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WEAK NULL SINGULARITIES IN GENERAL RELATIVITY

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1. Introduction

In this paper we study the existence and stability of weak null singularities in general relativity without symmetry assumptions. More precisely, a weak null singularity is a singular null boundary of a spacetime $(\mathcal{M}, g)$ solving the Einstein equations

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$$

such that the Christoffel symbols blow up and are not square integrable while the metric is continuous up to the boundary. This can be interpreted as a *terminal* singularity of the spacetime as it cannot be made sense of as a weak solution[1] to the Einstein equations along the singular boundary. While the singularity is sufficiently strong to be terminal, it is at the same time sufficiently weak such that the metric in an appropriate coordinate system is continuous up to the boundary.

The study of weak null singularities began with the attempts to understand the (in)stability of the Cauchy horizon in the black hole interior of Reissner–Nordström spacetimes. Reissner–Nordström spacetimes are the unique two-parameter family of asymptotically flat (with two ends), spherically symmetric, static solutions to the Einstein–Maxwell equations. Their Penrose diagrams[2] are given by Figure 1. As seen in the Penrose diagram, the Reissner–Nordström solution possesses a smooth Cauchy horizon $\mathcal{CH}^+$ in the interior of the black hole such that the spacetime can be extended nonuniquely as a smooth solution to the Einstein–Maxwell system. This feature is also shared[3] by the Kerr family of solutions to the vacuum Einstein equations, which can also be depicted by a Penrose diagram given by Figure 1. According to the strong cosmic censorship conjecture (see section 1.1 below), the Reissner–Nordström and Kerr spacetimes are expected to be nongeneric and the smooth Cauchy horizons are expected to be unstable.

In a seminal work Dafermos [7, 8] showed that for a spacetime solution to the spherically symmetric Einstein–Maxwell–real scalar field system, if an appropriate upper and lower bound for the scalar field is *assumed* on the event horizon, then in

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[1]One can define a weak solution to the Einstein equations by requiring $\int_{\mathcal{M}} \mathrm{Ric}(X,Y) - \frac{1}{2} R g(X,Y) - T(X,Y) d\mathrm{Vol} = 0$ in the weak sense for all compactly supported smooth vector fields X and Y. After integration by parts, the minimal regularity required for the spacetime for this to be defined is that the Christoffel symbols are square integrable; see the discussion in [5, p.13].

[2]for $0 < |Q| < M$

[3]for $0 < |a| < M$

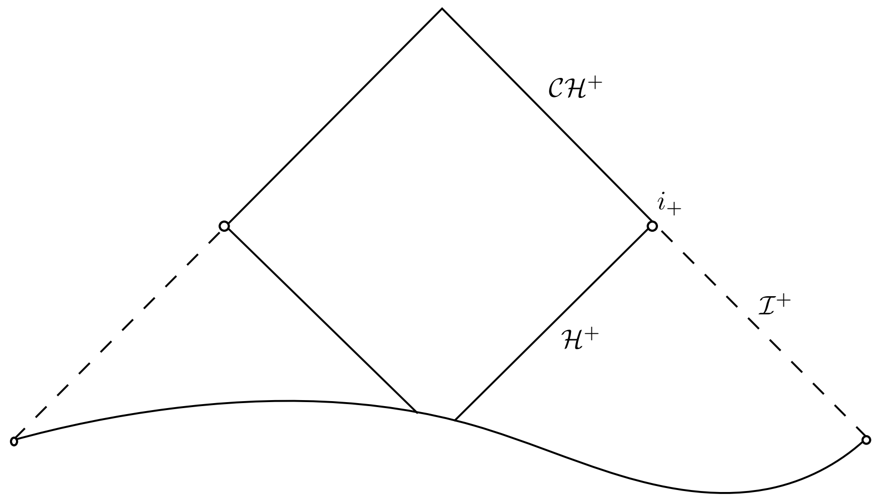

FIGURE 1. The Penrose diagram of Reissner–Nordström spacetimes

a neighborhood of timelike infinity, the black hole terminates in a weak null singularity. The necessary upper bound was shown to hold for nonextremal black hole spacetimes arising from asymptotically flat initial data by Dafermos and Rodnianski [10]. In particular this implies that near timelike infinity, the terminal boundary of the Cauchy development does not contain a spacelike portion.

In a more recent work [9], Dafermos showed that if, in addition to assuming the two black hole exterior regions settle to Reissner–Nordström with appropriate rates, the initial data are moreover globally close to that of Reissner–Nordström, then the maximal Cauchy development of the data possesses the same Penrose diagram as Reissner–Nordström. In particular the spacetime terminates in a global bifurcate weak null singularity and the singular boundary does not contain any spacelike portion.

The works [7–9] were in part motivated by the physics literature on the instability of Cauchy horizons, weak null singularities and the strong cosmic censorship conjecture. It will be discussed below in section 1.1.

While the works of Dafermos [7–9] are restricted to the class of spherically symmetric spacetimes, they nonetheless suggest the genericity of weak null singularities in the black hole interior, at least "in a neighborhood of timelike infinity". In particular they motivate the following conjecture for the vacuum Einstein equations,

$$\mathrm{Ric}_{\mu\nu} = 0. \tag{1}$$

Conjecture 1.

(1) *Consider the characteristic initial value problem with smooth characteristic initial data on a pair of null hypersurfaces H_0 and $\underline{H}_0$ intersecting on a 2-sphere. Suppose that H_0 is an affine complete null hypersurface on which the data approach that of the event horizon of a Kerr solution (with $0 < |a| < M$) at a sufficiently fast polynomial rate.*[4] *Then the development*

[4]In particular this applies if an asymptotically flat spacetime has an exterior region which approaches a subextremal Kerr solution at a sufficiently fast polynomial rate. This also holds

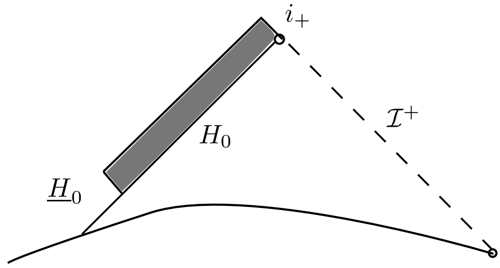

FIGURE 2. Region of existence in Conjecture 1

$(\mathcal{M}, g)$ of the initial data possesses a null boundary "emanating from timelike infinity i_+" through which the spacetime is extendible with a continuous metric (see shaded region in Figure 2). Moreover, given an appropriate "lower bound" on the H_0, this piece of null boundary is generically a weak null singularity with nonsquare-integrable Christoffel symbols.

(2) *(Ori, see discussion in* [9]*) If the data for $(\mathcal{M}, g)$ on a complete two-ended asymptotically flat Cauchy hypersurface are globally a small perturbation of two-ended Kerr initial data (with $0 < |a| < M$), then the maximal Cauchy development possesses a global bifurcate future null boundary $\partial\mathcal{M}$. Moreover, for generic such perturbations of Kerr, $\partial\mathcal{M}$ is a global bifurcate weak null singularity which intersects every futurely causally incomplete geodesic.*

If Conjecture 1 is true, then in particular there exist *local stable* weak null singularities for the *vacuum* Einstein equations *without symmetry assumptions*. We show in this paper that there is in fact a large class of such singularities, parameterized by singular initial data. More specifically, we solve a characteristic initial value problem with singular initial data and construct a class of stable bifurcate weak null singularities.

To motivate the strength of the singularity considered in this paper, we first recall the strength of the spherically symmetric weak null singularities in a neighborhood of Reissner–Nordström studied in [8]. The instability of the Reissner–Nordström Cauchy horizon is in fact already suggested by a linear analysis (see [4,20,23]). For a spherically symmetric solution to the linear wave equation which has a polynomially decaying (in the Eddington–Finkelstein coordinates) tail[5] along the event horizon, there is a singularity in a (C^0)-regular coordinate system near the Cauchy horizon of the strength[6]

$$|\partial_{\underline{u}}\phi| \sim (\underline{u}_* - \underline{u})^{-1}\log^{-p}\left(\frac{1}{\underline{u}_* - \underline{u}}\right), \tag{2}$$

for some $p > 1$ as $\underline{u} \to \underline{u}_*$. In particular along an outgoing null curve, $\partial_{\underline{u}}\phi$ is integrable but not L^q-integrable for any $q > 1$. In the spacetimes constructed by

in the case where the Cauchy hypersurface has only one asymptotically flat end. In that case, numerical work in spherical symmetry [13] suggests that the singular boundary may also contain a nonempty spacelike portion, in addition to the null portion.

[5]with upper and lower bounds

[6]This statement regarding the *linear* wave equation can be inferred using the methods in [7] for the nonlinear coupled Einstein–Maxwell–scalar field system.

Dafermos [7, 8], it was shown moreover that even in the nonlinear setting, $\partial_{\underline{u}}\phi$ is also singular but remains integrable. A more precise analysis will show that in fact the spherically symmetric scalar field in the nonlinear setting of [8] also blows up at a rate given by (2).

Returning to the problem of constructing stable weak null singularities in vacuum, our construction is based on solving a characteristic initial value problem with singular data. We will in fact construct spacetimes not only with one weak null singularity, but instead they will contain two weak null singularities terminating at a bifurcate sphere. More precisely, the data on the initial characteristic hypersurface H_0 (resp. $\underline{H}_0$) is determined by the traceless part of the null second fundamental form $\hat{\chi}$ (resp. $\underline{\hat{\chi}}$). We consider singular initial data satisfying in particular

$$|\hat{\chi}| \sim (\underline{u}_* - \underline{u})^{-1}\log^{-p}\left(\frac{1}{\underline{u}_* - \underline{u}}\right), \quad \text{for some } p > 1,$$

and

$$|\underline{\hat{\chi}}| \sim (u_* - u)^{-1}\log^{-p}\left(\frac{1}{u_* - u}\right), \quad \text{for some } p > 1.$$

This singularity is consistent with the strength of the weak null singularities in (2).

The following is a first version of the main result of this paper (see Figure 3). We refer the readers to the statement of Theorems 2, 3 and 4 for a more precise formulation of the theorem.

Theorem 1 (Main theorem, first version)**.** *For a class of singular characteristic initial data without any symmetry assumptions for the vacuum Einstein equations*

$$\mathrm{Ric}_{\mu\nu} = 0$$

with the singular profile as above (see precise requirements on the data in section 1.3*) and for ϵ sufficiently small and u_*, $\underline{u}_* \leq \epsilon$, there exists a unique smooth spacetime $(\mathcal{M}, g)$ endowed with a double null foliation $(u, \underline{u})$ in $0 \leq u < u_*$, $0 \leq \underline{u} < \underline{u}_*$, which satisfies the vacuum Einstein equations with the given data. Associated to $(\mathcal{M}, g)$, there exists a coordinate system $(u, \underline{u}, \theta^1, \theta^2)$ such that the metric extends continuously to the boundary but the Christoffel symbols are not in L^2.*

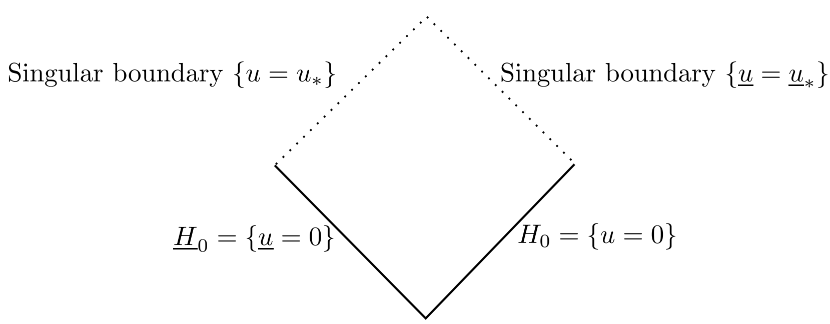

FIGURE 3. Region of existence in Theorem 1

Remark 1. This class of stable local weak null singularities that we construct in particular provides the first construction of weak null singularities of such strength for the vacuum Einstein equations.[7]

Theorem 1 allows singularities on both initial null hypersurface and is valid in the region where u_* and $\underline{u}_*$ are sufficiently small. In the context of the interior of black holes, this corresponds to the darker shaded region in Figure 4. The existence theorem clearly implies an existence result when the data are only singular on one of the initial null hypersurfaces. In that context, we can in fact combine the methods in this paper with that in [17] to show that the domain of existence can be extended so that only one of the characteristic length scales is required to be small. More precisely, we allow that data on H_0 such that

$$|\hat{\chi}| \sim (\underline{u}_* - \underline{u})^{-1}\log^{-p}\left(\frac{1}{\underline{u}_* - \underline{u}}\right), \quad \text{for some } p > 1,$$

on $0 \leq \underline{u} < \underline{u}_* \leq C$ and the data on $\underline{H}_0$ are smooth on $0 \leq u \leq u_* \leq \epsilon$. Then for ϵ sufficiently small, the spacetime $(\mathcal{M}, g)$ remains smooth in $0 \leq u < u_*$, $0 \leq \underline{u} < \underline{u}_*$ (see for example the lightly shaded region in Figure 4). We will omit the details of the proof of this result.

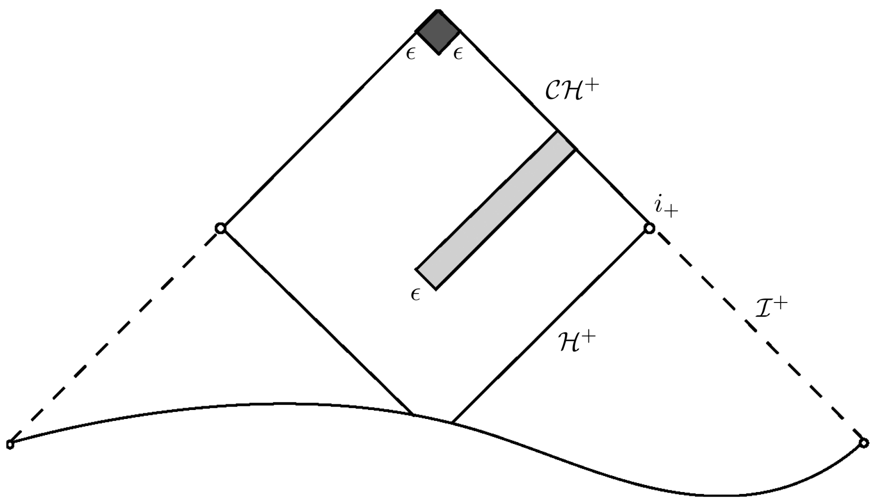

FIGURE 4. Domains of existence

Theorem 1, which proves the existence and stability of the conjecturally generic weak null singularities, can be viewed as a first step toward Conjecture 1. A next step is an analogue of [8] for the vacuum Einstein equations without symmetry assumptions, i.e., to solve the characteristic initial value problem inside the black hole with data prescribed on the event horizon that is approaching Kerr at appropriate rates. This requires an understanding of the *formation* of weak null singularities

[7]We recall Birkhoff's theorem which states that the only spherically symmetric vacuum spacetimes are the Minkowski and Schwarzschild solutions. Thus to construct stable examples of weak null singularities in vacuum, one necessarily works outside the class of spherically symmetric spacetimes.

from smooth data on the event horizon (see part (1) of Conjecture 1). A full resolution of Conjecture 1, part (2), however, requires in addition an understanding of the decay rates of gravitational radiation along the event horizon for generic perturbations of Kerr spacetime. This latter problem is intimately tied to the problem of the nonlinear stability of Kerr spacetimes, which continues to be one of the most important and challenging open problems in mathematical general relativity. Nevertheless, significant progress has been made for the corresponding *linear* problem in the past decade. We refer the readers to the survey of Dafermos and Rodnianski [11] for more about this linear problem.

The approach for the main theorem applies equally well to the Einstein–Maxwell–scalar field system without symmetry assumptions.[8] Thus, we show that the weak null singularity of Dafermos [8], which arises from appropriately decaying data on the event horizon, is stable against *nonspherically symmetric* perturbations on the hypersurface Σ sufficiently far within the black hole region (see Figure 5).

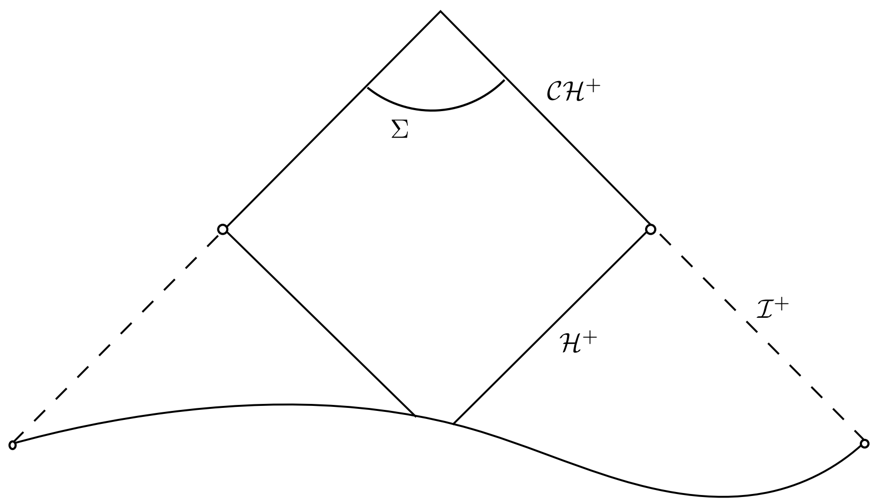

FIGURE 5. Perturbations in the black hole interior of Dafermos spacetimes

1.1. **Weak null singularities and strong cosmic censorship conjecture.** The study of weak null singularities can be viewed in the larger context of Penrose's celebrated strong cosmic censorship conjecture in general relativity. The conjecture states that for *generic* asymptotically flat initial data for "reasonable" Einstein-matter systems, the maximal Cauchy development is future inextendible as a suitably regular Lorentzian manifold. This would guarantee general relativity to be a deterministic theory.

