigma; \mathbb{Z})/W_{\mathbb{Z}}$. If the projection of $v - \nu \operatorname{Im} \omega$ would be non-zero, then the projection of $A_n(u) \cdot (v - \nu \operatorname{Im}(\omega))$ would be a non-zero integer vector as well, and hence far away from zero.

For $u \in \mathcal{C}$ with no saddle connection, let $E^s(u)$ be the stable space of the Kontsevich-Zorich cocycle restricted to W , i.e. the set of vectors that are asymptotically shrunk by the Kontsevich-Zorich cocycle.

Notice that if $v = 0$ and $v - \nu \operatorname{Im}(\omega) \in E^s(u)$, then $\nu = 0$ (since $\operatorname{Im}(\omega)$ generates the strongest unstable subspace for the Kontsevich-Zorich cocycle). Any measurable eigenfunction with eigenvalue 0 must be constant by ergodicity (which follows from the recurrence of $g_t u$ in \mathcal{C}).

Thus, the set of $u \in \mathcal{C}$ with no saddle connection and that admits an eigenvalue is contained in

$$\mathcal{E} = \bigcup_{v \in W_{\mathbb{Z}} \setminus \{0\}} \bigcap_{\sigma \neq \operatorname{id}} \mathcal{E}(\pi_\sigma(v)),$$

where

$$\mathcal{E}(w) = \{u \in \mathcal{C} : w \in E^s(u)\}.$$

Then, the exceptionality of eigenvalues for non-arithmetic surfaces (i.e., the ones for which we have at least one $\sigma \neq \operatorname{id}$) follows from the following result.

Theorem 22. *Let (S, Σ, ω) be a Veech surface and \mathcal{C} its $\operatorname{SL}(2, \mathbb{R})$ orbit. Let k be its holonomy field and V^σ the conjugates of the tautological space $V^{\operatorname{id}} = \mathbb{R} \operatorname{Re}(\omega) \oplus \mathbb{R} \operatorname{Im}(\omega)$.*

Then, for any $\sigma : k \rightarrow \mathbb{R}$ and any non-zero vector $w \in V^\sigma$, the set $\mathcal{E}^\sigma(w) = \{u \in \mathcal{C} : w \in (E^s(u) \cap V^\sigma)\}$ has measure zero with respect to the Haar measure on \mathcal{C} .

In other words, the harmonic measure of the Kontsevich-Zorich cocycle on V^σ has no atom.

Using that $W_{\mathbb{Z}}$ is only countable, the above theorem implies that $\{u \in \mathcal{C} : u \text{ has an eigenvalue}\}$ has zero measure. But as shown in Section 5.3, this is equivalent to the fact that $\{\theta \in S^1 : r_\theta u \text{ has an eigenvalue}\}$ has zero Lebesgue measure.

We will not prove Theorem 22 here. It will be refined by our bound on the Hausdorff dimension. It can be proved using the fact that the stable space of the Kontsevich-Zorich cocycles can be studied through random walks (see, e.g., the work of Chaika-Eskin [CE] or Eskin-Matheus [EM]) and the fact that the Veech group Γ and its conjugates Γ^σ are non-elementary.

4.2. Bound on Hausdorff dimension. Now, assuming the results of Sections 5 and 6, we derive an upper bound on the Hausdorff dimension of $\mathcal{E}(w)$.

Let $x = (S, \Sigma, \omega)$ be a Veech surface and \mathcal{C} its $SL(2, \mathbb{R})$ -orbit. We consider a small segment of unstable horocycle Δ through x and build a Poincaré section Q for the Teichmüller flow that contains Δ as in Section 5.2. The return map to Q induces a map $T : \Delta \rightarrow \Delta$ and an associated Kontsevich-Zorich cocycle $A : \Delta \rightarrow \text{Sp}(H^1(S, \Sigma; \mathbb{R}))$. We will show that in this horocycle segment, the set of surfaces for which the vertical flow is not weak mixing has the Hausdorff dimension $d < 1$. By Lemma 24, it will follow that for any surface in \mathcal{C} , the set of directions for which the directional flow is not weak mixing has Hausdorff dimension d as well.

Actually the map T is only a partial map defined on $\Delta^\infty \subset \Delta$ which is the set of all $u \in \Delta$ which are in the domain of T^n for all $n \in \mathbb{N}$ (equivalently, the set of surfaces for which the forward Teichmüller geodesic returns infinitely often to the Poincaré section Q). For any $u \in \Delta^\infty$, we may define the stable space $E^{\sigma, s}(u)$ of the Kontsevich-Zorich restricted to V^σ , i.e., the set of all $v \in V^\sigma$ such that $A_n(u) \cdot v \rightarrow 0$ as $n \rightarrow \infty$. Let similarly as above

$$\mathcal{E}^\sigma(w) = \{u \in \Delta \setminus \Delta^\infty : w \in E^{\sigma, s}(u)\}.$$

By Theorem 29 we have $\text{HD}(\Delta \setminus \Delta^\infty) < 1$. Hence our main result (Theorem 2) will follow once we show that for each $\sigma : k \rightarrow \mathbb{R}$ there exists $d_\sigma < 1$ so that $\text{HD}(\mathcal{E}^\sigma(w)) \leq d_\sigma$.

Recall that for any σ , the Lyapunov exponents of $\bar{\tau}\lambda^\sigma$ of $A|V^\sigma$ is positive (see Lemma 13). Here $\bar{\tau}$ is the normalization due to the time change induced by considering a first return map to Q instead of the geodesic flow (by definition, $\bar{\tau}$ is also the inverse of the Lyapunov exponent of $A|V^{id}$ since $\lambda^{id} = 1$).

By Lemma 26 we know that the expansion constant of $A|V^\sigma$ is given by the Lyapunov exponent. We can then apply Theorem 25 and Theorem 27 to conclude.

4.3. On the group of eigenvalues in exceptional directions. Using Theorem 16, we will show that the Kronecker factor (the maximal measurable almost periodic factor) of the translation flow of a Veech surface is always small. For arithmetic Veech surfaces (square tiled surfaces), we will see that, in any minimal direction, this factor actually identifies with a maximal torus quotient of that surface. For a non-arithmetic one, we obtain that the dimension of the Kronecker factor is at most the degree of the holonomy field.

Let (S, Σ, ω) be a Veech surface, $V = \mathbb{R} \operatorname{Re}(\omega) \oplus \mathbb{R} \operatorname{Im}(\omega) \subset H^1(S; \mathbb{R})$ the tau-topological bundle, and k its trace field. For each embedding $\sigma : k \rightarrow \mathbb{R}$ we note V^σ the Galois conjugate of V . The subspace $W = \bigoplus V^\sigma \subset H^1(S; \mathbb{R})$ is defined over \mathbb{Q} and has dimension $2[k : \mathbb{Q}]$.

The field k acts by multiplication on $H^1(S; \mathbb{R})$ preserving $H^1(S; \mathbb{Q})$ as follows: for $\lambda \in k$ consider the endomorphism of $H^1(S; \mathbb{R})$ that acts by multiplication by λ^σ in V^σ . In particular, the set \mathcal{O}_k of elements $\lambda \in k$ that preserves $H^1(S; \mathbb{Z})$ forms an order (a \mathbb{Z} -module of rank $[k : \mathbb{Q}]$, stable under multiplication). This phenomenon is actually much deeper as the action of k preserves the complex structure on $H^1(S; \mathbb{C})$ and the Hodge decomposition $H^1(S; \mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$ into a holomorphic and an anti-holomorphic one forms: Veech surfaces belong to so-called *real multiplication loci*; see [McM03] and [BM10].

Now we turn to the case of arithmetic surfaces and describe their maximal tori. Let (S, Σ, ω) be a square tiled surface. Let x be a point in S and Λ the subgroup of \mathbb{C} generated by the integration of ω along closed loops. Then we have a well defined map $f : S \rightarrow \mathbb{C}/\Lambda$ defined by $f(y) = \int_\gamma \omega \pmod{\Lambda}$ where γ is any path that joins x to y . The intersection $H^1(S; \mathbb{Z}) \cap V$ is naturally identified with $H^1(\mathbb{C}/\Lambda; \mathbb{Z})$ through f^* , and we call \mathbb{C}/Λ together with the projection f the *maximal torus* of S . Note that f is not necessarily ramified over only one point.

Theorem 23. *Let (S, Σ, ω) be a Veech surface, and let k be its holonomy field. Then in each minimal direction the group of eigenvalues is finitely generated. Moreover,*

- *If (S, Σ, ω) is arithmetic ($k = \mathbb{Q}$), then, in each minimal direction, all eigenfunctions of the flow of S are lifts from the maximal torus of S . In particular, there are exactly 2 rationally independent continuous eigenvalues.*
- *If (S, Σ, ω) is non-arithmetic ($k \neq \mathbb{Q}$), then the ratio of any two non-zero eigenvalues for the translation flow of S belongs to k . In particular, in each minimal direction, there are at most $[k : \mathbb{Q}]$ rationally independent eigenvalues.*

Proof. Let us assume that $\nu \in \mathbb{R}$ is an eigenvalue of the flow of (S, Σ, ω) .

Let $W = \bigoplus V^\sigma$ and $W_{\mathbb{Z}} = W \cap H^1(S; \mathbb{Z})$. Let $E^s \subset W$ be the stable space of the Kontsevich-Zorich cocycle restricted to W , and denote $E^{s,\sigma} = E^s \cap V^\sigma$. Note that $E^{s,\sigma}$ has dimension at most 1. From Theorem 16, if ν is an eigenvalue of the flow, there exists $v \in W_{\mathbb{Z}}$ such that $\nu \operatorname{Im} \omega - v \in E^s$. The map $\nu \mapsto v$ provides an isomorphism between the group of eigenvalues and a subgroup of $W_{\mathbb{Z}}$, so the group of eigenvalues is finitely generated.

Decomposing $v = \sum v_\sigma$ with respect to the direct sum $W = \bigoplus V^\sigma$ we get

- $\nu \operatorname{Im} \omega - v_{id} \in E^{s,id}$;
- for any $\sigma \neq id$, $v_\sigma \in E^{s,\sigma}$.

In particular, if the dimension of E^s is not maximal, then there is no eigenvalue.

The action of \mathcal{O}_k preserves the set of lines in each V^σ and hence preserves (globally) the stable space E^s . In particular, if $\nu \operatorname{Im} \omega - v \in E^s$, then for any $\lambda \in \mathcal{O}_k$ we have $\lambda \nu \operatorname{Im} \omega - \sum \sigma(\lambda) v_\sigma \in E^s$. So the set of potential eigenvalues

$$\Theta = \{ \mu \in \mathbb{R} : \exists v \in W_{\mathbb{Z}}, \quad \mu \operatorname{Im} \omega - v \in E^s \}$$

is stable under multiplication by \mathcal{O}_k .

If $k = \mathbb{Q}$, then we saw that $W_{\mathbb{Z}}$ is naturally identified with the cohomology of the maximal torus of S . As all eigenvalues are contained in Θ , they all come from the maximal torus.

Now, the rank of Λ is the rank of $H^1(S, \mathbb{Z}) \cap (\mathbb{R} \operatorname{Im}(\omega) \oplus E^s(\omega))$. We know that for non-arithmetic surfaces this rank cannot be maximal and is hence at most $[k : \mathbb{Q}]$. But, as Θ is stable under multiplication by \mathcal{O}_k , its rank is a multiple of $[k : \mathbb{Q}]$ and hence is 0 or $[k : \mathbb{Q}]$. As the ratio of any two eigenvalues is the ratio of two elements of Θ , it belongs to k . □

5. MARKOV MODEL

We introduce a Markov model to study the geodesic flow on the $\operatorname{SL}(2, \mathbb{R})$ -orbits of Veech surfaces. It will be used in Section 6 to get a bound on the Hausdorff dimension of exceptional sets. Our large deviation results mimic the case of random independent variables. The condition that ensures enough independence is the so-called *bounded distortion* property.

5.1. Locally constant cocycles. Let Δ be a measurable space, and let μ be a finite probability (reference) measure on Δ . Let $\Delta^{(l)}$, $l \in \mathbb{Z}$ be a partition μ -mod 0 of Δ into sets of positive μ -measure. Let $T : \Delta \rightarrow \Delta$ be a measurable map such that $T|_{\Delta^{(l)}} : \Delta^{(l)} \rightarrow \Delta$ is a bimeasurable map (i.e., a map that admits an almost everywhere defined measurable inverse).

Let Ω be the space of all finite sequences of integers. The length of $\underline{l} \in \Omega$ will be denoted by $|\underline{l}|$.

For $\underline{l} = (l_1, \dots, l_n) \in \Omega$, we let $\Delta^{\underline{l}}$ be the set of all $x \in \Delta$ such that $T^{j-1}(x) \in \Delta^{l_j}$ for $1 \leq j \leq n$. We say that T has a *bounded distortion* if there exists $C_0 > 0$ such that every $\Delta^{\underline{l}}$ has a positive μ -measure and $\mu^{\underline{l}} = \frac{1}{\mu(\Delta^{\underline{l}})} T_*^{|\underline{l}|}(\mu|_{\Delta^{\underline{l}}})$ satisfies $\frac{1}{C_0} \mu \leq \mu^{\underline{l}} \leq C_0 \mu$. In particular, for every $n \geq 1$, $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} T_*^k \mu$ satisfies $\frac{1}{C_0} \mu \leq \mu_n \leq C_0 \mu$. Taking a weak limit, one sees that there exists an invariant measure ν satisfying $\frac{1}{C_0} \mu \leq \nu \leq C_0 \mu$. It is easy to see that this invariant measure is ergodic provided the σ -algebra of μ -measurable sets is generated (mod 0) by $\{\Delta^{\underline{l}} : \underline{l} \in \Omega\}$.

The bounded distortion condition is a weakened form of independence. It says that the conditional measures with respect to some past events are comparable to the initial measure up to multiplicative constants. Indeed, it can be seen that if $C_0 = 1$ in the definition of bounded distortion, then the measure is invariant and is a Bernoulli measure; i.e., it does not depend on the past (or equivalently if $\underline{l} = (l_1, l_2, \dots, l_n)$, then $\mu(\Delta^{\underline{l}}) = \mu(\Delta^{(l_1)})\mu(\Delta^{(l_2)}) \dots \mu(\Delta^{(l_n)})$).

Let H be a finite dimensional (real or complex) vector space, and let $\operatorname{SL}(H)$ denote the space of linear automorphisms of H with determinant 1. Given $T : \Delta \rightarrow \Delta$ as above, we can define a locally constant $\operatorname{SL}(H)$ -cocycle over T by specifying a sequence $A^{(l)} \in \operatorname{SL}(H)$, $l \in \mathbb{Z}$: Take $A(x) = A^{(l)}$ for $x \in \Delta^{(l)}$ and $(T, A) : (x, w) \mapsto (T(x), A(x)w)$. Then the cocycle iterates are given by $(T, A)^n = (T^n, A_n)$ where $A_n(x) = A(T^{n-1}(x)) \dots A(x)$. Notice that if $\underline{l} = (l_1, \dots, l_n)$, then $A_n(x) = A^{\underline{l}}$ for $x \in \Delta^{\underline{l}}$ with $A^{\underline{l}} = A^{(l_n)} \dots A^{(l_1)}$.

For a matrix A , let $\|A\|_+ = \max(\|A\|, \|A^{-1}\|)$. We say that T is *fast decaying* if there exists $C_1 > 0, \alpha_1 > 0$ such that

$$\sum_{l: \mu(\Delta^{(l)}) \leq \varepsilon} \mu(\Delta^{(l)}) \leq C_1 \varepsilon^{\alpha_1}, \quad \text{for } 0 < \varepsilon < 1,$$

and we say that A is *fast decaying* if there exists $C_2 > 0, \alpha_2 > 0$ such that

$$\sum_{l: \|A^{(l)}\|_+ \geq n} \mu(\Delta^{(l)}) \leq C_2 n^{-\alpha_2}.$$

The fast decaying property of the map T says that most of the measure is supported on pieces $\Delta^{(l)}$ of large measures. In other words, it is a full-shift on an infinite alphabet but close to a full-shift on a finite alphabet.

The fast decaying property of cocycles is a crucial hypothesis for large deviations. It is equivalent to the integrability of $\exp(\delta \log \|A\|_+)$ for δ small enough. It implies in particular that A is a log-integrable cocycle with respect to the invariant measure ν , i.e., $\int \ln \|A\|_+ d\nu < \infty$.

Note that these two conditions would be automatically satisfied if instead of the infinite alphabet \mathbb{Z} we had a finite one.

In our applications, Δ will be a simplex in $\mathbb{P}\mathbb{R}^p$, i.e., the image of $\mathbb{P}\mathbb{R}_+^p$ by a projective transformation, μ is the Lebesgue measure, and $T|\Delta^{(l)}$ is a projective transformation for every $l \in \mathbb{Z}$. In fact, we will be mostly interested in the case $p = 2$, and we will use freely the identification of $\mathbb{P}\mathbb{R}^2$ with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Note that if there exists a simplex $\Delta \Subset \Delta'$ such that the projective extension of $(T|\Delta^{(l)})^{-1}$ maps Δ' into itself for every $l \in \mathbb{Z}$, then the bounded distortion property holds; see section 2 of [AF07]. In this case we will say that T is a *projective expanding map*.

5.2. The Markov model for the geodesic flow on $\text{SL}(2, \mathbb{R})/\Gamma$. Now we construct a map $T : \Delta \rightarrow \Delta$ which is a first return map of the geodesic flow. The bounded distortion and fast decaying properties will follow from hyperbolic properties of the geodesic flow. The associated Kontsevich-Zorich cocycle $A : \Delta \rightarrow \text{Sp}(H^1(S, \Sigma; \mathbb{R}))$ is shown to have the fast decaying property.

Let $\mathcal{C} = \text{SL}(2, \mathbb{R})/\Gamma$ where Γ has a finite covolume. Let us fix any point $x \in \mathcal{C}$. Then we can find a small smooth rectangle Q through x , which is transverse to the geodesic flow and provides us with a nice Poincaré section, in the sense that the first return map to Q under the geodesic flow has a particularly simple structure. All these properties will follow from the following commutation rules in $\text{SL}(2, \mathbb{R})$:

$$g_t h_s^- g_t = h_{e^{-2t}s}^- \quad \text{and} \quad g_t h_u^+ g_{-t} = h_{e^{2t}u}^+.$$

More precisely, let $p : \mathbb{R}^2 \rightarrow \mathcal{C}$ be given by $p(u, s) = h_s^-(h_u^+(x))$. Then for any $\varepsilon > 0$, we can find $u_- < 0 < u_+$ and $s_- < 0 < s_+$ with $u_+ - u_- < \varepsilon$ and $s_+ - s_- < \varepsilon$ such that, letting $\Delta = (u_-, u_+) \subset \mathbb{R}$ and $\widehat{\Delta} = \{(u, s) \in \Delta \times \mathbb{R}; s_- < \frac{s}{1+su} < s_+\}$, we can take $Q = p(\widehat{\Delta})$. It is clear that Q is transverse to the geodesic flow. Let F denote the first return map to Q . Then

- (1) There exist countably many disjoint open intervals $\Delta^{(l)} \subset \Delta$, such that the domain of F is the union of the $p(\widehat{\Delta}^{(l)})$, where $\widehat{\Delta}^{(l)} = \widehat{\Delta} \cap (\Delta^{(l)} \times \mathbb{R})$.

- (2) There exists a function $r : \bigcup \Delta^{(l)} \rightarrow \mathbb{R}_+$ such that if (u, s) belongs to some $\widehat{\Delta}^{(l)}$, then the return time of $p(u, s)$ to Q is $r(u)$. Moreover, r is globally bounded away from zero, and its restriction to each $\Delta^{(l)}$ is given by the logarithm of the restriction of a projective map $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$.
- (3) There exist functions $T : \bigcup \Delta^{(l)} \rightarrow \Delta$ and $S : \bigcup \Delta^{(l)} \rightarrow \mathbb{R}$ such that if (u, s) belongs to some $\widehat{\Delta}^{(l)}$, then $F(p(u, s)) = p(T(u), S(u) - e^{-2r(u)}s)$. Moreover, the restriction of T to each $\Delta^{(l)}$ coincides with the restriction of a projective map $T_l : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$, and the restriction of S to each $\Delta^{(l)}$ coincides with the restriction of an affine map $S_l : \mathbb{R} \rightarrow \mathbb{R}$.
- (4) There exists a bounded open interval Δ' containing $\overline{\Delta}$ such that $T_l^{-1}(\Delta') \subset \Delta'$ for every $l \in \mathbb{Z}$.