As pointed out above, the Kerr and Reissner–Nordström families of solutions (of the Einstein vacuum and Einstein–Maxwell equations, respectively) have maximal Cauchy developments that are extendible as larger smooth spacetimes unless the

[8]This can be easily seen by decomposing the Maxwell field and the gradient of the scalar field in terms of the null frame below. The components in this decomposition obey equations that can be put in the same schematic form as in section 2.4. Therefore, the Maxwell field and the scalar field and their derivatives satisfy estimates similar to those for the Ricci coefficients and curvature components.

angular momentum or the charge vanishes. This is connected with the existence of a smooth Cauchy horizon in the black hole interior such that the spacetime can be extended beyond as a smooth solution. According to the strong cosmic censorship conjecture, this is expected to be nongeneric.

On the other hand, the situation for the Schwarzschild spacetime is more preferable from the point of view of the deterministic nature of the theory. The maximal development of the Schwarzschild spacetime terminates with a spacelike singularity at which the Hawking mass and the curvature scalar invariants blow up. In particular the spacetime cannot be extended in C^2.

The early motivation for the strong cosmic censorship conjecture, besides the desirability of a deterministic theory, is a linear heuristic argument by Penrose [23] suggesting that the Reissner–Nordström Cauchy horizon is unstable. This was also confirmed by the numerical work by Simpson and Penrose [27]. It is thus conjectured that a small global perturbation would lead to a singularity in the interior of the black hole in such a way that the maximal Cauchy development is future inextendible.

However, the nature of the singular boundary in the interior of black holes was not well understood[9] until the first study of weak null singularity carried out by Hiscock [12]. In an attempt to understand the instability of the Reissner–Nordström Cauchy horizon, he considered the Vaidya model allowing for a self-gravitating ingoing null dust. In this model, an explicit solution can be found, and he showed that various components of the Christoffel symbols blow up. This, however, was called a whimper singularity as the Hawking mass and the curvature scalar invariants remain bounded.

In subsequent works, Poisson and Israel [25, 26] added an outgoing null dust to the model considered by Hiscock. While explicit solutions were not available, they were able to deduce that the second outgoing null dust would cause the Hawking mass to blow up at the null singularity. It was then thought of as a stronger singularity than that of Hiscock.

However, from the point of view of partial differential equations, it is more natural to view this singularity at the level of the nonsquare-integrability of the Christoffel symbols, which is exactly the threshold such that the spacetime cannot be defined as a weak solution to the Einstein equations. From this perspective, the singularity of Poisson and Israel is as strong as that of Hiscock, and both singularities can be viewed as terminal boundaries for the spacetimes in question.

While the Christoffel symbols blow up at the Cauchy horizon, one can also think that the Cauchy horizon is "stable" in the sense that no singularity arises before the "original Cauchy horizon". In particular there is no spacelike portion of the singular boundary in a neighborhood of timelike infinity. Thus, this is contrary to the case of the Schwarzschild spacetime. This weak null singularity picture has been further explored and justified in many numerical works (see [1–3]).

As we described before, the aforementioned picture of the interior of black holes was finally established by Dafermos in the context of the spherically symmetric Einstein–Maxwell–scalar field system [7]. This is the main motivation for our present work in which we initiate the study of weak null singularities of similar strength in vacuum without any symmetry assumptions.

[9] In particular it was believed that a perturbation of the Reissner–Nordström Cauchy horizon would lead to a Schwarzschild type singularity.

Finally, we note that a class of *analytic* spacetimes with slightly weaker singularities have been previously constructed in [22]. While this class of spacetime is more restrictive, as discussed in [22], it nonetheless admits the full "functional degrees of freedom" of the Einstein equations.

1.2. **Comparison with impulsive gravitational waves.** As pointed out by Dafermos [9], the weak null singularities that we consider in this paper share many similarities with impulsive gravitational waves. The latter are vacuum spacetimes admitting null hypersurfaces which support delta function singularities in the Riemann curvature tensor. Explicit examples were first constructed by Penrose [24], Khan and Penrose [14], and Szekeres [28]. In these spacetimes, while the Christoffel symbols are not continuous, they remain bounded. Therefore, in contrast with the weak null singularities that we consider here, these impulsive gravitational waves are not terminal singularities. In fact, the solution to the vacuum Einstein equation extends beyond the singularity and is smooth except across the singular hypersurface. Nevertheless, both scenarios represent singularities propagating along null hypersurfaces and from a mathematical point of view, the proofs of the existence theory for these singularities share many common features.

In recent joint works with Rodnianski [18, 19], we initiated the rigorous mathematical study for general impulsive gravitational waves without symmetry assumptions. We constructed the impulsive gravitational waves via solving the characteristic initial problem such that the initial data admit curvature delta singularities supported on an embedded 2-sphere. One of the new ideas in the proof is the use of renormalized energy estimates for the curvature components; i.e., instead of controlling the spacetime curvature components in L^2, we subtract off an L^∞ correction from some curvature components. This allowed us to derive a closed system of L^2 estimates which is completely independent of the singular curvature components.

In [18], when the interaction of impulsive gravitational waves was studied, we also extended the analysis to include a class of spacetimes such that when measured in the worst direction, the Christoffel symbols are merely in L^2. We proved an existence and uniqueness theorem for spacetimes with such low regularity and showed that the spacetime solution can be extended beyond the singularities. Notice that this result is in fact sharp: this is because if the Christoffel symbols fail to be square integrable, the spacetime cannot be extended as a weak solution to the Einstein equations (see footnote 1).

By contrast, the spacetimes considered in this paper have Christoffel symbols which are[10] not in L^2. Even though the weak null singularities are terminal singularities in the sense that there cannot be an existence theory beyond them, the theory developed in [18, 19] can be extended to control the spacetime *up to the singularity.* Moreover, our main theorem, which allows for two weak null singularities terminating at their intersection, can be viewed as an extension of the result in [18] on the interaction of two impulsive gravitational waves. In particular the renormalized energy of [18, 19] plays an important role in the proof of our main theorem. However, even after renormalization, the renormalized curvature is still singular (i.e., not in L^2) and has to be dealt with using an additional weighted estimate.

[10] In fact, we allow initial data to be in L^p only for $p = 1$, but not for any $p > 1$.

1.3. **Description of the main results.** Our setup is the characteristic initial value problem with initial data given on two null hypersurfaces H_0 and $\underline{H}_0$ intersecting at a 2-sphere $S_{0,0}$ (see Figure 6). We will follow the general notations in [5, 15, 16].

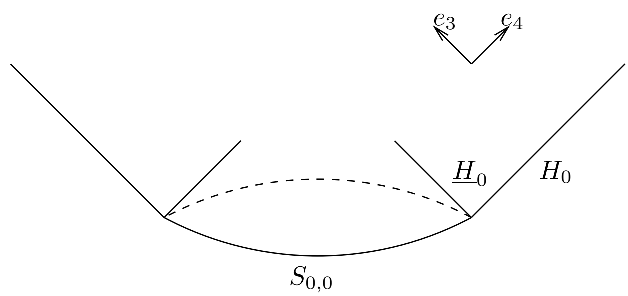

FIGURE 6. The basic setup

We introduce a null frame $\{e_1, e_2, e_3, e_4\}$ adapted to a double null foliation $(u, \underline{u})$ (see section 2.1). Denote the constant u hypersurfaces by H_u, the constant $\underline{u}$ hypersurfaces by $\underline{H}_{\underline{u}}$ and their intersections by $S_{u,\underline{u}} = H_u \cap \underline{H}_{\underline{u}}$. Decompose the Riemann curvature tensor with respect to the null frame $\{e_1, e_2, e_3, e_4\}$:

$$\alpha_{AB} = R(e_A, e_4, e_B, e_4), \qquad \underline{\alpha}_{AB} = R(e_A, e_3, e_B, e_3),$$
$$\beta_A = \frac{1}{2}R(e_A, e_4, e_3, e_4), \qquad \underline{\beta}_A = \frac{1}{2}R(e_A, e_3, e_3, e_4),$$
$$\rho = \frac{1}{4}R(e_4, e_3, e_4, e_3), \qquad \sigma = \frac{1}{4}{}^*R(e_4, e_3, e_4, e_3).$$

We also define the Gauss curvature of the 2-spheres associated to the double null foliation to be K. Define also the following Ricci coefficients with respect to the null frame:

$$\chi_{AB} = g(D_A e_4, e_B), \qquad \underline{\chi}_{AB} = g(D_A e_3, e_B),$$
$$\eta_A = -\frac{1}{2}g(D_3 e_A, e_4), \qquad \underline{\eta}_A = -\frac{1}{2}g(D_4 e_A, e_3),$$
$$\omega = -\frac{1}{4}g(D_4 e_3, e_4), \qquad \underline{\omega} = -\frac{1}{4}g(D_3 e_4, e_3),$$
$$\zeta_A = \frac{1}{2}g(D_A e_4, e_3).$$

Let $\hat{\chi}$ (resp. $\hat{\underline{\chi}}$) be the traceless part of χ (resp. $\underline{\chi}$).

The data on H_0 are given on $0 \le \underline{u} < \underline{u}_*$ such that χ becomes singular as $\underline{u} \to \underline{u}_*$. Similarly, the data on $\underline{H}_0$ is given on $0 \le u < u_*$ such that $\underline{\chi}$ becomes singular as $u \to u_*$.

More precisely, let $f_1 : [0, \underline{u}_*) \to \mathbb{R}$ be a smooth function such that $f_1(x) \ge 0$ is decreasing and

$$\int_0^{\underline{u}_*} \frac{1}{f_1(x)^2} dx < \infty$$

(resp. let $f_2 : [0, u_*) \to \mathbb{R}$ be a smooth function such that $f_2(x) \geq 0$ is decreasing and

$$\int_0^{u_*} \frac{1}{f_2(x)^2} dx < \infty).$$

For example, f_1 can be taken to be $f_1(x) = (\underline{u}_* - x)^{\frac{1}{2}} \log^p(\frac{1}{\underline{u}_* - x})$ for $p > \frac{1}{2}$.

Our main theorem shows local existence for a class of singular initial data with[11]

$$|\chi(0, \underline{u})| \lesssim f_1(\underline{u})^{-2}, \qquad |\underline{\chi}(u, 0)| \lesssim f_2(u)^{-2}.$$

We construct a (unique) solution $(\mathcal{M}, g)$ to the vacuum Einstein equations in the region $u < u_*$, $\underline{u} < \underline{u}_*$, where u_*, $\underline{u}_* \leq \epsilon$, and

$$\int_0^{\underline{u}_*} f_1(\underline{u})^{-2} d\underline{u}, \qquad \int_0^{u_*} f_2(u)^{-2} du \leq \epsilon^2. \tag{3}$$

Here, $(u, \underline{u})$ is a double null foliation for $(\mathcal{M}, g)$ and the metric g takes the form

$$g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du)$$

in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system (to be defined in section 2.2). Define also ∇ to be the induced Levi-Cevita connection on the 2-spheres of constant u and $\underline{u}$, i.e., $S_{u,\underline{u}}$, and ∇_3, ∇_4 to be the projections of the covariant derivatves D_3, D_4 to the tangent space of $S_{u,\underline{u}}$. Our main theorem (Theorem 1) can be stated precisely as a combination of Theorems 2, 3 and 4. The first main result is the following theorem, which shows the existence of a spacetime up to the (potentially singular) null boundaries:

Theorem 2. *Consider the characteristic initial value problem for*

$$\mathrm{Ric}_{\mu\nu} = 0 \tag{4}$$

with data that are smooth on $H_0 \cap \{0 \leq \underline{u} < \underline{u}_*\}$ *and* $\underline{H}_0 \cap \{0 \leq u < u_*\}$ *such that the following hold.*

- *There exists an atlas such that in each coordinate chart with local coordinates* (θ^1, θ^2)*, the initial metric* γ_0 *on* $S_{0,0}$ *obeys*

$$d \leq \det \gamma_0 \leq D$$

and

$$\sum_{i_1+i_2 \leq 6} \left| \left(\frac{\partial}{\partial \theta^1}\right)^{i_1} \left(\frac{\partial}{\partial \theta^2}\right)^{i_2} \gamma_{BC} \right| \leq D.$$

- *The metric on* H_0 *and* $\underline{H}_0$ *satisfies the gauge conditions*

$$\Omega = 1 \quad \text{on } H_0 \text{ and } \underline{H}_0$$

and

$$b^A = 0 \quad \text{on } \underline{H}_0.$$

[11] We assume also bounds for the angular derivatives that are consistent with this singular profile (see the precise statement in Theorem 2).

- *The Ricci coefficients on the initial hypersurface H_0 verify*

$$\sum_{i\leq 5}\sup_{\underline{u}}\left\|{f_1}^2(\underline{u})\nabla^i\chi\right\|_{L^2(S_{0,\underline{u}})}\leq D,$$

$$\sum_{i\leq 4}\sup_{\underline{u}}\left\|\nabla^i\zeta\right\|_{L^2(S_{0,\underline{u}})}\leq D,$$

$$\sum_{i\leq 4}\sup_{\underline{u}}\left\|\nabla^i\mathrm{tr}\,\underline{\chi}\right\|_{L^2(S_{0,\underline{u}})}\leq D.$$

- *The Ricci coefficients on the initial hypersurface $\underline{H}_0$ verify*

$$\sum_{i\leq 5}\sup_{u}\left\|{f_2}^2(u)\nabla^i\underline{\chi}\right\|_{L^2(S_{u,0})}\leq D,$$

$$\sum_{i\leq 4}\sup_{u}\left\|\nabla^i\zeta\right\|_{L^2(S_{u,0})}\leq D,$$

$$\sum_{i\leq 4}\sup_{u}\left\|\nabla^i\mathrm{tr}\,\chi\right\|_{L^2(S_{u,0})}\leq D.$$

Then for ϵ sufficiently small (depending only on d and D) and

$$u_*,\,\underline{u}_*\leq\epsilon,\quad \left\|f_1(\underline{u})^{-1}\right\|_{L^2_{\underline{u}}},\,\left\|f_2(u)^{-1}\right\|_{L^2_u}<\epsilon,$$

there exists a unique spacetime $(\mathcal{M},g)$ endowed with a double null foliation $(u,\underline{u})$ in $0\leq u<u_*$ and $0\leq\underline{u}<\underline{u}_*$, which is a solution to the vacuum Einstein equations (4) *with the given data. Moreover, the spacetime remains smooth in $0\leq u<u_*$ and $0\leq\underline{u}<\underline{u}_*$.*

Remark 2. In the following, we will only prove a priori estimates for spacetimes arising from these initial data (see Theorem 5). The existence of a spacetime and the propagation of regularity follow from standard arguments. (For an example of this argument in low regularity, see [19, Sections 4 and 5]. See also [5, Chapter 16].)

Remark 3. In order to simplify notations, we will omit the subscripts 1 and 2 in the weight functions f_1 and f_2. They can be inferred from whether f is a function of u or $\underline{u}$.

Remark 4. In section 4, we will construct a class of characteristic initial data which satisfies the assumptions of Theorem 2.

While the weight f in the spacetime norms allows the spacetime to be singular, the spacetime metric can be extended beyond the singular hypersurfaces H_{u_*} and $\underline{H}_{\underline{u}_*}$ continuously.

Theorem 3. *Under the assumptions of Theorem 2, the spacetime $(\mathcal{M},g)$ can be extended continuously up to and beyond the singular boundaries $\underline{H}_{\underline{u}_*}:=\{\underline{u}=\underline{u}_*\}$, $H_{u_*}:=\{u=u_*\}$. Moreover, the induced metric and null second fundamental form on the interior of the limiting hypersurfaces $\underline{H}_{\underline{u}_*}$ and H_{u_*} are regular. More precisely, for any coordinate chart U_i on $S_{0,0}$, the metric components γ, b, Ω satisfy* *the following estimates in the coordinate chart given by* $U_i(u,\underline{u}) := \underline{\Phi}_{\underline{u}} \circ \Phi_u(U_i)$*, where* Φ_u *and* $\underline{\Phi}_{\underline{u}}$ *are the diffeomorphisms generated by* $\underline{L}$ *and* L*, respectively:*[12]

$$\sum_{i_1+i_2\leq 4}\sup_{0\leq u\leq u_*}\left\|\left(\frac{\partial}{\partial\theta^1}\right)^{i_1}\left(\frac{\partial}{\partial\theta^2}\right)^{i_2}(\gamma,b,\Omega)\right\|_{L^2\left(U_i\left(u,\underline{u}_*\right)\right)}\leq C.$$

Moreover, for any fixed $U<u_*$*, we have the following bounds for the Ricci coefficients* $\underline{\hat{\chi}}, \mathrm{tr}\,\underline{\chi}, \underline{\omega}, \eta, \underline{\eta}$*:*

$$\sum_{j\leq 1}\sum_{i\leq 3-j}\sup_{0\leq u\leq U}\left\|\nabla_3^j\nabla^i\left(\underline{\hat{\chi}}, \mathrm{tr}\,\underline{\chi}, \underline{\omega}, \eta, \underline{\eta}\right)\right\|_{L^2\left(S_{u,\underline{u}_*}\right)}\leq C_U.$$

Similar regularity statements hold on H_{u_*}*.*

Remark 5. If we assume in addition that the higher angular derivatives of χ are bounded in $L^1_{\underline{u}}L^\infty(S)$, then the metric and the second fundamental form also inherit higher regularity in the interior of $\underline{H}_{\underline{u}_*}$. In particular if all angular derivatives of χ are bounded in $L^1_{\underline{u}}L^\infty(S)$, then the metric restricted to $\underline{H}_{\underline{u}_*}\cap\{0\leq u\leq U\}$ is smooth along the directions tangential to $\underline{H}_{\underline{u}_*}$. Similar statements hold on H_{u_*}. We will omit the details.