The basic idea of the construction is to guarantee that the forward orbit of the “unstable” frame $\delta_u(Q) = p\{(u, s) \in \partial\widehat{\Delta}; u = u_{\pm}\}$ and the backward orbit of the “stable” frame $\delta_s(Q) = p\{(u, s) \in \partial\widehat{\Delta}; \frac{s}{1+su} = s_{\pm}\}$ never come back to Q .⁷ This easily yields the Markovian structure, and the remaining properties follow from direct computation (or, for the last property, by shrinking ε).

Remark 5.1. Given $u_0 \in \Delta^{(l)}$, knowledge of $T(u_0)$, $S(u_0)$, and $r(u_0)$ allows one to easily compute T , S , and r restricted to $\Delta^{(l)}$. Indeed, for $u \in \Delta^{(l)}$, $g_{r(u_0)}(p(u, 0)) = h_{e^{2r(u_0)}(u-u_0)}^+ F(p(u_0, 0))$. To move it to Q , we must apply g_{-t} where t is bounded (indeed at most of order ε). Using that $F(p(u_0, 0)) = p(T(u_0), S(u_0))$ one gets $e^t = 1 + e^{2r(u_0)}(u - u_0)S(u_0)$, and then the formulas

$$e^{r(u)} = \frac{e^{r(u_0)}}{1 + e^{2r(u_0)}(u - u_0)S(u_0)},$$

$$T(u) = T(u_0) + \frac{e^{2r(u_0)}(u - u_0)}{1 + e^{2r(u_0)}(u - u_0)S(u_0)},$$

$$S(u) = S(u_0)(1 + e^{2r(u_0)}(u - u_0)S(u_0)).$$

Note that

$$(1) \quad DT(u) = e^{2r(u)},$$

and that for every $l \in \mathbb{Z}$, $r \circ T_l^{-1} : \Delta \rightarrow \mathbb{R}$ has a uniformly bounded derivative.

Note that T is a projective expanding map with bounded distortion, so it admits an ergodic invariant measure ν equivalent to Lebesgue measure. In order to obtain an upper bound on the Hausdorff dimension of the set of non-weak mixing directions, we will also need to use that T is fast decaying. Using (1), we see that fast

⁷This is easy enough to do when x is not a periodic orbit of a small period. From a segment of the unstable horocycle through x , remove all g_t pullback of the $C\varepsilon$ -neighborhood of x . Because of hyperbolicity, the remainder is a small Cantor set. Then u_{\pm} are chosen so that $h_{u_{\pm}}^+(x)$ lies in the complement of this set.

decay is implied by the following well known exponential tail estimate on return times: there exists $\delta > 0$ (depending on Q) such that

$$\int_{u_-}^{u_+} e^{-\delta r(u)} du < \infty.$$

Remark 5.2. The exponential tail estimate is usually proved using a finite Markov model for the full geodesic flow (as opposed to the infinite Markov model for a Poincaré return map that we consider here). However, it can also be proved using some more general information about the geodesic flow.

Details of this approach are carried out in [AD], where it is used to obtain exponential tails for the return time for a Markov model of an arbitrary affine $SL(2, \mathbb{R})$ -invariant measure in moduli space.

Remark 5.3. As remarked before, $r \circ T_l^{-1} : \Delta \rightarrow \mathbb{R}$ has a uniformly bounded derivative. This estimate can be iterated as follows. The equality $DT(u) = e^{2r(u)}$ does not depend on the fact that it was a first return. Let $r_n(u) = \sum_{k=0}^{n-1} r(T^k(u))$ be the n th return time to Q . Then we have $DT^n(u) = e^{2r_n(u)}$, and formulas similar to the one in Remark 5.1 for T^n and S^n hold. Writing $T_l = T_{l_n} \circ \dots \circ T_{l_1} : \Delta^l \rightarrow \Delta$ it implies that $r_n \circ T_l^{-1} : \Delta \rightarrow \mathbb{R}$ has a uniformly bounded derivative independent of n .

Let now $\mathcal{C} = SL(2, \mathbb{R})/\Gamma$ be the $SL(2, \mathbb{R})$ -orbit of some Veech surface. Then the Kontsevich-Zorich cocycle over \mathcal{C} gives rise to a locally constant cocycle over T as follows.

Recall from Section 2.5 that there is a natural trivialization of the Hodge bundle over Q that preserves the integer lattice, the tautological bundle $V = \mathbb{R} \operatorname{Re} \omega \oplus \mathbb{R} \operatorname{Im} \omega$, and its conjugates V^σ . The induced Kontsevich-Zorich cocycle $A : \Delta \rightarrow \operatorname{Sp}(H^1(S; \mathbb{R}))$ is such that $A(y)$ depends continuously on $y \in p(\widehat{\Delta}_j)$. Because it belongs to a discrete group, it must in fact be constant. We let $A^{(l)}$ denote its value on $\Delta^{(l)}$.

If $\underline{l} = (l_1, \dots, l_n)$ with $n \geq 1$, then F^n has a unique fixed point $(S, \Sigma, \omega) = p(u_{\underline{l}}, s_{\underline{l}})$ with $(u_{\underline{l}}, s_{\underline{l}}) \in \widehat{\Delta}^{\underline{l}} = (\Delta^{\underline{l}} \times \mathbb{R}) \cap \widehat{\Delta}$, and $A^{\underline{l}}|_{V_x}$ is hyperbolic with unstable direction $\operatorname{Im}(\omega)$, stable direction $\operatorname{Re}(\omega)$, and Lyapunov exponent $r_n(u_{\underline{l}})$. In particular, $\|A^{\underline{l}}|_V\|$ is of order $e^{r_n(u_{\underline{l}})}$ (up to uniformly bounded multiplicative constants), since the angle between $\operatorname{Re}(\omega)$ and $\operatorname{Im}(\omega)$ is uniformly bounded over Q .

Note that $\|A^{\underline{l}}\|$ is also of order $e^{r_n(u_{\underline{l}})}$ (this follows, for instance, from Remark 2.1). In particular ($n = 1$), the exponential tail estimate implies that A is fast decaying.

Since the geodesic flow on \mathcal{C} is ergodic, the Lyapunov exponents of the Kontsevich-Zorich cocycle on \mathcal{C} (with respect to the Haar measure) are the same as the Lyapunov exponents of the locally constant cocycle (T, A) , with respect to the invariant measure ν , up to the normalization factor $\bar{r} = \int r(u) d\nu(u)$.

5.3. Reduction to the Markov model. An eigenfunction $f : S \rightarrow \mathbb{C}$ with eigenvalue $\nu \in \mathbb{R}$ of a translation flow $\phi_t : S \rightarrow S$ is a measurable function such that $f \circ \phi_t = e^{2\pi i \nu t} f$. Note that if f is a measurable or continuous eigenfunction for the

vertical flow on a translation surface $z = (S, \Sigma, \omega) \in \mathcal{M}_{S, \Sigma}(\kappa)$, then f is also an eigenfunction for $g \cdot z$ for any $g \in \text{SL}(2, \mathbb{R})$ which fixes the vertical direction, and in particular for any g of the form $h_s^- g_t$, $s, t \in \mathbb{R}$.

Lemma 24. *Let \mathcal{C} be a closed $\text{SL}(2, \mathbb{R})$ orbit in some $\mathcal{M}_{S, \Sigma}(\kappa)$. Let I be a non-empty open subset of an $\text{SO}(2, \mathbb{R})$ orbit, and let J be a non-empty open subset of an unstable horocycle. Then*

- (1) *For every $z_0 \in I$, there exists a diffeomorphism taking z to x from an open neighborhood $I' \subset I$ of z_0 to a subinterval of J , such that the stable horocycle through z intersects the geodesic through x ;*
- (2) *For every $x_0 \in I$, there exists a diffeomorphism taking z to x from an open neighborhood $J' \subset J$ of x_0 to a subinterval of I , such that the stable horocycle through z intersects the geodesic through x .*

In particular, if Λ is any subset of \mathcal{C} which is invariant by the stable horocycle and geodesic flows (such as the set of translation surfaces for which the vertical flow admits a continuous eigenfunction, or a measurable but discontinuous eigenfunction), $\text{HD}(I \cap \Lambda) = \text{HD}(J \cap \Lambda)$.

Proof. We prove the first statement. Fix $z \in I$ and some compact segment $J_0 \subset J$. Then we can choose t_0 large such that there exists $y \in J_0$ with $g_{t_0}(y)$ close to z (indeed as $t \rightarrow \infty$, $g_t \cdot I_0$ is becoming dense in \mathcal{C}). Thus for every $\theta \in \mathbb{R}$ close to 0 we can write $r_\theta z = h_s^- h_u^+ g_{t_0+t} y$ in a unique way with s, t, u small, and moreover $\theta \mapsto u$ is a diffeomorphism.

The second statement is analogous. □

6. ANOMALOUS LYAPUNOV BEHAVIOR, LARGE DEVIATIONS AND HAUSDORFF DIMENSION

We have seen so far that weak mixing can be established by ruling out non-trivial intersections of $\text{Im}(\omega)$ with integer translates of the strong stable space. This criterion can be rephrased in terms of certain fixed vectors (projections of integer points on Galois conjugates of the tautological bundle) lying in the strong stable space. In particular, its iterate must see a non-positive rate of expansion, instead of the expected rate (given by one of the positive Lyapunov exponents).