Moreover, we show that if initially the data are indeed singular, then H_{u_*} and $\underline{H}_{\underline{u}_*}$ are terminal singularities of the spacetime in the following sense:

Theorem 4. *If, in addition to the assumptions of Theorem* 2*, we also have the following for the initial data,*

$$\int_0^{\underline{u}_*}\left|\hat{\chi}\left(0,\underline{u}\right)\right|^2 d\underline{u}=\infty,$$

along Lebesgue-almost every null generator on H_0*, then the Christoffel symbols in the coordinate system* $(u,\underline{u},\theta^1,\theta^2)$ *do not belong to* L^2 *in a neighborhood of any point on* $\underline{H}_{\underline{u}_*}$*.*

Similarly if the initial data satisfy

$$\int_0^{u_*}\left|\underline{\hat{\chi}}(u,0)\right|^2 du=\infty$$

along Lebesgue-almost every null generator on $\underline{H}_0$*, then the Christoffel symbols in the coordinate system* $(u,\underline{u},\theta^1,\theta^2)$ *do not belong to* L^2 *in a neighborhood of any point on* H_{u_*}*.*

Remark 6. Theorem 4 guarantees that if we extend the spacetime metric continuously in the obvious differentiable structure given by the coordinate system $(u,\underline{u},\theta^1,\theta^2)$, then the Christoffel symbols are nonsquare-integrable in the extension. However, it is an open problem whether the spacetime admits any continuous extensions with square integrable Christoffel symbols.

[12]See definition of L and $\underline{L}$ in section 2.1.

1.4. **Main ideas of the proof.** All the known proofs of regularity for the Einstein equations without symmetry assumptions rely on L^2 estimates on the metric and its derivatives or the Riemann curvature tensor and its derivatives. Let us denote schematically by Γ a general Ricci coefficient and by Ψ a general curvature component decomposed with respect to a null frame adapted to the double null foliation. In the double null foliation gauge (see, for example, [5, 15]), the standard approach to obtain a priori bounds is to couple the L^2 estimates for the curvature components

$$\int_H \Psi^2 + \int_{\underline{H}} \Psi^2 \leq \text{ Data } + \iint \Gamma\Psi\Psi$$

with the estimates for the Ricci coefficients obtained using the transport equations

$$\nabla_3\Gamma = \Psi + \Gamma\Gamma,$$

$$\nabla_4\Gamma = \Psi + \Gamma\Gamma.$$

However, in the setting of two weak null singularities, *none* of the spacetime curvature components $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$ are in L^2!

Nevertheless, while these curvature components are singular, the nature of their singularity is specific. More precisely, while the spacetime curvature components ρ and σ are not in L^2, they can be written as a sum of some regular intrinsic curvature components K and $\check{\sigma}$ (see further discussion in section 1.4.1) which belong to L^2 and terms which are quadratic in Γ. We therefore prove L^2 estimates for K and $\check{\sigma}$, which we will call the *renormalized curvature components* (see [18, 19]). Moreover, by considering $(K, \check{\sigma})$ instead of (ρ, σ), we remove all appearances of α and $\underline{\alpha}$ in the estimates and so that we do not have to deal with the singularities of α and $\underline{\alpha}$! It still remains to control the singular curvature components β and $\underline{\beta}$. Here, we make use of the fact that β and $\underline{\beta}$ are singular in a specific manner toward the singular boundary $\underline{H}_{\underline{u}_*}$ and H_{u_*}, respectively. We therefore introduce degenerate L^2 norms that incorporate these singularities. We will explain the renormalization and the degenerate estimates in more detail below.

1.4.1. *Renormalized energy estimates.* As described above, a main ingredient of the proof of the main theorem is the renormalized energy estimates introduced in [18, 19] in the study of impulsive gravitational waves. This can be seen as follows. For the class of weak null singularities that we consider, while the $\not{\mathcal{L}}_L$ derivative of the spacetime metric blows up, the metric restricted to the 2-sphere remains regular in the angular directions. Since the Gauss curvature K is intrinsic to the 2-spheres, it remains bounded. On the other hand, by the Gauss equation,

$$K = -\rho + \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} - \frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi},$$

and the fact that $\mathrm{tr}\,\chi$ and $\hat{\chi}$ blow up at $\underline{u} = \underline{u}_*$, ρ also blows up at $\underline{u} = \underline{u}_*$. In view of this, we estimate the Gauss curvature K instead of the spacetime curvature component ρ.

Indeed, we see that the Gauss curvature K satisfies equations such that the right-hand side contains terms that are less singular than the terms in the corresponding equation for ρ. More precisely, for the curvature component ρ, we have (up to lower-order terms) the Bianchi equation

$$\nabla_4\rho + \frac{3}{2}\mathrm{tr}\,\chi\rho = \mathrm{div}\,\beta - \frac{1}{2}\hat{\underline{\chi}}\cdot\alpha + \cdots,$$

which contains the nonintegrable curvature component α. On the other hand, the Gauss curvature obeys the equation (see (12))

$$\nabla_4 K + \mathrm{tr}\,\chi K = -\mathrm{div}\,\beta + \cdots ,$$

where there are no terms containing α or that are quadratic in $\mathrm{tr}\,\chi$, $\hat{\chi}$ and ω, i.e., every term on the right-hand side of the equation is integrable in the $\underline{u}$ direction.[13]

In a similar fashion, by considering the renormalized curvature component[14]

$$\check{\sigma} := \sigma + \frac{1}{2}\underline{\hat{\chi}} \wedge \hat{\chi}$$

instead of σ, we see that it satisfies an equation such that all the terms on the right-hand side are integrable in the $\underline{u}$ direction.

One consequence of the renormalization is that we have completely removed the appearances of the curvature component α in the equations. In fact, as in [18, 19], this allows us to derive a set of estimates for the renormalized curvature component without requiring any information on the curvature component α.

Moreover, when considering the equations for $\nabla_3 K$ and $\nabla_3 \check{\sigma}$ for the renormalized curvature components, one sees that $\underline{\alpha}$ does not appear and all the terms are integrable in the u direction. Therefore, although α or $\underline{\alpha}$ can be very singular near one of the singular boundaries, we do not need to derive any estimates for them!

1.4.2. *Degenerate L^2 estimates.* Since the renormalization above deals with the singularity in the ρ and σ components and avoids any information on α and $\underline{\alpha}$, it remains to derive appropriate L^2 estimates for β and $\underline{\beta}$.

The main observation is that while β and $\underline{\beta}$ are both singular and fail to be in L^2, their singularities can be captured quantitatively. Consider the curvature component β. Since the blow-up rate of $\mathrm{tr}\,\chi$ and $\hat{\chi}$ can be bounded above by $f(\underline{u})^{-2}$, in view of the Codazzi equations in (10), β is also bounded above by $f(\underline{u})^{-2}$. In particular while β is only in $L^1_{\underline{u}}$ but not in $L^p_{\underline{u}}$ for any $p > 1$, the assumptions on the initial data allow us to control $f(\underline{u})\beta$ in $L^2_{\underline{u}}$. We will thus incorporate this blowup in the norms and will be able to still use an L^2 based estimate.

The energy estimates will be obtained directly from two sets of Bianchi equations instead of using the Bel–Robinson tensor. Notice that since the energy estimates for $K, \check{\sigma}$ are obtained either together with that for β or that for $\underline{\beta}$, even though K and $\check{\sigma}$ are regular, their energy estimates degenerate. Therefore, at the highest level of derivatives, we have to be content with the weaker L^2 estimates for these curvature components.

A potentially more serious challenge is that the introduction of the degenerate weights in u and $\underline{u}$ would create terms that cannot be estimated by the energy estimates themselves. Nevertheless, since the weights are chosen to be decreasing toward the future, these uncontrollable terms in fact possess a good sign.

---

[13] The can be compared with the renormalization introduced in [19] and [18], where we estimated $\check{\rho} = \rho - \frac{1}{2}\hat{\chi} \cdot \underline{\hat{\chi}}$ instead of ρ. Whereas the renormalization using $\check{\rho}$ allows one to eliminate α in the estimates, it nonetheless introduces a term $\frac{1}{4}\mathrm{tr}\,\underline{\chi}|\hat{\chi}|^2$, which is not integrable in the $\underline{u}$ direction in the setting of the present paper. Instead, by studying the equation for K, we see none of these terms which are quadratic in $\mathrm{tr}\,\chi$, $\hat{\chi}$ or ω! This fact can also be derived directly by considering the equations for $\nabla_4 K$ using the intrinsic definition of the Gauss curvature.

[14] This is in fact related to the intrinsic curvature of the normal bundle to $S_{u,\underline{u}}$.

1.4.3. *Estimates for the Ricci coefficients.* As indicated above, the Ricci coefficients enter as error terms in the energy estimates. Thus, to close all the estimates, we need to control the Ricci coefficients Γ by using the transport equations which in turn have the curvature components in the source terms. Since the various Ricci coefficients have different singular behavior, we separate them according to the bounds that they obey. More precisely, denote by ψ_H the components that behave like $f(\underline{u})^{-2}$ as $\underline{u} \to \underline{u}_*$, by $\psi_{\underline{H}}$ the components that behave like $f(u)^{-2}$ as $u \to u_*$, and by ψ the components that are bounded.

For the singular Ricci coefficients ψ_H, we have the following schematic transport equations:

$$\nabla_3 \psi_H = K + \nabla\psi + \psi\psi + \psi_{\underline{H}}\psi_H.$$

The first three terms on the right-hand side of this equation are bounded while the last term is singular. Nevertheless, the singularity of $\psi_{\underline{H}}$ still allows it to be controlled in L^1 along the e_3 direction. Thus, this equation can be integrated to show that the initial (singular) bounds for ψ_H can be propagated. It is important that the terms of the form $\psi_H\psi_H$ and $\psi_{\underline{H}}\psi_{\underline{H}}$ do not appear in the equations. A similar structure can also be seen in the equation for the other singular Ricci coefficients $\psi_{\underline{H}}$, which takes the form

$$\nabla_4 \psi_{\underline{H}} = K + \nabla\psi + \psi\psi + \psi_{\underline{H}}\psi_H.$$

For the regular Ricci coefficients ψ, we have transport equations of the form

$$\nabla_4 \psi = \beta + \psi\psi_H \quad \text{or} \quad \nabla_3 \psi = \underline{\beta} + \psi\psi_{\underline{H}}.$$

The bounds that we prove show that the right-hand side is integrable, and therefore ψ remains bounded. For example, in the ∇_4 equation, it is important that we do not have terms of the form $\psi_H\psi_H$, $\psi\psi_{\underline{H}}$, $\psi_H\psi_{\underline{H}}$, and $\psi_{\underline{H}}\psi_{\underline{H}}$, which are not uniformly bounded after integrating along the e_4 direction.

1.4.4. *Null structure in the energy estimates.* A priori, the degenerate L^2 estimates that we introduce may not be sufficient to control the error terms. Nevertheless, the vacuum Einstein equations possess a remarkable null structure which allows one to close the estimates using only the degenerate L^2 estimates.

For example, in the energy estimates for the singular component β, we have

$$\|f(\underline{u})\beta\|^2_{L^2(H)} \leq \text{ Data } + \left\|f^2(\underline{u})\left(\beta\psi_{\underline{H}}\beta + \beta\psi_H\underline{\beta} + \beta\psi K\right)\right\|_{L^1_u L^1_{\underline{u}} L^1(S)}.$$

To estimate the first term, it suffices to note that $\psi_{\underline{H}}$, while singular, can be shown to be small after integrating along the u direction. Thus, the first term can be controlled using Gronwall's inequality. For the second term, since the singularity for β has the same strength as that for ψ_H (and similarly the singularity for $\underline{\beta}$ has the same strength as that for $\psi_{\underline{H}}$), the singularity in this term is similar to that in the first term and can also be bounded. The final term is less singular since ψ and K are both uniformly bounded.[15] Notice that if other combinations of curvature terms and Ricci coefficients such as $\beta\psi_H\beta$, $\beta\psi_{\underline{H}}\underline{\beta}$, or $\beta\psi_H K$ appear in the error terms, the degenerate energy will not be strong enough to close the bounds!

In order to close all the estimates, we need to commute also with higher derivatives. As in [18, 19], we will only commute with angular covariant derivatives. These commutations will not introduce terms that are more singular. Moreover,

[15] Although, as pointed out before, the highest derivative estimates for K in the energy norm suffer a loss as one approaches the singular boundaries, this term can nevertheless be controlled.

the null structure of the estimates indicated above is also preserved under these commutations.

Similar to [18, 19], the renormalization introduces error terms in the energy estimates such that the Ricci coefficients have one more derivative compared to the curvature components. These terms cannot be estimated via transport equations alone but are controlled using also elliptic estimates on the spheres. A form of null structure similar to that described above also makes an appearance in these elliptic estimates, allowing all the bounds to be closed.

1.5. **Outline of the paper.** We end the introduction with an outline of the remainder of the paper. In section 2, we introduce the basic setup of the paper, including the double null foliation, the coordinate system, and the Einstein vacuum equations recast in terms of the geometric quantities associated to the double null foliation. In section 3, we introduce the norms used in the paper and state a theorem on a priori estimates (Theorem 5) which imply our main existence theorem (Theorem 2). In section 4, we construct a class of characteristic initial data satisfying the assumptions of Theorem 2. In sections 5–8, we prove Theorem 5. In section 5, we obtain the estimates for the metric components and derive functional inequalities useful in our setting. Then in sections 6 and 7, we prove bounds for the Ricci coefficients assuming control of the curvature components. In section 8, we close all the estimates by obtaining bounds for the curvature components. Finally, in section 9, we discuss the nature of the singular boundary and prove Theorems 3 and 4.

2. Basic setup

2.1. **Double null foliation.** For a smooth[16] spacetime in a neighborhood of $S_{0,0}$, we define a double null foliation as follows: Let u and $\underline{u}$ be solutions to the eikonal equation

$$g^{\mu\nu}\partial_\mu u\partial_\nu u = 0, \qquad g^{\mu\nu}\partial_\mu \underline{u}\partial_\nu \underline{u} = 0,$$

such that $u = 0$ on H_0 and $\underline{u} = 0$ on $\underline{H}_0$. Let

$$L'^\mu = -2g^{\mu\nu}\partial_\nu u, \qquad \underline{L}'^\mu = -2g^{\mu\nu}\partial_\nu \underline{u}.$$

These are null and geodesic vector fields. Let

$$2\Omega^{-2} = -g\left(L', \underline{L}'\right).$$

Define

$$e_3 = \Omega\underline{L}', \qquad e_4 = \Omega L'$$

to be the normalized null pair such that

$$g(e_3, e_4) = -2$$

and

$$\underline{L} = \Omega^2\underline{L}',\ L = \Omega^2 L'$$

to be the so-called equivariant vector fields.

In this paper, we will consider spacetime solutions to the vacuum Einstein equations (1) in the gauge such that

$$\Omega = 1, \quad \text{on } H_0 \text{ and } \underline{H}_0.$$

[16]The spacetimes considered in this paper are not smooth at $u = u_*$ or $\underline{u} = \underline{u}_*$. However, since we first construct the spacetime in the region $\{u < u_*\} \cap \{\underline{u} < \underline{u}_*\}$ in which the spacetime is smooth (see Theorem 2), it suffices to define the double null foliation for smooth spacetimes.

The level sets of u (resp. $\underline{u}$) are denoted by H_u (resp. $\underline{H}_{\underline{u}}$). The eikonal equations imply that H_u and $\underline{H}_{\underline{u}}$ are null hypersurfaces. The intersections of the hypersurfaces H_u and $\underline{H}_{\underline{u}}$ are topologically 2-spheres, which we denote by $S_{u,\underline{u}}$. Note that the integral flows of L and $\underline{L}$ respect the foliation $S_{u,\underline{u}}$.

2.2. **The coordinate system.** We define a coordinate system $(u, \underline{u}, \theta^1, \theta^2)$ in a neighborhood of $S_{0,0}$ as follows. On the sphere $S_{0,0}$, we have an atlas such that in the local coordinate system (θ^1, θ^2) in each coordinate chart, the metric γ is smooth, bounded, and positive definite. Recall that in a neighborhood of $S_{0,0}$, u and $\underline{u}$ are solutions to the eikonal equations,

$$g^{\mu\nu}\partial_\mu u \partial_\nu u = 0, \qquad g^{\mu\nu}\partial_\mu \underline{u} \partial_\nu \underline{u} = 0.$$

We then require the coordinates to satisfy

$$\not{\mathcal{L}}_{\underline{L}}\theta^A = 0$$

on the initial hypersurface $\underline{H}_0$ and

$$\not{\mathcal{L}}_{L}\theta^A = 0$$

in the spacetime region. Here, $\not{\mathcal{L}}_L$ and $\not{\mathcal{L}}_{\underline{L}}$ denote the restriction of the Lie derivative to $TS_{u,\underline{u}}$ (See [5, Chapter 1].) and L and $\underline{L}$ are defined as in section 2.1. Relative to the coordinate system $(u, \underline{u}, \theta^1, \theta^2)$, the null pair e_3 and e_4 can be expressed as

$$e_3 = \Omega^{-1}\left(\frac{\partial}{\partial u} + b^A \frac{\partial}{\partial \theta^A}\right), \qquad e_4 = \Omega^{-1}\frac{\partial}{\partial \underline{u}},$$

for some b^A such that $b^A = 0$ on $\underline{H}_0$, while the metric g takes the form

$$g = -2\Omega^2\left(du \otimes d\underline{u} + d\underline{u} \otimes du\right) + \gamma_{AB}\left(d\theta^A - b^A du\right) \otimes \left(d\theta^B - b^B du\right).$$

2.3. **Equations.** We will recast the Einstein equations as a system for Ricci coefficients and curvature components associated to a null frame e_3, e_4 defined above and an orthonormal frame[17] $\{e_A\}_{A=1,2}$ tangent to the 2-spheres $S_{u,\underline{u}}$. We define the Ricci coefficients relative to the null fame,

$$\begin{aligned} \chi_{AB} &= g\left(D_A e_4, e_B\right), & \underline{\chi}_{AB} &= g\left(D_A e_3, e_B\right), \\ \eta_A &= -\frac{1}{2} g\left(D_3 e_A, e_4\right), & \underline{\eta}_A &= -\frac{1}{2} g\left(D_4 e_A, e_3\right), \\ \omega &= -\frac{1}{4} g\left(D_4 e_3, e_4\right), & \underline{\omega} &= -\frac{1}{4} g\left(D_3 e_4, e_3\right), \\ \zeta_A &= \frac{1}{2} g\left(D_A e_4, e_3\right), \end{aligned} \tag{5}$$

where $D_A = D_{e_{(A)}}$. We also introduce the null curvature components,

$$\begin{aligned} \alpha_{AB} &= R\left(e_A, e_4, e_B, e_4\right), & \underline{\alpha}_{AB} &= R\left(e_A, e_3, e_B, e_3\right), \\ \beta_A &= \frac{1}{2} R\left(e_A, e_4, e_3, e_4\right), & \underline{\beta}_A &= \frac{1}{2} R\left(e_A, e_3, e_3, e_4\right), \\ \rho &= \frac{1}{4} R\left(e_4, e_3, e_4, e_3\right), & \sigma &= \frac{1}{4} {}^*R\left(e_4, e_3, e_4, e_3\right). \end{aligned} \tag{6}$$

[17]Of course the orthonormal frame is only defined locally. Alternatively, the capital Latin indices can be understood as abstract indices.

Here ${}^*R$ denotes the Hodge dual of R. We denote by ∇ the induced covariant derivative operator on $S_{u,\underline{u}}$ and by ∇_3, ∇_4 the projections to $S_{u,\underline{u}}$ of the covariant derivatives D_3, D_4 (see precise definitions in [15, Chapter 3.1]).

Observe that

$$\begin{aligned}\omega &= -\frac{1}{2}\nabla_4(\log\Omega), &\quad \underline{\omega} &= -\frac{1}{2}\nabla_3(\log\Omega),\\ \eta_A &= \zeta_A + \nabla_A(\log\Omega), &\quad \underline{\eta}_A &= -\zeta_A + \nabla_A(\log\Omega).\end{aligned} \tag{7}$$

Define the following contractions of the tensor product $\phi^{(1)}$ and $\phi^{(2)}$ with respect to the metric γ:

$$\phi^{(1)}\cdot\phi^{(2)} := (\gamma^{-1})^{AC}(\gamma^{-1})^{BD}\phi^{(1)}_{AB}\phi^{(2)}_{CD} \quad \text{for symmetric 2-tensors } \phi^{(1)}_{AB},\ \phi^{(2)}_{AB},$$

$$\phi^{(1)}\cdot\phi^{(2)} := (\gamma^{-1})^{AB}\phi^{(1)}_{A}\phi^{(2)}_{B} \quad \text{for 1-forms } \phi^{(1)}_{A},\ \phi^{(2)}_{A},$$

$$(\phi^{(1)}\cdot\phi^{(2)})_A := (\gamma^{-1})^{BC}\phi^{(1)}_{AB}\phi^{(2)}_{C} \quad \text{for a symmetric 2-tensor } \phi^{(1)}_{AB} \text{ and a 1-form } \phi^{(2)}_{A},$$

$$(\phi^{(1)}\widehat{\otimes}\phi^{(2)})_{AB} := \phi^{(1)}_{A}\phi^{(2)}_{B} + \phi^{(1)}_{B}\phi^{(2)}_{A} - \gamma_{AB}(\phi^{(1)}\cdot\phi^{(2)}) \quad \text{for 1-forms } \phi^{(1)}_{A},\ \phi^{(2)}_{A},$$

$$\phi^{(1)}\wedge\phi^{(2)} := \not\epsilon^{AB}(\gamma^{-1})^{CD}\phi^{(1)}_{AC}\phi^{(2)}_{BD} \quad \text{for symmetric 2-tensors } \phi^{(1)}_{AB},\ \phi^{(2)}_{AB},$$

where $\not\epsilon$ is the volume form associated to the metric γ. We also define by * for 1-forms and symmetric 2-tensors, respectively, as follows (note that on 1-forms this is the Hodge dual on $S_{u,\underline{u}}$):

$$\begin{aligned}{}^*\phi_A &:= \gamma_{AC}\not\epsilon^{CB}\phi_B,\\ {}^*\phi_{AB} &:= \gamma_{BD}\not\epsilon^{DC}\phi_{AC}.\end{aligned}$$

Define the operator $\nabla\widehat{\otimes}$ on a 1-form ϕ_A by

$$\left(\nabla\widehat{\otimes}\phi\right)_{AB} := \nabla_A\phi_B + \nabla_B\phi_A - \gamma_{AB}\mathrm{div}\,\phi.$$

For totally symmetric tensors, define the div and curl operators as follows

$$(\mathrm{div}\,\phi)_{A_1\cdots A_r} := \nabla^B\phi_{BA_1\cdots A_r},$$

$$(\mathrm{curl}\,\phi)_{A_1\cdots A_r} := \not\epsilon^{BC}\nabla_B\phi_{CA_1\cdots A_r}.$$

Define also the trace of totally symmetric tensors to be

$$(\mathrm{tr}\,\phi)_{A_1\cdots A_{r-1}} := \left(\gamma^{-1}\right)^{BC}\phi_{BCA_1\cdots A_{r-1}}.$$

We separate the trace and traceless part of χ and $\underline{\chi}$. Let $\hat{\chi}$ and $\underline{\hat{\chi}}$ be the traceless parts of χ and $\underline{\chi}$, respectively. Then χ and $\underline{\chi}$ satisfy the following null structure equations:

$$
\begin{aligned}
\nabla_4 \mathrm{tr}\,\chi + \frac{1}{2}(\mathrm{tr}\,\chi)^2 &= -|\hat{\chi}|^2 - 2\omega \mathrm{tr}\,\chi, \\
\nabla_4 \hat{\chi} + \mathrm{tr}\,\chi \hat{\chi} &= -2\omega\hat{\chi} - \alpha, \\
\nabla_3 \mathrm{tr}\,\underline{\chi} + \frac{1}{2}(\mathrm{tr}\,\underline{\chi})^2 &= -2\underline{\omega}\mathrm{tr}\,\underline{\chi} - |\underline{\hat{\chi}}|^2, \\
\nabla_3 \underline{\hat{\chi}} + \mathrm{tr}\,\underline{\chi}\,\underline{\hat{\chi}} &= -2\underline{\omega}\underline{\hat{\chi}} - \underline{\alpha}, \\
\nabla_4 \mathrm{tr}\,\underline{\chi} + \frac{1}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} &= 2\omega \mathrm{tr}\,\underline{\chi} + 2\rho - \hat{\chi}\cdot\underline{\hat{\chi}} + 2\mathrm{div}\,\underline{\eta} + 2|\underline{\eta}|^2, \\
\nabla_4 \underline{\hat{\chi}} + \frac{1}{2}\mathrm{tr}\,\chi \underline{\hat{\chi}} &= \nabla\widehat{\otimes}\underline{\eta} + 2\omega\underline{\hat{\chi}} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\hat{\chi} + \underline{\eta}\widehat{\otimes}\underline{\eta}, \\
\nabla_3 \mathrm{tr}\,\chi + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi &= 2\underline{\omega}\mathrm{tr}\,\chi + 2\rho - \hat{\chi}\cdot\underline{\hat{\chi}} + 2\mathrm{div}\,\eta + 2|\eta|^2, \\
\nabla_3 \hat{\chi} + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\hat{\chi} &= \nabla\widehat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\,\chi\underline{\hat{\chi}} + \eta\widehat{\otimes}\eta.
\end{aligned}
\tag{8}
$$

The other Ricci coefficients satisfy the following null structure equations:

$$
\begin{aligned}
\nabla_4 \eta &= -\chi\cdot(\eta - \underline{\eta}) - \beta, \\
\nabla_3 \underline{\eta} &= -\underline{\chi}\cdot(\underline{\eta} - \eta) + \underline{\beta}, \\
\nabla_4 \underline{\omega} &= 2\omega\underline{\omega} - \eta\cdot\underline{\eta} + \frac{1}{2}|\eta|^2 + \frac{1}{2}\rho, \\
\nabla_3 \omega &= 2\omega\underline{\omega} - \eta\cdot\underline{\eta} + \frac{1}{2}|\underline{\eta}|^2 + \frac{1}{2}\rho.
\end{aligned}
\tag{9}
$$

The Ricci coefficients also satisfy the following constraint equations:

$$
\begin{aligned}
\mathrm{div}\,\hat{\chi} &= \frac{1}{2}\nabla\mathrm{tr}\,\chi - \frac{1}{2}(\eta - \underline{\eta})\cdot\left(\hat{\chi} - \frac{1}{2}\mathrm{tr}\,\chi\right) - \beta, \\
\mathrm{div}\,\underline{\hat{\chi}} &= \frac{1}{2}\nabla\mathrm{tr}\,\underline{\chi} + \frac{1}{2}(\eta - \underline{\eta})\cdot\left(\underline{\hat{\chi}} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right) + \underline{\beta}, \\
\mathrm{curl}\,\eta &= -\mathrm{curl}\,\underline{\eta} = \sigma + \frac{1}{2}\underline{\hat{\chi}}\wedge\hat{\chi}, \\
K &= -\rho + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi},
\end{aligned}
\tag{10}
$$

with K the Gauss curvature of the spheres $S_{u,\underline{u}}$. The null curvature components satisfy the following null Bianchi equations:

$$
\begin{aligned}
\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\alpha&=\nabla\widehat{\otimes}\beta+4\underline{\omega}\alpha-3\left(\hat{\chi}\rho+^*\hat{\chi}\sigma\right)+(\zeta+4\eta)\,\widehat{\otimes}\beta,\\
\nabla_4\beta+2\mathrm{tr}\,\chi\beta&=\mathrm{div}\,\alpha-2\omega\beta+\big(2\zeta+\underline{\eta}\big)\cdot\alpha,\\
\nabla_3\beta+\mathrm{tr}\,\underline{\chi}\beta&=\nabla\rho+2\underline{\omega}\beta+^*\nabla\sigma+2\hat{\chi}\cdot\underline{\beta}+3\left(\eta\rho+^*\eta\sigma\right),\\
\nabla_4\sigma+\frac{3}{2}\mathrm{tr}\,\chi\sigma&=-\mathrm{div}\,^*\beta+\frac{1}{2}\underline{\hat{\chi}}\wedge\alpha-\zeta\wedge\beta-2\underline{\eta}\wedge\beta,\\
\nabla_3\sigma+\frac{3}{2}\mathrm{tr}\,\underline{\chi}\sigma&=-\mathrm{div}\,^*\underline{\beta}-\frac{1}{2}\hat{\chi}\wedge\underline{\alpha}+\zeta\wedge\underline{\beta}-2\eta\wedge\underline{\beta},\\
\nabla_4\rho+\frac{3}{2}\mathrm{tr}\,\chi\rho&=\mathrm{div}\,\beta-\frac{1}{2}\underline{\hat{\chi}}\cdot\alpha+\zeta\cdot\beta+2\underline{\eta}\cdot\beta,\\
\nabla_3\rho+\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho&=-\mathrm{div}\,\underline{\beta}-\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}+\zeta\cdot\underline{\beta}-2\eta\cdot\underline{\beta},\\
\nabla_4\underline{\beta}+\mathrm{tr}\,\chi\underline{\beta}&=-\nabla\rho+^*\nabla\sigma+2\omega\underline{\beta}+2\underline{\hat{\chi}}\cdot\beta-3\left(\underline{\eta}\rho-^*\underline{\eta}\sigma\right),\\
\nabla_3\underline{\beta}+2\mathrm{tr}\,\underline{\chi}\underline{\beta}&=-\mathrm{div}\,\underline{\alpha}-2\underline{\omega}\underline{\beta}-(-2\zeta+\eta)\cdot\underline{\alpha},\\
\nabla_4\underline{\alpha}+\frac{1}{2}\mathrm{tr}\,\chi\underline{\alpha}&=-\nabla\widehat{\otimes}\underline{\beta}+4\omega\underline{\alpha}-3\left(\underline{\hat{\chi}}\rho-^*\underline{\hat{\chi}}\sigma\right)+(\zeta-4\underline{\eta})\widehat{\otimes}\underline{\beta},
\end{aligned}
\tag{11}
$$

where * denotes the Hodge dual on $S_{u,\underline{u}}$.

We now rewrite the Bianchi equations in terms of the Gauss curvature K of the spheres $S_{u,\underline{u}}$ and the renormalized curvature component $\check{\sigma}$ defined by

$$\check{\sigma}=\sigma+\frac{1}{2}\underline{\hat{\chi}}\wedge\hat{\chi}.$$

The Bianchi equations take the following form:

$$
\begin{aligned}
\nabla_3\beta+\mathrm{tr}\,\underline{\chi}\beta=&-\nabla K+^*\nabla\check{\sigma}+2\underline{\omega}\beta+2\hat{\chi}\cdot\underline{\beta}-3\left(\eta K-^*\eta\check{\sigma}\right)\\
&+\frac{1}{2}\left(\nabla\left(\hat{\chi}\cdot\underline{\hat{\chi}}\right)+^*\nabla\left(\hat{\chi}\wedge\underline{\hat{\chi}}\right)\right)+\frac{3}{2}\left(\eta\hat{\chi}\cdot\underline{\hat{\chi}}+^*\eta\hat{\chi}\wedge\underline{\hat{\chi}}\right)\\
&-\frac{1}{4}\left(\nabla\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+\mathrm{tr}\,\chi\nabla\mathrm{tr}\,\underline{\chi}\right)-\frac{3}{4}\eta\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi},\\
\nabla_4\check{\sigma}+\frac{3}{2}\mathrm{tr}\,\chi\check{\sigma}=&-\mathrm{div}\,^*\beta-\zeta\wedge\beta-2\underline{\eta}\wedge\beta-\frac{1}{2}\hat{\chi}\wedge\left(\nabla\widehat{\otimes}\underline{\eta}\right)-\frac{1}{2}\hat{\chi}\wedge\left(\underline{\eta}\widehat{\otimes}\underline{\eta}\right),\\
\nabla_4K+\mathrm{tr}\,\chi K=&-\mathrm{div}\,\beta-\zeta\cdot\beta-2\underline{\eta}\cdot\beta+\frac{1}{2}\hat{\chi}\cdot\nabla\widehat{\otimes}\underline{\eta}+\frac{1}{2}\hat{\chi}\cdot\left(\underline{\eta}\widehat{\otimes}\underline{\eta}\right)\\
&-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{div}\,\underline{\eta}-\frac{1}{2}\mathrm{tr}\,\chi|\underline{\eta}|^2,\\
\nabla_3\check{\sigma}+\frac{3}{2}\mathrm{tr}\,\underline{\chi}\check{\sigma}=&-\mathrm{div}\,^*\underline{\beta}+\zeta\wedge\underline{\beta}-2\eta\wedge\underline{\beta}+\frac{1}{2}\underline{\hat{\chi}}\wedge\left(\nabla\widehat{\otimes}\eta\right)+\frac{1}{2}\underline{\hat{\chi}}\wedge\left(\eta\widehat{\otimes}\eta\right),\\
\nabla_3K+\mathrm{tr}\,\underline{\chi}K=&\ \mathrm{div}\,\underline{\beta}-\zeta\cdot\underline{\beta}+2\eta\cdot\underline{\beta}+\frac{1}{2}\underline{\hat{\chi}}\cdot\nabla\widehat{\otimes}\eta+\frac{1}{2}\underline{\hat{\chi}}\cdot\left(\eta\widehat{\otimes}\eta\right)\\
&-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{div}\,\eta-\frac{1}{2}\mathrm{tr}\,\underline{\chi}|\eta|^2,\\
\nabla_4\underline{\beta}+\mathrm{tr}\,\chi\underline{\beta}=&\ \nabla K+^*\nabla\check{\sigma}+2\omega\underline{\beta}+2\underline{\hat{\chi}}\cdot\beta+3\left(\underline{\eta}K+^*\underline{\eta}\check{\sigma}\right)\\
&-\frac{1}{2}\left(\nabla\left(\hat{\chi}\cdot\underline{\hat{\chi}}\right)-^*\nabla\left(\hat{\chi}\wedge\underline{\hat{\chi}}\right)\right)-\frac{3}{2}\left(\underline{\eta}\hat{\chi}\cdot\underline{\hat{\chi}}-^*\underline{\eta}\hat{\chi}\wedge\underline{\hat{\chi}}\right)\\
&+\frac{1}{4}\left(\nabla\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+\mathrm{tr}\,\chi\nabla\mathrm{tr}\,\underline{\chi}\right)+\frac{3}{4}\underline{\eta}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}.
\end{aligned}
\tag{12}
$$

Notice that we have obtained a system for the renormalized curvature components in which the curvature components α and $\underline{\alpha}$ do not appear.[18]

From now on, we will use capital Latin letters $A \in \{1, 2\}$ for indices on the spheres $S_{u,\underline{u}}$ and Greek letters $\mu \in \{1, 2, 3, 4\}$ for indices in the whole spacetime.