In this section we introduce techniques to bound anomalous Oseledets behavior in the setting of locally constant cocycles with bounded distortion. The Oseledets theorem states that for a typical orbit, any vector will expand precisely at the rate of some Lyapunov exponent. For a given vector, one can consider the minimum expansion rate which can be seen with positive probability. We will first show a (finite time) upper bound on the probability of seeing less than such minimum expansion. Then we will show that such an estimate can be converted into an upper bound on the Hausdorff dimension of orbits exhibiting an exceptionally small expansion.

6.1. Large deviations. Let (T, A) be a locally constant log-integrable cocycle over a map $T : \Delta \rightarrow \Delta$ preserving a measure μ . The *expansion constant* of (T, A) is the maximal $c \in \mathbb{R}$ such that for all $v \in \mathbb{R}^d \setminus \{0\}$ and for μ -almost every $x \in \Delta$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \|A_n(x) \cdot v\| \geq c.$$

(The limit exists by Oseledets theorem applied to μ .)

Theorem 25. *Let $T : \Delta \rightarrow \Delta$ be a countable shift endowed with a measure μ with bounded distortion and which is fast decaying. Let $A : \Delta \rightarrow \text{SL}(d, \mathbb{R})$ be a fast decaying cocycle. Then for every c' smaller than the expansion constant of A , there exist $C_3 > 0, \alpha_3 > 0$ such that for every unit vector $v \in \mathbb{R}^d$,*

$$\mu\{u \in \Delta : \|A_n(x) \cdot v\| \leq e^{c'n}\} \leq C_3 e^{-\alpha_3 n}.$$

Actually, the expansion constant is intimately related to Lyapunov exponents. Recall that the cocycle $A : \Delta \rightarrow \text{SL}(d, \mathbb{R})$ is *irreducible* if there are no non-trivial subspaces of \mathbb{R}^d that are invariant under the group generated by the matrices $(A^{(l)})_{l \in \mathbb{Z}}$.

Lemma 26. *Let $T : \Delta \rightarrow \Delta$ be a countable shift with bounded distortion and let $A : \Delta \rightarrow \text{SL}(d, \mathbb{R})$ be an irreducible and log-integrable cocycle. Then the expansion constant of A is its maximal Lyapunov exponent.*

Let us first prove the lemma.

Proof. Let ν be the invariant measure of T and λ the maximal Lyapunov exponent of A , i.e., the a.e. limit

$$\lambda = \lim_{n \rightarrow \infty} \frac{\ln \|A_n(u)\|}{n}.$$

Let

$$E_2(u) = \left\{ w \in \mathbb{R}^d : \lim_{n \rightarrow \infty} \frac{\ln \|A_n(u) \cdot w\|}{n} < \lambda \right\}.$$

By the Oseledets theorem, for any $w \in \mathbb{R}^d \setminus \{0\}$ for ν -almost every $u \in \Delta$ we have that

$$\lim_{n \rightarrow \infty} \frac{\ln \|A_n(u) \cdot w\|}{n} = \lambda$$

unless w belongs to $E_2(u)$.

Assume that there exists a vector $w \in \mathbb{R}^d \setminus \{0\}$ and a subset of Δ of positive μ -measure such that $w \in E_2(u)$. Let $H \subset \mathbb{R}^d$ be a subspace of maximal dimension so that $H \subset E_2(u)$ for a positive measure set of u and let $u_0 \in \Delta$ be a density point in that set. Then applying T^n for n large enough and using bounded distortion, one can see that the space $(A_n(u_0)H) \subset E_2(u)$ for a subset of Δ of measure arbitrarily close to 1. By compactness of the grassmanian of \mathbb{R}^d , one can chose H' of the same dimension so that $H' \subset E_2(u)$ for a.e. $u \in \Delta$.

Now consider a finite word l and the subspace $H'' = (A^l)^{-1}H'$. Then, by construction for a.e. $u \in \Delta^l$ the set H'' is contained in $E_2(u)$. If $H'' \neq H'$, then the subspace $H'' \oplus H'$ would be contained in $E_2(u)$ on a positive measure set and of larger dimension. This contradicts our initial choice of H' . Hence the space H' is invariant under all matrices A^l which contradicts the irreducibility assumption since $0 < \dim H' \leq \dim E_2(u) < \dim H$. □

Proof of Theorem 25. Let c be the expansion constant of A , and let $c' < c$. For $v \in \mathbb{R}^d \setminus \{0\}$, let $I(x, n, v) = \frac{1}{n} \ln_+ \frac{\|A_n(x) \cdot v\|}{\|v\|}$ and $I(l, v) = \frac{1}{|l|} \frac{\ln_+ \|A^l \cdot v\|}{\|v\|}$ where $\ln_+(t) = \max(0, \ln(t))$.

Recall that because $T : \Delta \rightarrow \Delta$ has bounded distortion, the invariant measures μ comes with a family of measures $\mu^{\underline{l}}$ for $\underline{l} \in \Omega$ defined by

$$\mu(\Delta_{\underline{l}}^{\underline{l}'}) = \mu(\Delta_{\underline{l}}) \mu^{\underline{l}}(\Delta_{\underline{l}'}) .$$

They satisfy $1/C\mu \leq \mu^{\underline{l}} \leq C\mu$ for some constant C uniform in \underline{l} (see Section 5.1).

The proof basically proceeds in two steps. We first prove that there exists an integer n_0 for which we have some decay (see Equation (3) which is uniform in $v \in \mathbb{R}^d \setminus \{0\}$ and $\underline{l} \in \Omega$). Then we use the bounded distortion property to extend it to all n .

We first claim that

$$(2) \quad \lim_{n \rightarrow \infty} \sup_{v \in \mathbb{R}^d \setminus \{0\}} \sup_{\underline{l} \in \Omega} \mu^{\underline{l}}\{x \in \Delta : I(x, n, v) < c'\} = 0 .$$

Indeed, the definition of the expansion constant c gives that for any $c'' < c$ we have

$$\sup_{v \in \mathbb{R}^d \setminus \{0\}} \lim_{n \rightarrow \infty} \mu\{x : I(x, n, v) < c''\} = 0 .$$

And we exchange quantifiers as follows (the argument comes from Lemma 3.3 of [AF07]). Let $C(x, v) = \inf_{n \geq 0} \frac{\|A_n(x) \cdot v\|}{e^{nc''} \|v\|}$. By hypothesis, $C(x, v) > 0$ for all $v \in \mathbb{R}^d \setminus \{0\}$ and almost every x . Moreover, if $v_k \in \mathbb{R}^d \setminus \{0\}$ is a sequence converging to v , then $C(x, v_k) \rightarrow C(x, v)$ for almost every x .

Let α be a positive real constant. Using the Fatou lemma, we have that

$$\begin{aligned} \limsup_{v_k \rightarrow v} \mu\{x : C(x, v_k) < \alpha\} &= \limsup_{v_k \rightarrow v} \int_{\Delta} 1_{C(x, v_k) < \alpha} d\mu(x) \\ &\leq \int_{\Delta} \left(\limsup_{v_k \rightarrow v} 1_{C(x, v_k) < \alpha} \right) d\mu(x) = \int_{\Delta} 1_{C(x, v)} d\mu(x) . \end{aligned}$$

That is to say, the function $v \mapsto \mu\{x : C(x, v) < \alpha\}$ is upper semi-continuous. Now for every v there exists $\alpha_\varepsilon(v)$ so that $\mu\{x : C(x, v) \leq \alpha_\varepsilon(v)\} < \varepsilon$. By upper semi-continuity, this is also true in a neighborhood of v , and we can conclude using compactness (and the fact that c'' was arbitrary).

The fact that the convergence is uniform in $\underline{l} \in \Omega$ in Equation (2) follows directly from bounded distortion. This concludes the proof of the claim.

Because $\int_{\Delta} I(x, n, v) d\mu^{\underline{l}}(x) \geq c'' \mu^{\underline{l}}\{x : I(x, m, v) \geq c''\}$ (Markov inequality), we can find n_0 and κ' so that

$$(3) \quad \sup_{v \in \mathbb{R}^d \setminus \{0\}} \sup_{\underline{l} \in \Omega} \int c' - I(x, n_0, v) d\mu^{\underline{l}} < -\kappa' .$$

Now, we transform this estimate into an exponential form. By fast decay of the cycle A , there exists $C' > 0, \delta' > 0$ such that for all $|s| < \delta', 1 \leq n \leq n_0$ we have

$$\sup_{v, \underline{l}} \int_{\Delta} |e^{sn(c' - I(x, n, v))}| d\mu^{\underline{l}}(x) \leq C' .$$

Hence $\phi_{n,\underline{l},v} : s \mapsto \int e^{sn(c'-I(x,n,v))} d\mu^{\underline{l}}(x)$ are uniformly bounded holomorphic functions of $|s| < \delta'$ for $n \leq n_0$, $\underline{l} \in \Omega$, and $v \in \mathbb{R}^d \setminus \{0\}$. Note that $\phi_{n,\underline{l},v}(0) = 1$ and $\phi'_{n,\underline{l},v}(0) = n(\int c' - I(x,n,v) d\mu^{\underline{l}}(x))$. Thus, using (3) that gives an upper bound on the derivative $\phi_{n_0,\underline{l},v}(0)$, there exists δ, κ (with $0 < \delta \leq \delta'$ and $0 < \kappa \leq \delta\kappa'$) so that

$$(4) \quad \sup_{v,\underline{l}} \int e^{\delta n_0(c'-I(x,n_0,v))} d\mu^{\underline{l}} < e^{-\kappa n_0},$$

while, for every $1 \leq n \leq n_0 - 1$,

$$(5) \quad \sup_{v,\underline{l}} \int e^{\delta n(c'-I(x,n,v))} d\mu^{\underline{l}} < 2e^{-\kappa n}.$$

We have proved so far that at the scale n_0 , we have an exponential decay. We now use the bounded distortion property to extend it for all times. Note that from submultiplicativity of the norm, we have that $(m+n)I(x,m+n,v) \leq nI(x,n,v) + mI(T^n x, m, A_n(x) \cdot v)$. Hence, for all positive integers m, n we have

$$(6) \quad \int e^{\delta(n+m)(c'-I(x,n+m,v))} d\mu(x) \leq \sum_{|\underline{l}|=n} \mu(\Delta^{\underline{l}}) e^{\delta n(c'-I(\underline{l},v))} \int e^{\delta m(c'-I(x,m,A^{\underline{l}} \cdot v))} d\mu^{\underline{l}}(x) \\ \leq \int e^{\delta n(c'-I(x,n,v))} d\mu(x) \\ \times \sup_{|\underline{l}|=m} \int e^{\delta m(c'-I(x,m,A^{\underline{l}} \cdot v))} d\mu^{\underline{l}}.$$

Now given any n , we do the euclidean division $n = kn_0 + r$. Using the product formula (6) several times together with the estimates (4) (for the parts in n_0) and (5) (for the rest r), we obtain that

$$\mu\{x : I(x,n,v) \leq c'\} \leq \int e^{\delta n(c'-I(x,n,v))} d\mu(x) \leq 2e^{-\kappa n}. \quad \square$$

Remark 6.1. The previous theorem can be somewhat refined: If A is fast decaying and for some vector $v \in \mathbb{R}^d \setminus \{0\}$ we have $\lim \frac{1}{n} \ln \|A_n(x) \cdot v\| > c'$ for a positive μ -measure set of $x \in \Delta$, then the μ -measure of the set of x such that $\frac{1}{n} \ln \|A_n(x) \cdot v\| \leq c'$ is exponentially small in n . This can be proved by reduction to the setting above after taking the quotient by an appropriate invariant subspace.

6.2. Hausdorff dimension. The next result shows how to convert Theorem 25 into an estimate on Hausdorff dimension. We will assume that $T : \Delta \rightarrow \Delta$ is a transformation with bounded distortion, Δ is a simplex in $\mathbb{P}\mathbb{R}^p$ for some $p \geq 2$, and $T|_{\Delta^{(l)}}$ is a projective transformation for every $l \in \mathbb{Z}$.

Recall that T is fast decaying if there exists constants $\alpha_1 > 0$ and $C_1 > 0$ so that for any $\varepsilon > 0$

$$\sum_{\mu(\Delta^{(l)}) \leq \varepsilon} \mu(\Delta^{(l)}) \leq C_1 \varepsilon_1^\alpha.$$

Theorem 27. *Assume that T is fast decaying. For $n \geq 1$, let $X_n \subset \Delta$ be a union of $\Delta^{\underline{l}}$ with $|\underline{l}| = n$, and let $X = \liminf X_n$. Let*

$$\delta = \limsup_{n \rightarrow \infty} \frac{-\ln \mu(X_n)}{n}.$$

Then $\text{HD}(X) \leq p - 1 - \min(\delta, \alpha_1)$ where α_1 is the fast decay constant of T .

We will need a preliminary result.

Lemma 28. *Assume that T has a bounded distortion and is fast decaying. Then for $0 < \alpha_4 < \alpha_1$, there exists $C_4 > 0$ such that for every $n \geq 1$, we have*

$$\sum_{|\underline{l}|=n} \mu(\Delta^{\underline{l}})^{1-\alpha_4} \leq C_4^n.$$

Proof. The fast decay of T implies that for any k ,

$$\sum_{2^{-k-1} \leq \mu(\Delta^{(l)}) \leq 2^{-k}} \mu(\Delta^{(l)}) \leq C_1 2^{-\alpha_1 k}.$$

Hence

$$\sum_{2^{-k-1} \leq \mu(\Delta^{(l)})^{1-\varepsilon} \leq 2^{-k}} \mu(\Delta^{(l)}) \leq 2^{-(1-\varepsilon)k} \frac{C_1 2^{-\alpha_1 k}}{2^{-k-1}}.$$

Summing over k we get

$$\sum_{l \in \mathbb{Z}} \mu(\Delta^{(l)})^{1-\varepsilon} \leq 2C_1 \sum_{k \geq 0} 2^{(\varepsilon-\alpha_1)k}.$$

It follows that for every \underline{l} ,

$$\sum_{l \in \mathbb{Z}} \mu^{\underline{l}}(\Delta^{(l)})^{1-\varepsilon} \leq C_4.$$

On the other hand, it is clear that

$$\sum_{|\underline{l}'|=n+1} \mu(\Delta^{\underline{l}'})^{1-\varepsilon} = \sum_{|\underline{l}|=n} \mu(\Delta^{\underline{l}})^{1-\varepsilon} \sum_{l \in \mathbb{Z}} \mu^{\underline{l}}(\Delta^{(l)})^{1-\varepsilon}.$$

The result follows by induction. □

Proof of Theorem 27. Notice that there exists $C' > 0$ such that if $0 < \rho \leq \rho'$, then any simplex with Lebesgue measure ρ' is contained in the union of $C' \frac{\rho'}{\rho^{p-1}}$ balls of diameter ρ .

Fix $0 < \delta' < \min(\delta, \alpha_1)$. By definition, there are infinitely many n so that $\mu(X_n) < e^{-\delta' C n}$. Fix such n and fix $\delta' < \alpha_4 < \alpha_1$. Let $C_4 > 0$ be as in the previous lemma, and let $C > 0$ be such that $C_4 e^{-C(\alpha_4 - \delta')}$ < 1. We are going to find a cover $\{B_i\}$ of X_n by balls of diameter at most $e^{-C n}$ satisfying

$$\sum_i \text{diam}(B_i)^{p-1-\delta'} \leq 2C',$$

showing that $\text{HD}(\liminf X_n) \leq p - 1 - \delta'$.

Let $X_n = Y_n \cup Z_n$, where Y_n is the union of those Δ^l with $|l| = n$ such that $\mu(\Delta^l) > e^{-Cn}$ and Z_n is the complement. It follows that Y_n can be covered with at most $C'\mu(Y_n)e^{(p-1)Cn}$ balls of diameter e^{-Cn} . This cover $\{B_i^Y\}$ satisfies

$$\sum_i \text{diam}(B_i^Y)^{p-1-\delta'} \leq C'\mu(X_n)e^{\delta' Cn} \leq C'.$$

Let us cover each $\Delta^l \subset Z_n$ by the smallest possible number of balls of diameter $\mu(\Delta^l)$. The resulting cover $\{B_i^Z\}$ of Z_n then satisfies

(7)

$$\begin{aligned} \sum_i \text{diam}(B_i^Z)^{p-1-\delta'} &\leq \sum_{|l|=n, \mu(\Delta^l) \leq e^{-Cn}} C'\mu(\Delta^l)^{1-\delta'} \\ &\leq \sum_{|l|=n} C'\mu(\Delta^l)^{1-\alpha_4} e^{-Cn(\alpha_4-\delta')} \leq C'C_4^n e^{-Cn(\alpha_4-\delta')} \leq C'. \end{aligned}$$

The result follows. □

The following simple result will allow us to control the set of escaping points as well.

Theorem 29. *Assume that T is fast decaying. Let $\Delta^n \subset \Delta$ be the domain of T^n , and let $\Delta^\infty = \bigcap_{n \in \mathbb{N}} \Delta^n$. Then $\text{HD}(\Delta \setminus \Delta^\infty) \leq p - 1 - \frac{\alpha_1}{1+\alpha_1}$, where α_1 is the fast decay constant of T .*

Proof. Note that $\Delta^n \setminus \Delta^{n+1} = T^{-n}(\Delta \setminus \Delta^1)$, so $\text{HD}(\Delta \setminus \Delta^\infty) = \text{HD}(\Delta \setminus \Delta^1)$.

For simplicity, let us map Δ to the interior of the cube $W = [0, 1]^{p-1}$ by a bi-Lipschitz map P . For $M \in \mathbb{N}$, let us partition W into $2^{M(p-1)}$ cubes of side $\delta = 2^{-M}$ in the natural way. Let us estimate the number N of cubes that are not contained in $P(\Delta^1)$. In order to do this, we estimate the total volume L of those cubes.

For fixed $\varepsilon > 0$, L is at most the sum L_0 of the volumes of all $P(\Delta^{(l)})$ with volume at most ε , plus the sum L_1 of the volumes of the $(\sqrt{p-1})\delta$ -neighborhood of the boundary of each $P(\Delta^{(l)})$ with volume at least ε .

By the fast decay of T , we obviously have $L_0 \leq C\varepsilon^{\alpha_1}$. On the other hand, the volume of the $(\sqrt{p-1})\delta$ -neighborhood of the boundary of each $P(\Delta^{(l)})$ is at most $C\delta$. Thus $L \leq C(\delta\varepsilon^{-1} + \varepsilon^{\alpha_1})$. Taking $\varepsilon = \delta^{\frac{1}{1+\alpha_1}}$, we get $L \leq 2C\delta^{\frac{\alpha_1}{1+\alpha_1}}$ and hence $N \leq 2C\delta^{-M+\frac{\alpha_1}{1+\alpha_1}}$. The result follows. □

7. CONSTRUCTION OF DIRECTIONS WITH NON-TRIVIAL EIGENFUNCTIONS

In this section, we provide a general construction of directional flows with non-trivial eigenfunctions in a Veech surface. This construction makes use of very particular elements in the Veech group called *Salem*. The presence of a single element will allow us to apply a somewhat more general geometric criterion for positivity of the Hausdorff dimension of certain exceptional Oseledets behavior, which we now describe in the setting of locally constant cocycles.

7.1. Lower bound on Hausdorff dimension. Let H be a finite dimensional (real or complex) vector space. We consider locally constant $\mathrm{SL}(H)$ -cocycles (T, A) where $T : \bigcup_{l \in \mathbb{Z}} \Delta^{(l)} \rightarrow \Delta$ restricts to projective maps $\Delta^{(l)} \rightarrow \Delta$ between simplices in $\mathbb{P}\mathbb{R}^p$, $p \geq 2$. We will assume that there exists some $\underline{l} \in \Omega$ such that $\Delta^{\underline{l}}$ is compactly contained in Δ , but we will not need to assume that T has bounded distortion or even that $\bigcup_{l \in \mathbb{Z}} \Delta^{(l)}$ has full measure in Δ .