2.4. **Schematic notation.** We define a schematic notation for the Ricci coefficients according to the estimates that they obey. Introduce the following conventions:[19]

$$\psi \in \left\{\eta, \underline{\eta}\right\}, \quad \psi_H \in \left\{\mathrm{tr}\,\chi, \hat{\chi}, \omega\right\}, \quad \psi_{\underline{H}} \in \left\{\mathrm{tr}\,\underline{\chi}, \underline{\hat{\chi}}, \underline{\omega}\right\}.$$

We will use this schematic notation only in the situations where the exact constant in front of the term is irrelevant to the argument. We will denote by $\psi\psi$ (or $\psi\psi_H$, etc.) an arbitrary contraction with respect to the metric γ and by $\nabla\psi$ an arbitrary angular covariant derivative. $\nabla^i\psi^j$ will be used to denote the sum of all terms which are products of j factors, such that each factor takes the form $\nabla^{i_k}\psi$ and that the sum of all i_k's is i, i.e.,

$$\nabla^i\psi^j = \sum_{i_1+i_2+\cdots+i_j=i} \underbrace{\nabla^{i_1}\psi\nabla^{i_2}\psi\cdots\nabla^{i_j}\psi}_{\text{j factors}}.$$

We will use brackets to denote terms with one of the components in the brackets. For instance, the notation $\psi(\psi, \psi_H)$ denotes the sum of all terms of the form $\psi\psi$ or $\psi\psi_H$.

In this schematic notation, the Ricci coefficients ψ_H satisfy

$$\nabla_3\psi_H = K + \nabla\psi + \psi\psi + \psi_H\psi_{\underline{H}}.$$

The Ricci coefficients $\psi_{\underline{H}}$ similarly obey

$$\nabla_4\psi_{\underline{H}} = K + \nabla\psi + \psi\psi + \psi_H\psi_{\underline{H}}.$$

The Ricci coefficients ψ obey either one of the following equations:

$$\nabla_3\psi = \underline{\beta} + \psi\psi_{\underline{H}}$$

or

$$\nabla_4\psi = \beta + \psi\psi_H.$$

[18] Moreover, compared to the renormalization in [19], this system does not contain the terms $\mathrm{tr}\,\chi|\underline{\hat{\chi}}|^2$ and $\mathrm{tr}\,\underline{\chi}|\hat{\chi}|^2$, which would be uncontrollable in the context of this paper.

[19] Notice that this definition is different form that in [19], since in the context of the present paper $\mathrm{tr}\,\chi$ and $\mathrm{tr}\,\underline{\chi}$ verify different bounds compared to [19].

We also rewrite the Bianchi equations in the following schematic notation:

$$
\begin{aligned}
\nabla_3\beta+\nabla K -^* \nabla\check{\sigma} &= \sum_{i_1+i_2=1} \psi_{\underline{H}}\psi^{i_1}\nabla^{i_2}\psi_H + \psi K + \sum_{i_1+i_2=1} \psi^{i_1}\psi\nabla^{i_2}\psi,\\
\nabla_4\check{\sigma} + \mathrm{div}\,^*\beta &= \psi_H\check{\sigma} + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_H,\\
\nabla_4 K+\mathrm{div}\,\beta &= \psi_H K + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_H,\\
\nabla_3\check{\sigma} + \mathrm{div}\,^*\underline{\beta} &= \psi_{\underline{H}}\check{\sigma} + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_{\underline{H}},\\
\nabla_3 K-\mathrm{div}\,\underline{\beta} &= \psi_{\underline{H}} K + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_{\underline{H}},\\
\nabla_4\underline{\beta}-\nabla K -^* \nabla\check{\sigma} &= \sum_{i_1+i_2=1} \psi_H\psi^{i_1}\nabla^{i_2}\psi_{\underline{H}} + \psi K + \sum_{i_1+i_2=1} \psi^{i_1}\psi\nabla^{i_2}\psi.
\end{aligned}
\tag{13}
$$

3. Norms

In this section we define the norms that we will use to control the geometric quantities. We will in particular use the schematic notation defined in section 2.4. Our norms will be of the form $L^p_u L^q_{\underline{u}} L^r(S)$, where L^p_u and $L^q_{\underline{u}}$ are defined with respect to the measures du and $d\underline{u}$, respectively, and $L^r(S)$ is defined for any tensors ϕ on $S_{u,\underline{u}}$ by

$$\|\phi\|_{L^r(S_{u,\underline{u}})} := \left(\int_{S_{u,\underline{u}}} \left(\phi_{A_1A_2\cdots A_n}\phi^{A_1A_2\cdots A_n}\right)^{\frac{r}{2}}\right)^{\frac{1}{r}},$$

where the integral is with respect to the volume form induced by γ.

We define the following norms for the Ricci coefficients ψ for $p\in[1,\infty]$, $i\in\mathbb{N}$:

$$\mathcal{O}_{i,p}\left[\psi\right] := \left\|\nabla^i\psi\right\|_{L^\infty_u L^\infty_{\underline{u}} L^p(S)}. \tag{14}$$

Define the following norms for the Ricci coefficients ψ_H for $p\in[1,\infty]$, $i\in\mathbb{N}$:

$$\mathcal{O}_{i,p}\left[\psi_H\right] := \left\|f(\underline{u})\nabla^i\psi_H\right\|_{L^2_{\underline{u}} L^\infty_u L^p(S)}. \tag{15}$$

Similarly, we define the following norms for the Ricci coefficients $\psi_{\underline{H}}$ for $p\in[1,\infty]$, $i\in\mathbb{N}$:

$$\mathcal{O}_{i,p}\left[\psi_{\underline{H}}\right] := \left\|f(u)\nabla^i\psi_{\underline{H}}\right\|_{L^2_u L^\infty_{\underline{u}} L^p(S)}. \tag{16}$$

As a shorthand, we define the following norm combining all of the norms above:

$$\mathcal{O}_{i,p} := \sum_{\psi\in\{\eta,\underline{\eta}\}} \mathcal{O}_{i,p}\left[\psi\right] + \sum_{\psi_H\in\{tr\chi,\hat{\chi},\omega\}} \mathcal{O}_{i,p}\left[\psi_H\right] + \sum_{\psi_{\underline{H}}\in\{tr\underline{\chi},\underline{\hat{\chi}},\underline{\omega}\}} \mathcal{O}_{i,p}\left[\psi_{\underline{H}}\right].$$

We make two remarks concerning these norms.

Remark 7. While the norms for ψ_H and $\psi_{\underline{H}}$ are based on L^2 in $\underline{u}$ and u, respectively, by virtue of the weights $f(\underline{u})$ and $f(u)$, they actually control the L^1 norms. More precisely, since $\int_0^{\underline{u}_*} \frac{1}{f^2(\underline{u}')}d\underline{u}' < \epsilon^2$ and $\int_0^{u_*} \frac{1}{f^2(u')}du' < \epsilon^2$, by the Cauchy–Schwarz inequality we have

$$\left\|\nabla^i\psi_H\right\|_{L^1_{\underline{u}} L^\infty_u L^p(S)} \leq C\epsilon\mathcal{O}_{i,p}\left[\psi_H\right]$$

and

$$\left\|\nabla^i\psi_{\underline{H}}\right\|_{L^1_u L^\infty_{\underline{u}} L^p(S)} \leq C\epsilon\mathcal{O}_{i,p}\left[\psi_{\underline{H}}\right].$$

Remark 8. The norm $\mathcal{O}_{i,p}[\psi_H]$ (resp. $\mathcal{O}_{i,p}[\psi_{\underline{H}}]$) allows us to first take L^∞ along the u direction (resp. $\underline{u}$ direction) before the L^2 norm in $\underline{u}$ (resp. u) is taken. This is stronger than the norms such that the order is reversed; i.e., we have

$$\left\|f(\underline{u})\nabla^i\psi_H\right\|_{L^\infty_u L^2_{\underline{u}} L^p(S)} \leq C\mathcal{O}_{i,p}\left[\psi_H\right]$$

and

$$\left\|f(u)\nabla^i\psi_{\underline{H}}\right\|_{L^\infty_{\underline{u}} L^2_{\underline{u}} L^p(S)} \leq C\mathcal{O}_{i,p}\left[\psi_{\underline{H}}\right].$$

In addition to the above norms, we need to define norms for the highest derivatives for the Ricci coefficients. Let

$$\begin{aligned}\tilde{\mathcal{O}}_{4,2} := &\left\|f(\underline{u})^2\nabla^4\mathrm{tr}\,\chi\right\|_{L^\infty_u L^\infty_{\underline{u}} L^2(S)} + \left\|f(u)^2\nabla^4\mathrm{tr}\,\underline{\chi}\right\|_{L^\infty_u L^\infty_{\underline{u}} L^2(S)}\\ &+\left\|f(\underline{u})\nabla^4(\hat{\chi},\omega)\right\|_{L^\infty_u L^2_{\underline{u}} L^2(S)} + \left\|f(u)\nabla^4(\eta,\underline{\eta})\right\|_{L^\infty_u L^2_{\underline{u}} L^2(S)}\\ &+\left\|f(u)\nabla^4(\underline{\hat{\chi}},\underline{\omega})\right\|_{L^\infty_{\underline{u}} L^2_u L^2(S)} + \left\|f(\underline{u})\nabla^4(\eta,\underline{\eta})\right\|_{L^\infty_{\underline{u}} L^2_u L^2(S)}.\end{aligned} \tag{17}$$

Remark 9. Here, note that for the norms for $\hat{\chi}$, ω, η, $\underline{\eta}$, $\underline{\hat{\chi}}$, and ω, L^∞ in $\underline{u}$ (or u) is taken after L^2 in u (or $\underline{u}$). According to Remark 8, this is weaker than the $\mathcal{O}_{i,2}$ norms defined above.

Remark 10. Notice that the norms for the fourth derivatives of η and $\underline{\eta}$ come with a weight $f(u)$ or $f(\underline{u})$. This is in contrast to the lower-order derivatives for η and $\underline{\eta}$, which can be estimated in $L^\infty_u L^\infty_{\underline{u}}$ without any degeneration. The degeneration here arises from the fact that these higher-order derivatives are recovered from the energy estimates for $\nabla^3 K$. These energy estimates for $\nabla^3 K$, which are derived simultaneously with the estimates for the singular components $\nabla^3\beta$ or $\nabla^3\underline{\beta}$, have a degeneration either in $\underline{u}$ or u.

We also define the curvature norms for the curvature components. For $i\in\mathbb{N}$, let

$$\begin{aligned}\mathcal{R}_i := &\left\|f(\underline{u})\nabla^i\beta\right\|_{L^\infty_u L^2_{\underline{u}} L^2(S)} + \left\|f(u)\nabla^i(K,\check{\sigma})\right\|_{L^\infty_u L^2_{\underline{u}} L^2(S)}\\ &+\left\|f(\underline{u})\nabla^i(K,\check{\sigma})\right\|_{L^\infty_{\underline{u}} L^2_u L^2(S)} + \left\|f(u)\nabla^i\underline{\beta}\right\|_{L^\infty_{\underline{u}} L^2_u L^2(S)}.\end{aligned} \tag{18}$$

As a shorthand, we also let

$$\mathcal{R} := \sum_{i\leq 3}\mathcal{R}_i.$$

Finally, let $\mathcal{O}_{\mathrm{ini}}$ and $\mathcal{R}_{\mathrm{ini}}$ denote the corresponding norms for the initial data, i.e.,

$$\begin{aligned}\mathcal{O}_{\mathrm{ini}} := &\sum_{i\leq 3}\Big(\left\|\nabla^i\psi\right\|_{L^\infty_{\underline{u}} L^2(S_{0,\underline{u}})} + \left\|\nabla^i\psi\right\|_{L^\infty_u L^2(S_{u,0})}\\ &\qquad+\left\|f(\underline{u})\nabla^i\psi_H\right\|_{L^2_{\underline{u}} L^2(S_{0,\underline{u}})} + \left\|f(u)\nabla^i\psi_{\underline{H}}\right\|_{L^2_u L^2(S_{u,0})}\Big)\\ &+\left\|f(\underline{u})^2\nabla^4\mathrm{tr}\,\chi\right\|_{L^\infty_{\underline{u}} L^2(S_{0,\underline{u}})} + \left\|f(u)^2\nabla^4\mathrm{tr}\,\underline{\chi}\right\|_{L^\infty_u L^2(S_{u,0})}\\ &+\left\|\nabla^4\mathrm{tr}\,\underline{\chi}\right\|_{L^\infty_{\underline{u}} L^2(S_{0,\underline{u}})} + \left\|\nabla^4\mathrm{tr}\,\chi\right\|_{L^\infty_u L^2(S_{u,0})}\\ &+\left\|f(\underline{u})\nabla^4\left(\hat{\chi},\omega\right)\right\|_{L^2_{\underline{u}} L^2(S_{0,\underline{u}})} + \left\|\nabla^4\left(\eta,\underline{\eta}\right)\right\|_{L^2_{\underline{u}} L^2(S_{0,\underline{u}})}\\ &+\left\|f(u)\nabla^4\left(\underline{\hat{\chi}},\underline{\omega}\right)\right\|_{L^2_u L^2(S_{u,0})} + \left\|\nabla^4\left(\eta,\underline{\eta}\right)\right\|_{L^2_u L^2(S_{u,0})}\end{aligned}$$

and

$$\mathcal{R}_{\text{ini}} := \sum_{i\leq 3}\Big(\|f(\underline{u})\nabla^i\beta\|_{L^2_{\underline{u}}L^2(S_{0,\underline{u}})} + \|\nabla^i(K,\check{\sigma})\|_{L^2_{\underline{u}}L^2(S_{0,\underline{u}})} + \|\nabla^i(K,\check{\sigma})\|_{L^2_u L^2(S_{u,0})} + \|f(u)\nabla^i\underline{\beta}\|_{L^2_u L^2(S_{u,0})}\Big).$$

In order to prove Theorem 2, we will establish a priori estimates for the geometric quantities in the above norms:

Theorem 5. *Assume that the initial data for the characteristic initial value problem satisfy the assumptions of Theorem 2 with ϵ sufficiently small. Then there exists B depending only on D and d such that*

$$\sum_{i\leq 3}\mathcal{O}_{i,2} + \tilde{\mathcal{O}}_{4,2} + \mathcal{R} \leq B.$$

In the remainder of the paper, we will focus on the proof of Theorem 5 (after constructing initial data sets in the next section). Standard methods show that Theorem 5 implies Theorem 2. We will omit the details and refer the readers to [5, 19] for a proof that the a priori estimates imply the existence theorem.

Remark 11. The assumptions of Theorem 2 imply the boundedness of the following weighted L^2 norms of the curvature components:

$$\sum_{i\leq 3}\|f(\underline{u})\nabla^i\beta\|_{L^2_{\underline{u}}L^2(S_{0,\underline{u}})} + \sum_{i\leq 3}\|\nabla^i(K,\check{\sigma})\|_{L^2_{\underline{u}}L^2(S_{0,\underline{u}})} \leq \tilde{D}$$

and

$$\sum_{i\leq 3}\|f(u)\nabla^i\underline{\beta}\|_{L^2_u L^2(S_{u,0})} + \sum_{i\leq 3}\|\nabla^i(K,\check{\sigma})\|_{L^2_u L^2(S_{u,0})} \leq \tilde{D}$$

for some $\tilde{D}$ depending only on D and d. These estimates for β, $\check{\sigma}$, and $\underline{\beta}$ follow immediately from the constraint equations on the 2-spheres (see (10)). The bound for K follows after integrating the null Bianchi equations for K on each of the initial null hypersurfaces (see (12)).[20] In particular the assumptions of Theorem 2 imply that

$$\mathcal{O}_{\text{ini}} + \mathcal{R}_{\text{ini}} \leq \tilde{D}.$$

4. Construction of initial data set

In this section we construct initial data sets satisfying the assumptions of Theorems 2 and 4. In particular we show that the constraint equations can be solved for $|\hat{\chi}(0,\underline{u})| \sim (f(\underline{u}))^{-2}$ and $|\underline{\hat{\chi}}(u,0)| \sim (f(u))^{-2}$. Our approach in this section follows closely that of Christodoulou in [5, Chapter 2].