Theorem 30. *Let (T, A) be a cocycle as above. Assume that for every $v \in H \setminus \{0\}$, there exists $\underline{l} \in \Omega$ such that $\|A^{\underline{l}} \cdot v\| < \|v\|$. Then there exists a finite subset $J \subset \mathbb{Z}$ such that for every $v \in H \setminus \{0\}$, there exists a compact set $K_v \subset \Delta$ with positive Hausdorff dimension such that for every $x \in K_v$ we have $T^n(x) \in \bigcup_{j \in J} \Delta^{(j)}$, $n \geq 0$, and $\limsup \frac{1}{n} \ln \|A_n(x) \cdot v\| < 0$.*

Proof. Fix two words $\underline{l}', \underline{l}'' \in \Omega$ such that $\Delta^{\underline{l}'}$ and $\Delta^{\underline{l}''}$ have disjoint closures contained in Δ . Two such words exist. Indeed, there is a word \underline{l}_0 so that $\Delta^{\underline{l}_0}$ is compactly contained in Δ . We can then take $\underline{l}' = (i, \underline{l}_0)$ and $\underline{l}'' = (j, \underline{l}_0)$ for two distinct integers i and j .

By compactness, there exists $\varepsilon > 0$ and a finite subset $F \subset \Omega$ such that for every $v \in H \setminus \{0\}$, there exists $\underline{l}(v) \in F$ such that $\|A^{\underline{l}(v)} \cdot v\| < e^{-\varepsilon} \|v\|$. Let $J \subset \mathbb{Z}$ be a finite subset containing all entries of words in F , as well as all entries of \underline{l}' and \underline{l}'' .

Let $F^n \subset \Omega$ be the subset consisting of the concatenation of n words (not necessarily distinct) in F . By induction, we see that for every $v \in H \setminus \{0\}$, there exists $\underline{l}^n(v) \in F^n$ such that $\|A^{\underline{l}^n(v)} \cdot v\| < e^{-n\varepsilon} \|v\|$ (just take $\underline{l}^1(v) = \underline{l}(v)$ and for $n \geq 2$ take $\underline{l}^n(v)$ as the concatenation of $\underline{l}(v)$ and $\underline{l}^{n-1}(A^{\underline{l}(v)} \cdot v)$).

Choose n such that $e^{-n\varepsilon} < \frac{1}{2} \max\{\|A^{\underline{l}'}\|, \|A^{\underline{l}''}\|\}$.

For $k \geq 1$ and a sequence $(t_0, \dots, t_{k-1}) \in \{0, 1\}^k$, let us define a word $\underline{l}(v, t)$ as follows. For $k = 1$, we let $\underline{l}(v, t) = \underline{l}^n(v)\underline{l}'$ if $t = (0)$ and $\underline{l}(v, t) = \underline{l}^n(v)\underline{l}''$ if $t = (1)$. For $k \geq 2$ and $t = (t_0, \dots, t_{k-1})$, denoting $\sigma(t) = (t_1, \dots, t_{k-1})$, we let $\underline{l}(v, t) = \underline{l}(v, t_0)\underline{l}(A^{\underline{l}(v, t_0)} \cdot v, \sigma(t))$.

Recall that the simplex Δ can be endowed with its Hilbert metric. It has the property that for any matrix P so that $P\Delta$ is compactly contained in Δ , the map $P : \Delta \rightarrow \Delta$ is a contraction for the Hilbert metric.

Note that the diameter of $\Delta^{\underline{l}(v, t)}$ in the Hilbert metric of Δ is exponentially small in k : indeed, the diameter of $\Delta^{\underline{l}(v, t)}$ in $\Delta^{\underline{l}(v, t_0)}$ is equal to the diameter of $\Delta^{\underline{l}(v, \sigma(t))}$ in Δ , and the Hilbert metric of $\Delta^{\underline{l}(v, (t_0))}$ is strictly stronger than the Hilbert metric of Δ . Thus given an infinite sequence $t \in \{0, 1\}^{\mathbb{N}}$, the sequence $\Delta^{\underline{l}(v, (t_0, \dots, t_{k-1}))}$ decreases to a point denoted by $\gamma_v(t) \in \Delta$. The map γ_v then provides a homeomorphism between $\{0, 1\}^{\mathbb{N}}$ and a Cantor set $K_v \subset \Delta$.

By definition, if $t \in \{0, 1\}^k$, then $\|A^{\underline{l}(v, t)} \cdot v\| < 2^{-k} \|v\|$. It thus follows that for $x \in K_v$ we have $\limsup \frac{1}{n} \ln \|A_n(x) \cdot v\| \leq -\frac{\ln 2}{M}$, where M is the maximal length of all possible words $\underline{l}(v, t)$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in \{0, 1\}$.

Let us endow $\{0, 1\}^{\mathbb{N}}$ with the usual 2-adic metric d_2 , where for $t \neq t'$ we let $d_2(t, t') = 2^{-k}$ where k is maximal such that $t_j = t'_j$ for $j < k$. With respect to this metric, $\{0, 1\}^{\mathbb{N}}$ has Hausdorff dimension 1. To conclude, it is enough to show that $\gamma_v^{-1} : K \rightarrow \{0, 1\}$ is α -Hölder for some $\alpha > 0$, as this will imply that the Hausdorff dimension of K is at least α .

Let d be the spherical metric on $\mathbb{P}H$. Let $\varepsilon_0 > 0$ be such that for every $x \in \partial\Delta$ and $y \in \bigcup_{v \in H \setminus \{0\}} \bigcup_{t \in \{0, 1\}} \Delta^{\underline{l}(v, t)}$ we have $d(x, y) > \varepsilon_0$. Such ε_0 exists since all

$\Delta^{l(v,t)}$ are contained in $\Delta^{\underline{l}'} \cup \Delta^{\underline{l}''}$ which is compactly contained in Δ . Let $\Lambda > 1$ be an upper bound on the derivative of the projective actions of any $A^{l(v,t)}$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in \{0, 1\}$. For $k \in \mathbb{N}$, and $t \in \{0, 1\}^{\mathbb{N}}$, $\gamma_v(t)$ is contained in $\Delta^{l(v,(t_0,\dots,t_k))}$ and hence at a distance at least $\varepsilon\Lambda^{-k}$ from $\partial\Delta^{l(v,(t_0,\dots,t_{k-1}))}$. It follows that if $d_2(t,t') \geq 2^{1-k}$, then $d(\gamma_v(t), \gamma_v(t')) \geq \varepsilon_0\Lambda^{-k}$. The result then follows with $\alpha = \frac{\ln 2}{\ln \Lambda}$. □

The previous result would have been enough to construct continuous eigenfunctions. In order to construct discontinuous eigenfunctions as well, we will need the following more precise result.

Theorem 31. *Let (T, A) be a cocycle as above. Assume that for every $v \in H \setminus \{0\}$, there exist $\underline{l}, \tilde{\underline{l}} \in \Omega$ such that $\|A^{\underline{l}} \cdot v\| < \|v\| < \|A^{\tilde{\underline{l}}} \cdot v\|$. Then there exists a finite subset $J \subset \mathbb{Z}$ such that for every $v \in H \setminus \{0\}$, and for every sequence $a_k \in \mathbb{R}_+$, $k \in \mathbb{N}$, such that $\sup_k |\ln a_k - \ln a_{k+1}| < \infty$, there exists a compact set $K_v \subset \Delta$ with positive Hausdorff dimension such that for every $x \in K_v$ we have $T^n(x) \in \bigcup_{l \in J} \Delta^{(l)}$, $n \geq 0$, and there exists a strictly increasing subsequence m_k , $k \in \mathbb{N}$, such that $\sup_k m_{k+1} - m_k < \infty$ and $\sup_k |\ln \|A_{m_k}(x) \cdot v\| - \ln a_k| < \infty$.*

Proof. Fix two words $\underline{l}', \underline{l}'' \in \Omega$ such that $\Delta^{\underline{l}'}$ and $\Delta^{\underline{l}''}$ have disjoint closures contained in Δ .

Let C_0 be an upper bound for $|\ln a_j - \ln a_{j+1}|$.

As in the proof of the previous theorem, define a finite set $F \subset \Omega$ such that for every $v \in \mathbb{R}^d \setminus \{0\}$, there exist $\underline{l}^c(v), \underline{l}^e(v) \in F$, such that

$$\max_{\underline{l} \in \{\underline{l}', \underline{l}''\}} \|A^{\underline{l}^c(v)\underline{l}} \cdot v\| < e^{-C_0} \|v\|,$$

$$\min_{\underline{l} \in \{\underline{l}', \underline{l}''\}} \|A^{\underline{l}^e(v)\underline{l}} \cdot v\| > e^{C_0} \|v\|.$$

Given $k \geq 1$ and a sequence $t = (t_0, \dots, t_{k-1}) \in \{0, 1\}^k$, define $\underline{l}(v, t)$ by induction as follows. If $k = 1$, then we let $\underline{l}(v, t) = \underline{l}^a \underline{l}^b$ where $\underline{l}^a = \underline{l}^c(v)$ if $\|v\| > a_0$, $\underline{l}^a = \underline{l}^e(v)$ if $\|v\| \leq a_0$, $\underline{l}^b = \underline{l}'$ if $t = 0$, and $\underline{l}^b = \underline{l}''$ if $t = 1$. If $k \geq 2$, we let $\underline{l}(v, t) = \underline{l}(v, t_0) \underline{l}(A^{l(v,t_0)} \cdot v, \sigma(t))$, where $\sigma(t_0, \dots, t_{k-1}) = (t_1, \dots, t_{k-1})$.

Notice that the set $G \subset \Omega$ of possible words $\underline{l}(v, t)$ with $v \in \mathbb{R}^d \setminus \{0\}$ and $t \in \{0, 1\}$ is finite.

By induction, we get $|\ln \|A^{\underline{l}(v,t)} \cdot v\| - \ln a_k| \leq |\ln \|v\| - \ln a_0| + C_1$, where

$$C_1 = \max_{\underline{l} \in G} \{\ln \|A^{\underline{l}}\|, \ln \|(A^{\underline{l}})^{-1}\|\}.$$

As in the proof of the previous theorem, we define $\gamma_v : \{0, 1\}^{\mathbb{N}} \rightarrow \Delta$ so that $\gamma_v(t)$ is the intersection of the $\Delta^{\underline{l}(v,(t_0,\dots,t_{k-1}))}$ and conclude that $K_v = \gamma_v(\{0, 1\}^{\mathbb{N}})$ is a Cantor set of positive Hausdorff dimension.

By construction, if $x = \gamma_v(t)$, then for every $n \in \mathbb{N}$ we have $A_n(x) \in \Delta^{(j)}$ for some entry j of some word in G . Moreover, $|\ln \|A_{m_k}(x) \cdot v\| - \ln a_k| \leq |\ln \|v\| - \ln a_0| + C_1$ where m_k is the length of $\underline{l}(v, (t_0, \dots, t_{k-1}))$. In particular, m_k is strictly increasing and $m_{k+1} - m_k$ is bounded by the maximal length of the words in G . □

7.2. Salem elements and eigenfunctions. A real number λ is a *Salem* number if it is an algebraic integer greater than 1, all its conjugates have absolute values not greater than 1, and at least one has absolute value 1. These conditions imply that the minimal polynomial of a Salem number is reciprocal and that all conjugates have modulus one except λ and $1/\lambda$. For $M \in \mathrm{SL}(2, \mathbb{R})$, we say that M is a *Salem* matrix if its dominant eigenvalue is a Salem number.

Let (S, Σ, ω) be a Veech surface, Γ its Veech group, and k the trace field of Γ . We recall that the action of the Veech group on the tautological subspace $V = \mathbb{R} \operatorname{Re}(\omega) \oplus \mathbb{R} \operatorname{Im}(\omega)$ is naturally identified with the Veech group (see Section 2.2). For each $\sigma \in \operatorname{Gal}(k/\mathbb{Q})$ there is a well defined conjugate V^σ of V which is preserved by the affine group of (S, Σ, ω) . These actions identify to conjugates of the Veech group (see Section 2.4). Salem elements in the Veech group have an alternative definition: an element of a Veech group is Salem if and only if it is direct hyperbolic⁸ and its Galois conjugates are elliptic.

Theorem 32. *Let (S, Σ, ω) be a non-arithmetic Veech surface and assume that its Veech group contains a Salem element. Then*

- (1) *the set of angles whose directional flow has a continuous eigenfunction has a positive Hausdorff dimension;*
- (2) *the set of angles whose directional flow has a measurable discontinuous eigenfunction has a positive Hausdorff dimension.*

To build directions with eigenvalues, we use a criterion proved in [BDM1] which is a partial converse of the Veech criterion (see Section 3). An earlier version of this criterion appears in the paper of Veech [Ve84]. The criterion of [BDM1] concerns only *linearly recurrent systems*: the translation flow of (S, ω) is linearly recurrent if there exists a constant K such that for any horizontal interval J embedded in (S, ω) the maximum return time to J is bounded by $K/|J|$. Equivalently, a translation surface is linearly recurrent if and only if the associated forward Teichmüller geodesic is bounded in the moduli space of translation surfaces. For these equivalences we refer to the so-called *Vorobets's identity* in the paper of Marchese-Hubert-Ulcigrai [MHU].

Theorem 33 ([BDM1]). *Let U be a relatively compact open subset in the moduli space $\mathcal{M}_g(\kappa)$ in which the Hodge bundle admits a trivialization and let A_n be the associated Kontsevich-Zorich cocycle. Let $(S, \Sigma, \omega) \in U$ be such that the return times to U have bounded gaps, and then*

- (1) *ν is a continuous eigenvalue of (S, ω) if and only if there exists an integer vector $v \in H^1(S; \mathbb{Z}) \setminus \{0\}$ such that*

$$\sum_{n \geq 0} \|A_n(\omega) \cdot (\nu \operatorname{Im}(\omega) - v)\| < \infty.$$

⁸An hyperbolic matrix in $\mathrm{SL}(2, \mathbb{R})$ can either have a positive or negative dominant eigenvalue. The former case is called *direct hyperbolic*.

- (2) ν is an L^2 eigenvalue of (S, Σ, ω) if and only if there exists an integer vector $v \in H^1(S; \mathbb{Z}) \setminus \{0\}$ such that

$$\sum_{n \geq 0} \|A_n(\omega) \cdot (\nu \operatorname{Im}(\omega) - v)\|^2 < \infty.$$

Actually, the criterion applies to the Cantor space obtained from the translation surface where each point that belongs to a singular leaf is doubled. The continuity in that space is weaker than the continuity on the surface. For a continuous eigenfunction f , the cohomological equation $f(\phi_T(x)) = e^{2i\pi\nu T} f(x)$ allows to recover the value along a leaf from the value at one point on that leaf. But singular leaves either coincide in the past or in the future. Hence the values of the eigenfunctions on the doubled leaves must be identical. In other words, the function is well defined on the quotient which is the surface.

Remark 7.1. Theorem 33 allows one to strengthen the conclusion of Theorem 23 when the flow is linearly recurrent: if (S, Σ, ω) is a non-arithmetic Veech surface with trace field k and whose flow is linearly recurrent, then it admits either 0 or $[k : \mathbb{Q}]$ rationally independent eigenvalues. Moreover, they are simultaneously continuous or discontinuous. Indeed (following the proof of Theorem 23), any two non-zero “potential eigenvalues” $\mu, \mu' \in \mathbb{R}$ such that there exists $v, v' \in W_{\mathbb{Z}}$ with $\mu \operatorname{Im}(\omega) - v$ and $\mu' \operatorname{Im}(\omega') - v'$ belonging to E^s are such that $\frac{\|A_n(\omega) \cdot (\mu \operatorname{Im}(\omega) - v)\|}{\|A_n(\omega) \cdot (\mu' \operatorname{Im}(\omega) - v')\|}$ is uniformly bounded away from zero or infinity (independently of n). Indeed, writing $v = \sum v_\sigma$ each component v_σ belongs to $E^s(V^\sigma)$ and is non-zero. One concludes using the fact that $E^s(V^\sigma)$ is one dimensional.

By Theorem 33, μ is a continuous eigenvalue if and only if μ' is, and μ is an L^2 eigenvalue if and only if μ' is. Since the set Θ of potential eigenvalues either is $\{0\}$ or has dimension $[k : \mathbb{Q}]$ over \mathbb{Q} , the result follows.

Before going into details of the proof, we provide various examples of Veech surfaces which contain Salem elements. In particular, the next result shows that Theorem 3 follows from Theorem 32.

Proposition 34 ([BBH], Proposition 1.7). *A Veech surface with a quadratic trace field has a Salem element in its Veech group.*

Proof. We follow [BBH]. The Veech group has only one conjugate, and this conjugate is non-discrete (see Proposition 14). On the other hand, we know from a result of Beardon ([Be83], Theorem 8.4.1) that any non-discrete subgroup of $\operatorname{SL}(2, \mathbb{R})$ contains an elliptic element with an irrational angle. If g is an element of the Veech group whose conjugate is an irrational rotation, then g cannot be elliptic as it is of infinite order and the Veech group is discrete, and g cannot be parabolic because a conjugate of a parabolic element is again parabolic. Hence g is hyperbolic and g^2 is a Salem element of the Veech group. □

For more general Veech surfaces, we obtain examples through computational experiments (see Appendix A for explicit matrices).

Proposition 35. *For respectively odd $q \leq 15$ and any $q \leq 15$ the triangle groups $\Delta(2, q, \infty)$ and $\Delta(q, \infty, \infty)$ contain Salem elements.*

In particular, Theorem 32 holds for many billiards in regular polygons P_n defined in the Introduction.

Now, we proceed to the proof of Theorem 32.

Lemma 36. *Let λ be a Salem number and $\{\lambda, 1/\lambda, e^{i\alpha_1}, e^{-i\alpha_1}, \dots, e^{i\alpha_k}, e^{-i\alpha_k}\}$ its Galois conjugates, and then $\alpha_1, \dots, \alpha_k, \pi$ are rationally independent.*

Proof. Let n_1, \dots, n_k, m be an integer such that

$$(8) \quad n_1\alpha_1 + \dots + n_k\alpha_k = 2m\pi.$$

Then

$$(9) \quad (e^{i\alpha_1})^{n_1} \dots (e^{i\alpha_k})^{n_k} = 1.$$

Each element of the Galois group is a field homomorphism, and hence it preserves the partition $\{\lambda, 1/\lambda\}, \{e^{i\alpha_1}, e^{-i\alpha_1}\}, \dots, \{e^{i\alpha_k}, e^{-i\alpha_k}\}$. By definition, the Galois group acts transitively on $\{\lambda, 1/\lambda, e^{i\alpha_1}, e^{-i\alpha_1}, \dots, e^{i\alpha_k}, e^{-i\alpha_k}\}$, and for each $i = 1, \dots, k$ there exists a field homomorphism that maps $e^{i\alpha_j}$ to λ and all other $e^{i\alpha_{j'}}$ to some $e^{\pm i\alpha_{j''}}$. By applying this field homomorphism to the equality (9) and taking the absolute value we get that $n_i = 0$ because $|\lambda| > 1$. Hence the relation (8) is trivial. \square

We now show that the presence of Salem elements allows us to verify the hypothesis of Theorem 31.

Lemma 37. *Let (S, Σ, ω) be a Veech surface, k its holonomy field, $V = \mathbb{R} \operatorname{Re} \omega \oplus \operatorname{Im} \omega$ the tautological subspace, and $W^0 = \bigoplus_{\sigma \in \operatorname{Gal}(k/\mathbb{Q}) \setminus \{\operatorname{id}\}} V^\sigma$. Let γ be a Salem element of the Veech group and $\gamma_j, j \geq 1$, be such that for each $\sigma \neq \operatorname{id}$, the norm of the conjugates γ_j^σ grows to infinity. Denote by g and g_k their actions on W^0 . Then for any $v \in W^0 \setminus \{0\}$ there exist positive integers n_-, n_+ and k_-, k_+ such that elements $g_- = g_{k_-} g^{n_-}$ and $g_+ = g_{k_+} g^{n_+}$ satisfy $\|g_- v\| < \|v\| < \|g_+ v\|$.*

Proof. Let $v \in W^0$ be a unit vector, and for $\sigma \neq \operatorname{id}$, let $\pi_\sigma : W^0 \rightarrow V^\sigma$ be the projection on the V^σ -coordinate. Let $D \subset \operatorname{Gal}(k/\mathbb{Q}) \setminus \{\operatorname{id}\}$ be the set of all σ such that $\pi_\sigma(v) \neq 0$. We show that there exists positive integers n and k such that for all $\sigma \in D$, we have $\|g_k g^n \pi_\sigma(v)\| < \frac{\|v\|}{\#D}$, which implies the first inequality with $n = n_-$ and $k = k_-$. The other inequality may be obtained by the very same argument.