Assume for simplicity that $S_{0,0}$ is a standard sphere of radius 1. Introduce[21] the standard stereographic coordinates (θ^1,θ^2) such that the standard metric $\overset{\circ}{\gamma}$ on the

[20] Notice that it is precisely for the initial bound for K that we require an extra derivative for χ on H_0 (and $\underline{\chi}$ on $\underline{H}_0$) in the assumptions of the theorem. This is related to the intrinsic loss of derivatives for the characteristic initial value problem for second-order hyperbolic systems (see [21]).

[21] While we only write down one coordinate chart, it is implicit that we have two stereographic charts—the north pole chart and the south pole chart. In the following, when we derive the estimates for the geometric quantities, we only prove the bounds in a sufficiently large ball B_ρ in each of these charts.

sphere takes the form

$$\overset{\circ}{\gamma}_{AB} = \frac{\delta_{AB}}{\left(1+\frac{1}{4}|\theta|^2\right)^2}.$$

Clearly, it suffices to construct initial data on H_0 (with $0 \leq \underline{u} < \underline{u}_* $ for $\underline{u}_* \leq \epsilon$). The construction on $\underline{H}_0$ is similar. On H_0, we set $\Omega = 1$ and therefore $e_4 = \frac{\partial}{\partial \underline{u}}$. We will construct a metric on H_0 in the $(\underline{u}, \theta^1, \theta^2)$ coordinates taking the form

$$\gamma_{AB} = \Phi^2 \hat{\gamma}_{AB}, \quad \text{where } \hat{\gamma}_{AB} = \frac{m_{AB}}{\left(1+\frac{1}{4}|\theta|^2\right)^2}, \tag{19}$$

and $\det m_{AB} = 1$ and $\Phi \restriction_{S_{0,0}} = 1$. In order to ensure that m satisfies $\det m = 1$, we write

$$m = \exp \Psi,$$

with $\Psi \in \hat{S}$, where $\hat{S}$ denotes the set of all matrices taking the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

We will impose upper and lower bounds on Ψ. Since there are no smooth globally non-vanishing $\Psi \in \hat{S}$ on the 2-sphere, we use the convention that $\lesssim$ denotes that the quantity is bounded above by a uniform constant, while $\sim$ denotes that the quantity is bounded above by a uniform constant, and is bounded below at every (θ^1, θ^2) by a constant depending on (θ^1, θ^2) (where the constant is moreover allowed to vanish at finitely many isolated points). We require $\Psi \in \hat{S}$ to satisfy[22]

$$\sum_{|J| \leq N} \left| \left(\frac{\partial}{\partial\theta}\right)^J \Psi \right| \lesssim 1, \qquad \sum_{|J| \leq N} \left| \left(\frac{\partial}{\partial\theta}\right)^J \frac{\partial}{\partial \underline{u}} \Psi \right| \lesssim f(\underline{u})^{-2}, \qquad \left| \frac{\partial}{\partial \underline{u}} \Psi \right| \sim f(\underline{u})^{-2} \tag{20}$$

for some sufficiently large integer N. Following [5], we have

$$\hat{\chi}_{AB} = \frac{1}{2}\Phi^2 \frac{\partial}{\partial \underline{u}} \hat{\gamma}_{AB}, \quad \operatorname{tr}\chi = \frac{2}{\Phi}\frac{\partial \Phi}{\partial \underline{u}}. \tag{21}$$

We can also derive that

$$\|\hat{\chi}\|_\gamma^2 = \frac{1}{4}\left(\hat{\gamma}^{-1}\right)^{AC}\left(\hat{\gamma}^{-1}\right)^{BD} \frac{\partial}{\partial \underline{u}}\hat{\gamma}_{AB} \frac{\partial}{\partial \underline{u}}\hat{\gamma}_{CD}.$$

Thus by (20), we have

$$|\hat{\chi}|_\gamma^2 \sim f(\underline{u})^{-4}. \tag{22}$$

In particular this implies the requirement in Theorem 4 is satisfied if $\int_0^{\underline{u}_*} f(\underline{u})^{-4} d\underline{u} = \infty$. By the equation

$$\not{\mathcal{L}}_{\frac{\partial}{\partial \underline{u}}} \operatorname{tr}\chi = -\frac{1}{2}(\operatorname{tr}\chi)^2 - |\hat{\chi}|^2,$$

Φ can be solved from the ODE

$$\frac{\partial^2 \Phi}{\partial \underline{u}^2} + \frac{1}{8}\left(\left(\hat{\gamma}^{-1}\right)^{AC}\left(\hat{\gamma}^{-1}\right)^{BD} \frac{\partial}{\partial \underline{u}}\hat{\gamma}_{AB} \frac{\partial}{\partial \underline{u}}\hat{\gamma}_{CD}\right)\Phi = 0. \tag{23}$$

We prescribe $\operatorname{tr}\chi$ on $S_{0,0}$ to obey the initial conditions

$$\Phi \restriction_{S_{0,0}} = 1, \qquad \frac{\partial \Phi}{\partial \underline{u}} \restriction_{S_{0,0}} = \frac{1}{2}\operatorname{tr}\chi \restriction_{S_{0,0}} \lesssim 1. \tag{24}$$

[22]Here and in the rest of this section, we use the notation that $J = (j_1, j_2) \in (\mathbb{N}\cup\{0\}) \times (\mathbb{N}\cup\{0\})$ is a multi-index and $(\frac{\partial}{\partial\theta})^J = (\frac{\partial}{\partial\theta^1})^{j_1}(\frac{\partial}{\partial\theta^2})^{j_2}$. We moreover denote $|J| = j_1 + j_2$.

Finally, we prescribe ζ on $S_{0,0}$ such that

$$\sum_{|J|\leq N-1}\left|\left(\frac{\partial}{\partial\theta}\right)^J\zeta\right|_\gamma^2\lesssim 1. \tag{25}$$

We check that these initial data obey all the estimates required by Theorem 2:

Estimates for $\nabla^i\chi$ and the metric.

To satisfy the upper bounds in Theorem 2, we need to show that

$$\sum_{i\leq N}\left|\nabla^i\chi\right|_\gamma(0,\underline{u})\lesssim f(\underline{u})^{-2}. \tag{26}$$

We will show the estimates separately for $\operatorname{tr}\chi$ and $\hat{\chi}$. By (22), (26) holds for $\hat{\chi}$ when $i=0$. To derive this bound for $\operatorname{tr}\chi$, notice that by the ODE (23) for Φ, the initial conditions (24), and the bound (22) for $|\hat{\chi}|^2$, we have

$$\frac{1}{2}\leq\Phi\leq 1 \tag{27}$$

and

$$\left|\frac{\partial\Phi}{\partial\underline{u}}\right|\lesssim 1+\int_0^{\underline{u}}f\left(\underline{u}'\right)^{-4}d\underline{u}'\leq 1+f\left(\underline{u}\right)^{-2}\int_0^{\underline{u}_*}f\left(\underline{u}'\right)^{-2}d\underline{u}'\leq 1+\epsilon^2 f\left(\underline{u}\right)^{-2}$$

for ϵ sufficiently small. In the above estimate, we have used $\int_0^{\underline{u}_*}f(\underline{u}')^{-2}\,d\underline{u}'\leq\epsilon^2$. By (21), we thus have

$$\|\operatorname{tr}\chi\|\lesssim f(\underline{u})^{-2}.$$

We now move on to control the angular derivatives of χ. By (20),

$$\sum_{|J|\leq N}\left|\left(\frac{\partial}{\partial\theta}\right)^J\frac{\partial}{\partial\underline{u}}m_{AB}\right|\lesssim f\left(\underline{u}\right)^{-2}.$$

Using this bound and commuting the ODE (23) with $\frac{\partial}{\partial\theta}$, we also have that for up to N coordinate angular derivatives $\frac{\partial}{\partial\theta}$,

$$\sum_{|J|\leq N}\left|\left(\frac{\partial}{\partial\theta}\right)^J\Phi\right|\lesssim 1. \tag{28}$$

This implies via (19) and (20) that the metric γ obeys the bounds

$$\sum_{|J|\leq N}\left|\left(\frac{\partial}{\partial\theta}\right)^J\gamma_{AB}\right|\lesssim 1,\qquad\sum_{|J|\leq N}\left|\left(\frac{\partial}{\partial\theta}\right)^J(\gamma^{-1})^{AB}\right|\lesssim 1. \tag{29}$$

Together with (20) and (21), (28) implies

$$\sum_{|J|\leq N}\left|\left(\frac{\partial}{\partial\theta}\right)^J\hat{\chi}\right|\lesssim f(\underline{u})^{-2}. \tag{30}$$

By (21), we also have

$$\sum_{|J|\leq N}\left|\left(\frac{\partial}{\partial\theta}\right)^J\operatorname{tr}\chi\right|\lesssim f(\underline{u})^{-2}. \tag{31}$$

Finally, we notice that by (29), the angular *covariant* derivatives of $\mathrm{tr}\,\chi$ and $\hat{\chi}$ can be controlled by the angular coordinate derivatives of $\mathrm{tr}\,\chi$ and $\hat{\chi}$. Therefore, (26) follows from (30) and (31).

Estimates for $\nabla^i K$.
To control $\nabla^i K$, we simply notice that by (29), we have

$$\sum_{i\leq N-2} \left|\nabla^i K\right|_\gamma \lesssim 1.$$

Estimates for $\nabla^i \zeta$.
On H_0, since $\Omega = 1$, $\eta = \zeta$. Thus combining the transport equation for ζ in (9) and the Codazzi equation for β in (10), and rewriting in $\not{\mathcal{L}}$ (instead of ∇_4), we have

$$\not{\mathcal{L}}_{\frac{\partial}{\partial \underline{u}}}\zeta + \mathrm{tr}\,\chi\zeta = \mathrm{div}\,\chi - \nabla \mathrm{tr}\,\chi.$$

Recall from (25) that the initial data for ζ and its angular derivatives are bounded. Therefore, by the estimates for $\mathrm{tr}\,\chi$ and $\hat{\chi}$ (and their angular derivatives) above, we have

$$\sum_{|J|\leq N-1} \left\| \left(\frac{\partial}{\partial\theta}\right)^J \zeta \right\| \lesssim 1.$$

The bounds for the metric and Christoffel symbols on the sphere imply

$$\sum_{j\leq N-1} \left\|\nabla^j \zeta\right\|_{L^\infty_{\underline{u}} L^\infty(S_{0,\underline{u}})} \lesssim 1$$

as desired.

Estimates for $\nabla^i \mathrm{tr}\,\underline{\chi}$.
Similarly to ζ, $\mathrm{tr}\,\underline{\chi}$ obeys a transport equations along the null generators of H_0. More precisely, (9) and the Gauss equation in (10) imply that

$$\not{\mathcal{L}}_{\frac{\partial}{\partial \underline{u}}}\mathrm{tr}\,\underline{\chi} + \mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} = -2K - 2\mathrm{div}\,\zeta + 2|\zeta|^2.$$

Thus, the previous estimates imply

$$\sum_{j\leq N-2} \left\|\nabla^j \mathrm{tr}\,\underline{\chi}\right\|_{L^\infty_{\underline{u}} L^\infty(S_{0,\underline{u}})} \lesssim 1.$$

Now, combining all the estimates that we have obtained so far, requiring f to satisfy

$$\int_0^{\underline{u}_*} f(\underline{u})^{-4} d\underline{u} = \infty$$

and taking N to sufficiently large, we have thus constructed initial data set on H_0 that obeys the assumptions of Theorems 2 and 4 on H_0. As mentioned above, it is easy to construct initial data set analogously on $\underline{H}_0$ so that the full set of assumptions of Theorems 2 and 4 are satisfied.

5. The preliminary estimates

We now turn to the proof Theorem 5, which will form the content of sections 5–8. In this section we derive the necessary preliminary estimates. In section 6 (see Proposition 15), we will prove the bound

$$\sum_{i\leq 3}\mathcal{O}_{i,2}\leq C(\mathcal{O}_{\mathrm{ini}});$$

in section 7 (see Proposition 25), we will prove

$$\tilde{\mathcal{O}}_{4,2}\leq C(\mathcal{O}_{\mathrm{ini}})(1+\mathcal{R});$$

and in section 8 (see Proposition 32), we will derive the estimate

$$\mathcal{R}\leq C(\mathcal{O}_{\mathrm{ini}},\mathcal{R}_{\mathrm{ini}}).$$

Combining these estimates then implies the conclusion of Theorem 5.

We now begin with the preliminary estimates. All estimates in this section will be proved under the bootstrap assumption

$$\sum_{i\leq 1}\mathcal{O}_{i,\infty}+\sum_{i\leq 2}\mathcal{O}_{i,4}+\sum_{i\leq 3}\mathcal{O}_{i,2}\leq\Delta_1, \tag{A1}$$

where Δ_1 is a constant that will be chosen later.

5.1. **Estimates for metric components.** We first show that we can control Ω under the bootstrap assumption (A1):

Proposition 1. *There exists $\epsilon_0=\epsilon_0(\Delta_1)$ such that for every $\epsilon\leq\epsilon_0$,*

$$\frac{1}{2}\leq\Omega\leq 2.$$

Moreover, Ω is continuous up to $u=u_*$ and $\underline{u}=\underline{u}_*$.

Proof. Consider the equation

$$\omega=-\frac{1}{2}\nabla_4\log\Omega=\frac{1}{2}\Omega\nabla_4\Omega^{-1}=\frac{1}{2}\frac{\partial}{\partial\underline{u}}\Omega^{-1}. \tag{32}$$

Fix $\underline{u}$. Notice that both ω and Ω are scalars and therefore the L^∞ norm is independent of the metric. We can integrate equation (32) using the fact that $\Omega^{-1}=1$ on $\underline{H}_0$ to obtain

$$\begin{aligned}&\left\|\Omega^{-1}-1\right\|_{L^\infty(S_{u,\underline{u}})}\\&\quad\leq C\int_0^{\underline{u}}\|\omega\|_{L^\infty(S_{u,\underline{u}'})}\,d\underline{u}'\leq C\left\|f(\underline{u})^{-1}\right\|_{L^2_{\underline{u}}}\|f(\underline{u})\omega\|_{L^\infty_uL^2_{\underline{u}}L^\infty(S)}\leq C\Delta_1\epsilon.\end{aligned}$$

This implies both the upper and lower bounds for Ω for sufficiently small ϵ. To show continuity, take a sequence of points $(u_n,\underline{u}_n,\theta^1_n,\theta^2_n)$ such $u_n\to u_\infty$, $\underline{u}_n\to\underline{u}_\infty$,

$\theta_n^1 \to \theta_\infty^1$, and $\theta_n^2 \to \theta_\infty^2$. Then

$$\begin{aligned}
&\left|\Omega^{-1}\left(u_n, \underline{u}_n, \theta_n^1, \theta_n^2\right) - \Omega^{-1}\left(u_m, \underline{u}_m, \theta_m^1, \theta_m^2\right)\right| \\
&\quad \le \left|\Omega^{-1}\left(u_n, \underline{u}_n, \theta_n^1, \theta_n^2\right) - \Omega^{-1}\left(u_n, \underline{u}_n, \theta_m^1, \theta_m^2\right)\right| \\
&\qquad + \left|\Omega^{-1}\left(u_n, \underline{u}_n, \theta_m^1, \theta_m^2\right) - \Omega^{-1}\left(u_n, \underline{u}_m, \theta_m^1, \theta_m^2\right)\right| \\
&\qquad + \left|\Omega^{-1}\left(u_n, \underline{u}_m, \theta_m^1, \theta_m^2\right) - \Omega^{-1}\left(u_m, \underline{u}_m, \theta_m^1, \theta_m^2\right)\right| \\
&\quad \le C \left\|\nabla \log \Omega\right\|_{L^\infty\left(S_{u_n, \underline{u}_n}\right)} \mathrm{dist}_{S_{u_n, \underline{u}_n}}\left(\theta_n, \theta_m\right) + 2\left|\int_{\underline{u}_n}^{\underline{u}_m} \|\omega\|_{L^\infty\left(S_{u_n, \underline{u}'}\right)} d\underline{u}'\right| \\
&\qquad + 2\left|\int_{u_n}^{u_m} \|\underline{\omega}\|_{L^\infty\left(S_{u', \underline{u}_m}\right)} du'\right|.
\end{aligned}$$

Since by the bootstrap assumption (A1), $\nabla \log \Omega = \frac{1}{2}(\eta + \underline{\eta})$ is uniformly bounded, $||\omega||_{L^\infty(S_{u,\underline{u}})}$ is uniformly integrable in $\underline{u}$ for all u, and $||\underline{\omega}||_{L^\infty(S_{u,\underline{u}})}$ is uniformly integrable in u for all $\underline{u}$, the right-hand side can be made arbitrarily small by taking $n, m \ge N$ for N sufficiently large. The conclusion thus follows. ☐

We then show that we can control γ under the bootstrap assumption (A1):

Proposition 2. *There exists $\epsilon_0 = \epsilon_0(\Delta_1)$ such that for $\epsilon \le \epsilon_0$, in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system, we have*

$$c \le \det \gamma \le C, \quad |\gamma_{AB}|, \left|\left(\gamma^{-1}\right)^{AB}\right| \le C,$$

where the constants depend only on d and D. Moreover, γ remains continuous up to $u = u_*$ and $\underline{u} = \underline{u}_*$.