Let θ_k^σ be the norm of γ_k^σ . By hypothesis, $\theta_k^\sigma > 1$ for every σ and every k sufficiently large. Let $F_-^\sigma \in \mathbb{P}V^\sigma$ be the most contracted direction of γ_k^σ in V^σ .

Consider the action of g on the torus $\prod_{\sigma \in D} \mathbb{P}V^\sigma$. By Lemma 36, this action is minimal. In particular, there exists a sequence of non-negative integer n_j such that $g^{n_j} \pi_\sigma(v)$ converges to a vector w_σ in $F_-^\sigma \setminus \{0\}$ for every $\sigma \in D$. In particular, for fixed $k \in \mathbb{N}$ we have $\lim_{j \rightarrow \infty} \|g_k g^{n_j} \pi_\sigma(v)\| = \frac{\|w_\sigma\|}{\theta_k^\sigma}$ for every $\sigma \in D$. The result follows by taking k such that $\frac{\|w_\sigma\|}{\theta_k^\sigma} < \frac{\|v\|}{\#D}$ (recall that $\theta_k^\sigma \rightarrow \infty$ as $k \rightarrow \infty$). \square

Proof of Theorem 32. Let $V = \mathbb{R} \operatorname{Re}(\omega) \oplus \mathbb{R} \operatorname{Im}(\omega)$ be the tautological subspace of the cohomology $H^1(S; \mathbb{R})$, $W = \bigoplus_{\sigma: k \rightarrow \mathbb{R}} V^\sigma$, and $W^0 = \bigoplus_{\sigma \neq \operatorname{id}} V^\sigma$. We are going to construct a Markov model $(T : \Delta \rightarrow \Delta, A)$ for the Kontsevich-Zorich over the $\operatorname{SL}(2, \mathbb{R})$ orbit \mathcal{C} of (S, Σ, ω) , a point $x \in \Delta$ and a positive integer n , and a sequence

of points $y_k \in \Delta$ and positive integers n_k , such that $A_n(x)|V$ is a Salem element and $\lim_{k \rightarrow \infty} \inf_{\sigma \neq \text{id}} \|A_{n_k}|V^\sigma\| = \infty$. First, let us show how this construction implies the result.

By Lemma 37, this implies that the hypotheses of Theorem 31 are satisfied for the cocycle $(T, A|W^0)$, so for every $w \in W^0 \setminus \{0\}$, there exist subsets $Z_c, Z_m \subset \Lambda$ of positive Hausdorff dimension such that for $u \in Z_c$ we have $\sum \|A_n(u) \cdot w\| < \infty$ and for $u \in Z_m$ we have $\sum \|A_n(u) \cdot w\|^2 < \infty$ and $\sum \|A_n(u) \cdot w\| = \infty$. Moreover, since Theorem 31 provides also that $T^n(u)$ visits only finitely many distinct $\Delta^{(l)}$, the return times $r(T^n(u))$ remain bounded so that the forward Teichmüller geodesic starting at u is bounded in moduli space.

Let us take w as the projection on W^0 , along V , of a non-zero vector $v \in W \cap H^1(S; \mathbb{Z})$. Fix $u \in Z_c \cup Z_m$. Let $\omega = p(u, 0)$, and write $v = w + \nu \text{Im } \omega + \eta \text{Re } \omega$. Then $A_n(u) \cdot (\nu \text{Im } \omega - v) = -A_n(u) \cdot w - \eta A_n(u) \cdot \text{Re } \omega$. Note that $A_n(u) \cdot \text{Re } \omega$ decays exponentially fast, since $\text{Re } \omega$ is in the direction of the strongest contracting subbundle of the Kontsevich-Zorich cocycle. Notice that $\nu \neq 0$; otherwise the integer non-zero vectors $A_n(u) \cdot v$ would converge to 0. By Theorem 33, if $u \in Z_c$, then the vertical flow for ω admits a continuous eigenfunction with eigenvalue ν , and if $u \in Z_m$, then the vertical flow for ω admits a measurable eigenfunction, but no continuous eigenfunction, with eigenvalue ν .

We have thus obtained positive Hausdorff dimension subsets $p(Z_c \times \{0\})$ and $p(Z_m \times \{0\})$ of an unstable horocycle for which the vertical flow has continuous and measurable but discontinuous eigenfunctions. Using Lemma 24, we transfer the result to the directional flow in any surface in \mathcal{C} , giving the desired conclusion.

We now proceed with the construction of the Markov model. Recall that for each hyperbolic element γ in the Veech group, there exists a periodic orbit O_σ of the Teichmüller flow in the $\text{SL}(2, \mathbb{R})$ orbit and a positive integer n_γ , such that the restriction to V of the n_γ th iterate of the monodromy of the Kontsevich-Zorich cocycle along this periodic orbit is conjugate to γ .

Let γ be a Salem element in the Veech group, and let us consider the Markov model (T, A) for the Kontsevich-Zorich cocycle obtained by taking a small Poincaré section Q through some $x \in O_\gamma$. Then clearly $A_{n_\gamma}(x)|V$ is a Salem element.

On the other hand, by Lemma 13, $(T, A|V^\sigma)$ has a positive Lyapunov exponent for every σ . Thus for large n and for a set of y of probability close to 1, the norm of $\|A_n(y)|V^\sigma\|$ is large. In particular, for each $k \in \mathbb{N}$ there exists y_k and a positive integer n_k such that $\|A_{n_k}|V^\sigma\| > k$ for every $\sigma \neq \text{id}$. The result follows. □

APPENDIX A. SALEM ELEMENTS IN TRIANGLE GROUPS

In this appendix we provide explicit Salem matrices in triangle groups $\Delta(2, q, \infty)$ and $\Delta(q, \infty, \infty)$ for a trace field of degree greater than two and small values of q . The matrices are given in terms of the standard generators s, t of the triangle group $\Delta(p, q, r)$ that satisfy

$$s^p = t^q = (st)^r = \pm \text{id}.$$

Instead of writing down the minimal polynomial of the eigenvalue, we write it for half the trace. The roots of modulus less than one are the cosine of the angles of the corresponding elliptic matrices.

The array stops at the values $q = 17$ for $\Delta(2, q, \infty)$ and $q = 16$ for $\Delta(q, \infty, \infty)$ for which we were unable to find Salem elements. All these examples were obtained using the mathematical software Sage [Sa].

A.1. Salem elements in $\Delta(2, q, \infty)$.

q	degree	matrix m
		minimal polynomial of $\text{trace}(m)/2$ approximate conjugates of $\text{trace}(m)/2$
7	3	$t^3.s$
		$x^3 - 2x^2 - x + 1$ 2.247, 0.5550, -0.8019
9	3	$t^4.s$
		$x^3 - 3x^2 + 1$ 2.879, 0.6527, -0.5321
11	5	$t^5.s.t^4.s$
		$x^5 - \frac{39}{2}x^4 - 47x^3 - \frac{243}{8}x^2 - \frac{17}{16}x + \frac{89}{32}$ 21.73, 0.2425, -0.6156, -0.8781, -0.9764
13	6	$t^7.s.t^7.s.t^4.s$
		$x^6 - 227x^5 - 11x^4 + 318x^3 + 41x^2 - 110x - 25$ 227.0, 0.9072, 0.8412, -0.2464, -0.6697, -0.8746
15	4	$t^7.s$
		$x^4 - 4x^3 - 4x^2 + x + 1$ 4.783, 0.5112, -0.5473, -0.7472

A.2. Salem elements in $\Delta(q, \infty, \infty)$.

q	degree	matrix m
		minimal polynomial of $\text{trace}(m)/2$ approximate conjugates of $\text{trace}(m)/2$
7	3	$t.s^3$
		$x^3 - 3x^2 - 4x - 1$ 4.049, -0.3569, -0.6920
8	4	$t.s^2.t.s^3$
		$x^4 - 24x^3 + 15x^2 + 4x + \frac{1}{8}$ 23.35, 0.8571, -0.03655, -0.1709
9	3	$t.s^2$
		$x^3 - 3x^2 + 1$ 2.879, 0.6527, -0.5321
10	4	$t.s^3.t.s^7$
		$x^4 - 49x^3 - \frac{441}{4}x^2 - \frac{291}{4}x - \frac{199}{16}$ 51.18, -0.2644, -0.9504, -0.9672
11	5	$t.s^4.t.s^7$
		$x^5 - \frac{155}{2}x^4 - 122x^3 - \frac{459}{8}x^2 - \frac{173}{16}x - \frac{23}{32}$ 79.05, -0.1907, -0.2214, -0.2388, -0.9015
12	4	$t.s^2.t.s^3$
		$x^4 - 24x^3 - 61x^2 - 48x - \frac{191}{16}$ 26.38, -0.5254, -0.9096, -0.9468
13	6	$t.s^4.t.s^5.t^{-1}.s^4.t^{-1}.s^5$
		$x^6 - \frac{43107}{2}x^5 - \frac{188297}{4}x^4 - 26514x^3 + \frac{53979}{8}x^2 + \frac{304515}{32}x + \frac{124175}{64}$ 21560., 0.5373, -0.3375, -0.7022, -0.8374, -0.8440
14	6	$t.s^5.t.s^9$
		$x^6 - 125x^5 - \frac{955}{4}x^4 - \frac{45}{4}x^3 + \frac{1653}{8}x^2 + \frac{967}{8}x + \frac{1009}{64}$ 126.9, 0.9692, -0.1912, -0.6930, -0.9794, -0.9879
15	4	$t.s^3$
		$x^4 - 4x^3 - 4x^2 + x + 1$ 4.783, 0.5112, -0.5473, -0.7472

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