Proof. We first prove the bound for γ on the initial hypersurface $\underline{H}_0$. Using

$$\not{\mathcal{L}}_{\underline{L}} \gamma = 2\Omega \underline{\chi},$$

we get[23]

$$\frac{\partial}{\partial u} \gamma_{AB} = 2\Omega \underline{\chi}_{AB}, \qquad \frac{\partial}{\partial u} \log(\det \gamma) = \Omega \mathrm{tr}\, \underline{\chi}$$

on $\underline{H}_0$. We therefore have

$$\left|\frac{\det \gamma(u,0)}{\det \gamma(0,0)}\right| \le C \exp\left(\int_0^u \left|\mathrm{tr}\, \underline{\chi}\right|(u', 0)\, du'\right) \le C(D). \tag{33}$$

This implies that the $\det \gamma$ is bounded above and below. Let Λ be the larger eigenvalue of γ. Clearly,

$$\Lambda \le C \sup_{A,B=1,2} |\gamma_{AB}|, \qquad \sum_{A,B=1,2} \left|\underline{\chi}_{AB}\right|^2 \le C\Lambda^2 \left\|\underline{\chi}\right\|^2_{L^\infty\left(S_{u,\underline{u}}\right)}. \tag{34}$$

Then

$$\begin{aligned}
&|\gamma_{AB}(u,0) - (\gamma)_{AB}(0,0)| \\
&\quad \le C\epsilon \left(\sup_{u' \le u} \Lambda\right) \left(\int_0^u f(u')^2 \left\|\underline{\chi}\right\|^2_{L^\infty\left(S_{u',0}\right)} du'\right)^{\frac{1}{2}} \le C(D) \left(\sup_{u' \le u} \Lambda\right) \epsilon.
\end{aligned} \tag{35}$$

Using the first upper bound in (34), we thus obtain the upper bound for $|\gamma_{AB}|$ after choosing ϵ to be sufficiently small. The upper bound for $|(\gamma^{-1})^{AB}|$ follows from the upper bound for $|\gamma_{AB}|$ and the lower bound for $\det \gamma$.

[23]Note that $b^A = 0$ on $\underline{H}_0$.

Now, in order to obtain the bounds for γ_{AB} in the spacetime, we argue similarly but use the propagation equation in the $\underline{u}$ direction and compare $\gamma(u,\underline{u})$ with $\gamma(u,0)$. Here, we use bootstrap assumption (A1) instead of the assumptions on the initial data. More precisely, we have

$$\frac{\partial}{\partial \underline{u}}\gamma_{AB} = 2\Omega\chi_{AB}, \qquad \frac{\partial}{\partial \underline{u}}\log(\det\gamma) = \Omega \mathrm{tr}\,\chi. \tag{36}$$

We then derive as above that

$$\left|\frac{\det\gamma(u,\underline{u})}{\det\gamma(u,0)}\right| \leq C\exp(C\Delta_1\epsilon), \quad |\gamma_{AB}(u,\underline{u}) - \gamma_{AB}(u,0)| \leq C\left(\sup_{\substack{u'\leq u\\ \underline{u}'\leq\underline{u}}}\Lambda\right)\Delta_1\epsilon,$$

where Λ is the larger eigenvalue for γ_{AB}. As before, we thus obtain the upper bounds for $|\gamma_{AB}|$ and $|(\gamma^{-1})^{AB}|$. Finally, the continuity of γ up to the boundary follows as in the proof of continuity for Ω in Proposition 1. □

With the estimates on γ, it follows that the L^p norms defined with respect to the metric and the L^p norms defined with respect to the coordinate system are equivalent.

Proposition 3. *Given a covariant tensor $\phi_{A_1\cdots A_r}$ on $S_{u,\underline{u}}$, we have*

$$\int_{S_{u,\underline{u}}}\langle\phi,\phi\rangle_\gamma^{p/2} \sim \sum_{A_i=1,2}\iint |\phi_{A_1\cdots A_r}|^p\sqrt{\det\gamma}\,d\theta^1 d\theta^2.$$

We can also bound b under the bootstrap assumption, thus controlling the full spacetime metric:

Proposition 4. *In the coordinate system $(u,\underline{u},\theta^1,\theta^2)$,*

$$\left|b^A\right| \leq C\Delta_1\epsilon.$$

Moreover, b^A is continuous up to $u = u_*$ and $\underline{u} = \underline{u}_*$.

Proof. b^A satisfies the equation

$$\frac{\partial b^A}{\partial \underline{u}} = -4\Omega^2\zeta^A. \tag{37}$$

This can be derived from

$$[L,\underline{L}] = \frac{\partial b^A}{\partial \underline{u}}\frac{\partial}{\partial\theta^A}.$$

Now, integrating (37) and using Proposition 3 gives the bound on b. Continuity of b up to the boundary follows as in the proof of Proposition 1. □

5.2. **Estimates for transport equations.** In this subsection, we prove general propositions for obtaining bounds from the covariant null transport equations. Such estimates require the integrability of $\mathrm{tr}\,\chi$ and $\mathrm{tr}\,\underline{\chi}$, which is consistent with our bootstrap assumption (A1). This will be used in the following sections to derive some estimates for the Ricci coefficients and the null curvature components from the null structure equations and the null Bianchi equations, respectively. Below, we state two propositions which provide L^p estimates for general quantities satisfying transport equations either in the e_3 or e_4 direction.

Proposition 5. *There exists* $\epsilon_0 = \epsilon_0(\Delta_1)$ *such that for all* $\epsilon \le \epsilon_0$ *and for every* $2 \le p < \infty$*, we have*

$$||\phi||_{L^p(S_{u,\underline{u}})} \le C(||\phi||_{L^p(S_{u,\underline{u}'})} + \int_{\underline{u}'}^{\underline{u}} ||\nabla_4\phi||_{L^p(S_{u,\underline{u}''})} d\underline{u}''),$$

$$||\phi||_{L^p(S_{u,\underline{u}})} \le C(||\phi||_{L^p(S_{u',\underline{u}})} + \int_{u'}^{u} ||\nabla_3\phi||_{L^p(S_{u'',\underline{u}})} du'')$$

for any tensor ϕ *tangential to the spheres* $S_{u,\underline{u}}$*.*

Proof. The following identity[24] holds for any scalar f:

$$\frac{\partial}{\partial \underline{u}} \int_{\mathcal{S}_{u,\underline{u}}} f = \int_{\mathcal{S}_{u,\underline{u}}} \Omega \left(e_4(f) + \mathrm{tr}\,\chi f\right).$$

Similarly, we have

$$\frac{\partial}{\partial u} \int_{\mathcal{S}_{u,\underline{u}}} f = \int_{\mathcal{S}_{u,\underline{u}}} \Omega \left(e_3(f) + \mathrm{tr}\,\underline{\chi} f\right).$$

Hence, taking $f = |\phi|^p_\gamma$, we have

(38)

$$||\phi||^p_{L^p(S_{u,\underline{u}})} = ||\phi||^p_{L^p(S_{u,\underline{u}'})} + \int_{\underline{u}'}^{\underline{u}} \int_{S_{u,\underline{u}''}} p|\phi|^{p-2}\Omega \left(\langle \phi, \nabla_4\phi\rangle_\gamma + \frac{1}{p}\mathrm{tr}\,\chi |\phi|^2_\gamma \right) d\underline{u}'',$$

$$||\phi||^p_{L^p(S_{u,\underline{u}})} = ||\phi||^p_{L^p(S_{u',\underline{u}})} + \int_{u'}^{u} \int_{S_{u'',\underline{u}}} p|\phi|^{p-2}\Omega \left(\langle \phi, \nabla_3\phi\rangle_\gamma + \frac{1}{p}\mathrm{tr}\,\underline{\chi} |\phi|^2_\gamma \right) du''.$$

The bootstrap assumption (A1) implies that $\mathrm{tr}\,\chi$ and $\mathrm{tr}\,\underline{\chi}$ are integrable (and in fact it also implies that $||\mathrm{tr}\,\chi||_{L^1_{\underline{u}} L^\infty_u L^\infty(S)}$ and $||\mathrm{tr}\,\underline{\chi}||_{L^1_u L^\infty_{\underline{u}} L^\infty(S)}$ are small after choosing ϵ to be small depending on Δ_1). Thus the proposition can be proved by using Hölder's inequality and Gronwall's inequality, together with the bound for Ω given in Proposition 1. □

We also have the following bounds for the $p = \infty$ case by integrating along the integral curves of e_3 and e_4:

Proposition 6. *There exists* $\epsilon_0 = \epsilon_0(\Delta_1)$ *such that for all* $\epsilon \le \epsilon_0$*, we have*

$$||\phi||_{L^\infty(S_{u,\underline{u}})} \le C\left(||\phi||_{L^\infty(S_{u,\underline{u}'})} + \int_{\underline{u}'}^{\underline{u}} ||\nabla_4\phi||_{L^\infty(S_{u,\underline{u}''})} d\underline{u}''\right),$$

$$||\phi||_{L^\infty(S_{u,\underline{u}})} \le C\left(||\phi||_{L^\infty(S_{u',\underline{u}})} + \int_{u'}^{u} ||\nabla_3\phi||_{L^\infty(S_{u'',\underline{u}})} du''\right)$$

for any tensor ϕ *tangential to the spheres* $S_{u,\underline{u}}$*.*

Proof. This follows simply from integrating along the integral curves of L and $\underline{L}$, and the estimate on Ω in Proposition 1. □

[24] Here, $\frac{\partial}{\partial \underline{u}}$ on the left-hand side is to be understood as the coordinate vector field in the $(u, \underline{u})$-plane. Similarly for $\frac{\partial}{\partial u}$ below.

5.3. **Sobolev embedding.** Using the estimates for the metric γ in Proposition 2, we have the following Sobolev embedding theorem:

Proposition 7. *There exists* $\epsilon_0 = \epsilon_0(\Delta_1)$ *such that as long as* $\epsilon \leq \epsilon_0$*, we have*

$$||\phi||_{L^4(S_{u,\underline{u}})} \leq C \sum_{i=0}^{1} ||\nabla^i \phi||_{L^2(S_{u,\underline{u}})}$$

and

$$||\phi||_{L^\infty(S_{u,\underline{u}})} \leq C \left(||\phi||_{L^2(S_{u,\underline{u}})} + ||\nabla\phi||_{L^4(S_{u,\underline{u}})}\right)$$

for any tensor ϕ *tangential to the spheres* $S_{u,\underline{u}}$*. Combining the above estimates, we also have*

$$||\phi||_{L^\infty(S_{u,\underline{u}})} \leq C \sum_{i=0}^{2} ||\nabla^i \phi||_{L^2(S_{u,\underline{u}})}.$$

Proof. By (35) in the proof of Proposition 2, $|\gamma_{AB}(u,\underline{u}) - \gamma_{AB}(0,0)|$ can be made arbitrarily small by choosing ϵ to be small. Therefore, the isoperimetric constant

$$I(S_{u,\underline{u}}) = \sup_U \frac{\min\{\text{Area}(U), \text{Area}(U^c)\}}{\text{Perimeter}(\partial U)}$$

on every sphere $S_{u,\underline{u}}$ is controlled[25] up to a constant factor by the corresponding isoperimetric constant on $S_{0,0}$. Once the isoperimetric constants are uniformly controlled, the Sobolev embedding theorem follows from [5, Lemmas 5.1 and 5.2] and the fact that the volume of $S_{u,\underline{u}}$ is bounded uniformly above and below. □

5.4. **Commutation formulae.** We have the following formula from [6, Lemma 7.3.3]:

Proposition 8. *The commutator* $[\nabla_4, \nabla]$ *acting on a rank* r *tensor* ϕ *tangential to the spheres* $S_{u,\underline{u}}$ *is given by*

$$\begin{aligned}
&[\nabla_4, \nabla_B]\phi_{A_1\cdots A_r} \\
&\quad = (\nabla_B \log\Omega)\nabla_4\phi_{A_1\cdots A_r} - (\gamma^{-1})^{CD}\chi_{BD}\nabla_C\phi_{A_1\cdots A_r} \\
&\qquad + \sum_{i=1}^{r}((\gamma^{-1})^{CD}\chi_{A_iB}\underline{\eta}_D - (\gamma^{-1})^{CD}\chi_{BD}\underline{\eta}_{A_i} + \not\epsilon_{A_i}{}^{C*}\beta_B)\phi_{A_1\cdots\hat{A}_iC\cdots A_r}.
\end{aligned}$$

Similarly, the commutator $[\nabla_3, \nabla]$ *is given by*

$$\begin{aligned}
&[\nabla_3, \nabla_B]\phi_{A_1\cdots A_r} \\
&\quad = (\nabla_B \log\Omega)\nabla_3\phi_{A_1\cdots A_r} - (\gamma^{-1})^{CD}\underline{\chi}_{BD}\nabla_C\phi_{A_1\cdots A_r} \\
&\qquad + \sum_{i=1}^{r}((\gamma^{-1})^{CD}\underline{\chi}_{A_iB}\eta_D - (\gamma^{-1})^{CD}\underline{\chi}_{BD}\eta_{A_i} - \not\epsilon_{A_i}{}^{C*}\underline{\beta}_B)\phi_{A_1\cdots\hat{A}_iC\cdots A_r}.
\end{aligned}$$

Recall the schematic notation

$$\psi \in \{\eta, \underline{\eta}\}, \qquad \psi_H \in \{\mathrm{tr}\,\chi, \hat{\chi}, \omega\}, \qquad \psi_{\underline{H}} \in \{\mathrm{tr}\,\underline{\chi}, \hat{\underline{\chi}}, \underline{\omega}\}.$$

By induction and the schematic Codazzi equations

$$\beta = \nabla\chi + \psi\chi = \nabla\psi_H + \psi\psi_H, \qquad \underline{\beta} = \nabla\underline{\chi} + \psi\underline{\chi} = \nabla\psi_{\underline{H}} + \psi\psi_{\underline{H}},$$

[25] This argument is standard. We refer the readers for instance to [5, Lemma 5.4].

we get the following schematic formula for repeated commutations (see [19]):

Proposition 9. *Suppose* $\nabla_4\phi = F_0$ *for some tensors* ϕ *and* F_0*. Let* F_i *be the tensor defined by* $\nabla_4\nabla^i\phi = F_i$*. Then*

$$F_i \sim \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}F_0 + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\phi.$$

Similarly, suppose $\nabla_3\phi = G_0$ *for some tensors* ϕ *and* G_0*. Let* G_i *be the tensor defined by* $\nabla_3\nabla^i\phi = G_i$*. Then*

$$G_i \sim \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}G_0 + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}\phi.$$

5.5. **General elliptic estimates for Hodge systems.** We recall the definition of the divergence and curl of a symmetric covariant tensor of rank $r+1$:

$$(\operatorname{div}\phi)_{A_1\cdots A_r} = \nabla^B\phi_{BA_1\cdots A_r},$$
$$(\operatorname{curl}\phi)_{A_1\cdots A_r} = \not\in^{BC}\nabla_B\phi_{CA_1\cdots A_r},$$

where $\not\in$ is the volume form associated to the metric γ. Recall also that the trace is defined to be

$$(tr\phi)_{A_1\cdots A_{r-1}} = (\gamma^{-1})^{BC}\phi_{BCA_1\cdots A_{r-1}}.$$

The following elliptic estimate is standard (see, for example, [6, Lemmas 2.2.2, 2.2.3] or [5, Lemmas 7.1, 7.2, 7.3]):

Proposition 10. *Let* ϕ *be a symmetric* r *covariant tensor on a 2-sphere* $(\mathbb{S}^2,\gamma)$ *satisfying*

$$\operatorname{div}\phi = f, \qquad \operatorname{curl}\phi = g, \qquad tr\phi = h.$$

Suppose also that

$$\sum_{i\leq 2}||\nabla^i K||_{L^2(S)} < \infty.$$

Then for $i \leq 4$*, there exists a constant* C_E *depending only on* $\sum_{i\leq 2}||\nabla^i K||_{L^2(S)}$ *such that*

$$||\nabla^i\phi||_{L^2(S)} \leq C_E\left(\sum_{j=0}^{i-1}(||\nabla^j f||_{L^2(S)} + ||\nabla^j g||_{L^2(S)} + ||\nabla^j h||_{L^2(S)} + ||\nabla^j\phi||_{L^2(S)})\right).$$

For the special case that ϕ is a symmetric traceless 2-tensor, we only need to know its divergence:

Proposition 11. *Suppose* ϕ *is a symmetric traceless 2-tensor satisfying*

$$\operatorname{div}\phi = f.$$

Suppose moreover that

$$\sum_{i\leq 2}||\nabla^i K||_{L^2(S)} < \infty.$$

Then for $i \leq 4$*, there exists a constant* C_E *depending only on* $\sum_{i\leq 2}||\nabla^i K||_{L^2(S)}$ *such that*

$$||\nabla^i\phi||_{L^2(S)} \leq C_E\left(\sum_{j=0}^{i-1}(||\nabla^j f||_{L^2(S)} + ||\nabla^j\phi||_{L^2(S)})\right).$$

Proof. This follows from Proposition 10 and the fact that

$$\operatorname{curl}\phi =^* f.$$

□

6. Estimates for the Ricci coefficients via transport equations

In this section we prove L^2 estimates for the Ricci coefficients and their first, second, and third derivatives. We will assume bounds for $\mathcal{R}$ and $\tilde{\mathcal{O}}_{4,2}$ and show that for ϵ_0 chosen to be sufficiently small, $\sum_{i\leq 3}\mathcal{O}_{i,2}$ is likewise bounded. In order to achieve this, we continue to work under the bootstrap assumption (A1) and will show that the constant in (A1) can in fact be improved (see Proposition 15).

Recall that we will use the following notation: $\psi\in\{\eta,\underline{\eta}\}$, $\psi_{\underline{H}}\in\{\mathrm{tr}\,\underline{\chi},\hat{\underline{\chi}},\underline{\omega}\}$, and $\psi_H\in\{\mathrm{tr}\,\chi,\hat{\chi},\omega\}$.

We first show bounds for ψ.

Proposition 12. *Assume*

$$\mathcal{R}<\infty.$$

Then there exists $\epsilon_0=\epsilon_0(\Delta_1,\mathcal{R})$ such that whenever $\epsilon\leq\epsilon_0$,

$$\sum_{i\leq 3}\mathcal{O}_{i,2}[\psi]\leq C(\mathcal{O}_{\mathrm{ini}}),$$

i.e., the bounds depends only on the initial data norm $\mathcal{O}_{\mathrm{ini}}$. In particular $C(\mathcal{O}_{\mathrm{ini}})$ is independent of Δ_1.

Proof. We first estimate η; the estimates for $\underline{\eta}$ are similar after we replace u with $\underline{u}$ and 3 with 4. Using the null structure equations, we have a schematic equation of the type

$$\nabla_4\eta=\beta+\psi_H\psi.$$

We also commute the null structure equations with angular derivatives to get

$$\nabla_4\nabla^i\eta=\sum_{i_1+i_2+i_3=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\beta+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_H. \tag{39}$$

By Proposition 5 in order to estimate $||\nabla^i\eta||_{L^\infty_uL^\infty_{\underline{u}}L^2(S)}$, it suffices to estimate the initial data and the $||\cdot||_{L^\infty_uL^1_{\underline{u}}L^2(S)}$ norm of the right-hand side (39). Using the bootstrap assumption, we will show that the right-hand side is bounded in a weighted $L^2_{\underline{u}}$ norm. This in turn implies via an application of the Cauchy–Schwarz inequality that the $L^1_{\underline{u}}$ norm is also bounded. We now turn to the details.

We first estimate the curvature term

$$\sum_{i_1+i_2+i_3\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\beta.$$

For the terms such that at most 1 derivative falling on ψ, the bootstrap assumption (A1) allows us to control $\sum_{i\leq 1}\|\nabla^i\psi\|_{L^\infty_uL^\infty_{\underline{u}}L^\infty(S)}$ by Δ_1. We then need to control $\sum_{i\leq 3}\nabla^i\beta$ in $L^\infty_uL^1_{\underline{u}}L^2(S)$. By the Cauchy–Schwarz inequality, since the $L^2_{\underline{u}}$ norm of $f(\underline{u})^{-1}$ is smaller than ϵ, we can bound this by $\sum_{i\leq 3}\nabla^i\beta$ in the weighted L^2 norms. More precisely, we have

$$
\begin{aligned}
&\left\|\sum_{i_1\leq 1, i_2\leq 3, i_3\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\beta\right\|_{L^\infty_u L^1_{\underline{u}} L^2(S)}\\
&\leq C\left(\sum_{i_1\leq 1, i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_u L^\infty_{\underline{u}} L^\infty(S)}\right)\left(\sum_{i_3\leq 3}||f(\underline{u})\nabla^{i_3}\beta||_{L^\infty_u L^1_{\underline{u}} L^2(S)}\right)\\
&\leq C\left(\sum_{i_1\leq 1, i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_u L^\infty_{\underline{u}} L^\infty(S)}\right)\\
&\quad\times\left(\sum_{i_3\leq 3}||f(\underline{u})\nabla^{i_3}\beta||_{L^\infty_u L^2_{\underline{u}} L^2(S)}\right)||f(\underline{u})^{-1}||_{L^\infty_u L^2_{\underline{u}} L^\infty(S)}\\
&\leq C\epsilon(1+\Delta_1)^3\mathcal{R}.
\end{aligned}
\tag{40}
$$

For the term where exactly two derivatives fall on ψ (notice that this is the highest number of derivatives that can fall on ψ), we control $\nabla^2\psi$ in $L^\infty_u L^\infty_{\underline{u}} L^2(S)$ by Δ_1 (using (A1)). Thus we are left with β in $L^\infty_u L^1_{\underline{u}} L^\infty(S)$. By Sobolev embedding (Proposition 7), this can be bounded by $\sum_{i\leq 3}\|\nabla^i\beta\|_{L^\infty_u L^1_{\underline{u}} L^2(S)}$, which in turn can be controlled by $\mathcal{R}$ after applying the Cauchy–Schwarz inequality as in (40). More precisely,

$$
\begin{aligned}
&||\nabla^2\psi\beta||_{L^\infty_u L^1_{\underline{u}} L^2(S)}\\
&\leq C||\nabla^2\psi||_{L^\infty_u L^\infty_{\underline{u}} L^2(S)}||\beta||_{L^\infty_u L^1_{\underline{u}} L^\infty(S)}\\
&\leq C||\nabla^2\psi||_{L^\infty_u L^\infty_{\underline{u}} L^2(S)}\left(\sum_{i\leq 2}||\nabla^i\beta||_{L^\infty_u L^1_{\underline{u}} L^2(S)}\right)\\
&\leq C||\nabla^2\psi||_{L^\infty_u L^\infty_{\underline{u}} L^2(S)}\left(\sum_{i\leq 2}||f(\underline{u})\nabla^i\beta||_{L^\infty_u L^2_{\underline{u}} L^\infty(S)}\right)||f(\underline{u})^{-1}||_{L^\infty_u L^2_{\underline{u}} L^\infty(S)}\\
&\leq C\epsilon\Delta_1\mathcal{R}.
\end{aligned}
\tag{41}
$$

Combining (40) and (41), we have

$$
\left\|\sum_{i_1+i_2+i_3\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\beta\right\|_{L^\infty_u L^1_{\underline{u}} L^2(S)}\leq C\epsilon(1+\Delta_1)^3\mathcal{R}.
$$

We then estimate the second term in (39). We separate the terms where more derivatives fall on ψ_H and those where more derivatives fall on ψ:

(42)

$$\begin{aligned}
&\left\|\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_H\right\|_{L^\infty_u L^1_{\underline{u}}L^2(S)}\\
&\leq C\left(\sum_{i_1\leq 1, 1\leq i_2\leq 4}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_u L^\infty_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_3\leq 3}||\nabla^{i_3}\psi_H||_{L^\infty_u L^1_{\underline{u}}L^2(S)}\right)\\
&\quad + C(1+||\psi||_{L^\infty_u L^\infty_{\underline{u}}L^\infty(S)})\left(\sum_{i_1\leq 3}||\nabla^{i_1}\psi||_{L^\infty_u L^\infty_{\underline{u}}L^2(S)}\right)\left(\sum_{i_2\leq 1}||\nabla^{i_2}\psi_H||_{L^\infty_u L^1_{\underline{u}}L^\infty(S)}\right)\\
&\leq C\Delta_1(1+\Delta_1)^3\left(\sum_{i\leq 3}||\nabla^i\psi_H||_{L^\infty_u L^1_{\underline{u}}L^2(S)}+\sum_{i\leq 1}||\nabla^i\psi_H||_{L^\infty_u L^1_{\underline{u}}L^\infty(S)}\right)\\
&\leq C\Delta_1(1+\Delta_1)^3||f(\underline{u})^{-1}||_{L^\infty_u L^2_{\underline{u}}L^\infty(S)}\\
&\qquad\times\left(\sum_{i\leq 3}||f(\underline{u})\nabla^i\psi_H||_{L^\infty_u L^2_{\underline{u}}L^2(S)}+\sum_{i\leq 1}||f(\underline{u})\nabla^i\psi_H||_{L^\infty_u L^2_{\underline{u}}L^\infty(S)}\right)\\
&\leq C\Delta_1^2(1+\Delta_1)^3\epsilon.
\end{aligned}$$

Hence, by Proposition 5, we have

$$\sum_{i\leq 3}||\nabla^i\eta||_{L^\infty_u L^\infty_{\underline{u}}L^2(S)}\leq C(\mathcal{O}_{\text{ini}})+C\epsilon(\Delta_1^2(1+\Delta_1)^3+\mathcal{R}(1+\Delta_1)^3)\leq C(\mathcal{O}_{\text{ini}}),$$

after choosing ϵ to be sufficiently small. Similarly, we consider the equation for $\nabla_3\nabla^i\underline{\eta}$ to get

$$\sum_{i\leq 3}||\nabla^i\underline{\eta}||_{L^\infty_u L^\infty_{\underline{u}}L^2(S)}\leq C(\mathcal{O}_{\text{ini}}). \qquad \square$$

We now move to the terms that we denote by $\psi_{\underline{H}}$, i.e., $\operatorname{tr}\underline{\chi}$, $\hat{\underline{\chi}}$, and $\underline{\omega}$. All of them obey a ∇_4 equation. Unlike the previous estimates for ψ, the initial data for the quantities $\psi_{\underline{H}}$ are not in L^∞_u. We will therefore prove only a bound for $\psi_{\underline{H}}$ in the weighted norm $||f(u)\cdot||_{L^2_u L^\infty_{\underline{u}}L^\infty(S)}$.

Proposition 13. *Assume*

$$\mathcal{R}<\infty,\quad \tilde{\mathcal{O}}_{4,2}<\infty.$$

Then there exists $\epsilon_0=\epsilon_0(\Delta_1,\mathcal{R},\tilde{\mathcal{O}}_{4,2})$ such that whenever $\epsilon\leq\epsilon_0$,

$$\sum_{i\leq 3}\mathcal{O}_{i,2}[\psi_{\underline{H}}]\leq C(\mathcal{O}_{\text{ini}}).$$

In particular as before, this estimate is independent of Δ_1.

Proof. According to the definition of the $\mathcal{O}_{i,2}$ norm, we need to control the weighted $L^2_u L^\infty_{\underline{u}}L^2(S)$ norm of $\psi_{\underline{H}}$. Using the null structure equations, for each $\psi_{\underline{H}}\in\{\operatorname{tr}\underline{\chi},\hat{\underline{\chi}},\underline{\omega}\}$, we have an equation of the type

$$\nabla_4\psi_{\underline{H}}=K+\nabla\underline{\eta}+\psi\psi+\psi_H\psi_{\underline{H}}.$$

We also use the null structure equations commuted with angular derivatives:

$$\nabla_4 \nabla^i \psi_{\underline{H}} = \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(K+\nabla\underline{\eta}) + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}.$$

We estimate the curvature term using the curvature norm. Recall that the curvature norm for K along the H_u is weighted with $f(u)$. Using the Sobolev embedding theorem in Proposition 7, we have

$$\begin{aligned}
&\left\|\sum_{i_1+i_2+i_3\leq 3} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}K\right\|_{L^1_{\underline{u}}L^2(S)} \\
&\leq Cf(u)^{-1}\left(\sum_{i_1\leq 1, i_2\leq 3} ||\nabla^{i_1}\psi||^{i_2}_{L^2_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_3\leq 3}||f(u)\nabla^{i_3}K||_{L^2_{\underline{u}}L^2(S)}\right) \\
&\quad + Cf(u)^{-1}||\nabla^2\psi||_{L^2_{\underline{u}}L^4(S)}||f(u)K||_{L^2_{\underline{u}}L^4(S)} \\
&\leq C\epsilon^{\frac12} f(u)^{-1}\left(\sum_{i_1\leq 3, i_2\leq 3} ||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^2(S)}\right)\left(\sum_{i_3\leq 3}||f(u)\nabla^{i_3}K||_{L^2_{\underline{u}}L^2(S)}\right) \\
&\leq C\epsilon^{\frac12} f(u)^{-1}(1+\Delta_1)^3\mathcal{R}.
\end{aligned} \tag{43}$$

The term linear in $\nabla^4\eta$ can be estimated analogously but using the $\tilde{\mathcal{O}}_{4,2}$ norms instead of the $\mathcal{R}$ norms:

$$\begin{aligned}
&\left\|\sum_{i_1+i_2+i_3\leq 3} \nabla^{i_1}\psi^{i_2}\nabla^{i_3+1}\eta\right\|_{L^1_{\underline{u}}L^2(S)} \\
&\leq C\epsilon\left(\sum_{i_1\leq 3}\sum_{i_2\leq 4} ||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^2(S)}\right) + C\epsilon^{\frac12} f(u)^{-1}||f(u)\nabla^4\eta||_{L^2_{\underline{u}}L^2(S)} \\
&\leq C\epsilon(1+\Delta_1)^4 + C\epsilon^{\frac12} f(u)^{-1}\tilde{\mathcal{O}}_{4,2}.
\end{aligned} \tag{44}$$

We now move to control the terms that are nonlinear in the Ricci coefficients. First, we estimate the terms without ψ_H or $\psi_{\underline{H}}$:

$$\begin{aligned}
&\left\|\sum_{i_1+i_2+i_3+i_4\leq 3} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\right\|_{L^1_{\underline{u}}L^2(S)} \\
&\leq C\epsilon\left(\sum_{i_1\leq 1, i_2\leq 4} ||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_3\leq 3}||\nabla^{i_3}\psi||_{L^\infty_{\underline{u}}L^2(S)}\right) \\
&\leq C\epsilon(1+\Delta_1)^5.
\end{aligned} \tag{45}$$

We then control the term with both ψ_H and $\psi_{\underline{H}}$:

(46)

$$\begin{aligned}
&\left\|\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}\right\|_{L^1_{\underline{u}}L^2(S)}\\
\leq& C\left(\sum_{i_1\leq 1,i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_3\leq 1}||\nabla^{i_3}\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_4\leq 3}||\nabla^{i_4}\psi_H||_{L^1_{\underline{u}}L^2(S)}\right)\\
&+C\left(\sum_{i_1\leq 1,i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_3\leq 3}||\nabla^{i_3}\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2(S)}\right)\left(\sum_{i_4\leq 1}||\nabla^{i_4}\psi_H||_{L^1_{\underline{u}}L^\infty(S)}\right)\\
&+C||\nabla^2\psi||_{L^\infty_{\underline{u}}L^2(S)}||\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^\infty(S)}||\psi_H||_{L^1_{\underline{u}}L^\infty(S)}\\
\leq& C\left(\sum_{i_1\leq 1,i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_3\leq 1}||\nabla^{i_3}\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^\infty(S)}\right)\\
&\quad\times\left(\sum_{i_4\leq 3}||f(\underline{u})\nabla^{i_4}\psi_H||_{L^2_{\underline{u}}L^2(S)}\right)||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}\\
&+C\left(\sum_{i_1\leq 1,i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty(S)}\right)\left(\sum_{i_3\leq 3}||\nabla^{i_3}\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2(S)}\right)\\
&\quad\times\left(\sum_{i_4\leq 1}||f(\underline{u})\nabla^{i_4}\psi_H||_{L^2_{\underline{u}}L^\infty(S)}\right)||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}\\
&+C||\nabla^2\psi||_{L^\infty_{\underline{u}}L^2(S)}||\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^\infty(S)}||f(\underline{u})\psi_H||_{L^2_{\underline{u}}L^\infty(S)}||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}\\
\leq& C\epsilon(1+\Delta_1)^3\left(\sum_{i_1\leq 3}||f(\underline{u})\nabla^{i_1}\psi_H||_{L^2_{\underline{u}}L^2(S)}\right)\left(\sum_{i_2\leq 3}||\nabla^{i_2}\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2(S)}\right)\\
\leq& C\epsilon\Delta_1(1+\Delta_1)^3\left(\sum_{i\leq 3}||\nabla^i\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2(S)}\right).
\end{aligned}$$

Therefore, by the bounds (43), (44), (45), and (46), we have that for every fixed u,

$$\begin{aligned}
&\sum_{i\leq 3}||\nabla^i\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2(S)}\\
&\qquad\leq C\left(\sum_{i\leq 3}||\nabla^i\psi_{\underline{H}}||_{L^2(S_{u,0})}\right)+C\epsilon^{\frac{1}{2}}f(u)^{-1}(\mathcal{R}+\tilde{\mathcal{O}}_{4,2})+C\epsilon(1+\Delta_1)^5\\
&\qquad\quad+C\epsilon\Delta_1(1+\Delta_1)^3\left(\sum_{i\leq 3}||\nabla^i\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2(S)}\right).
\end{aligned}$$

We now multiply this inequality by $f(u)$ and take the L^2 norm in u to get

$$\begin{aligned}&\sum_{i\leq 3}||f(u)\nabla^i\psi_{\underline{H}}||_{L^2_u L^\infty_{\underline{u}} L^2(S)}\\ &\quad\leq C(\mathcal{O}_{\text{ini}})+C\epsilon^{\frac{1}{2}}||f(u)(f(u)^{-1})||_{L^2_u}(\mathcal{R}+\tilde{\mathcal{O}}_{4,2})+C\epsilon(1+\Delta_1)^5\\ &\qquad+C\epsilon\Delta_1(1+\Delta_1)^3\left(\sum_{i\leq 3}||f(u)\nabla^i\psi_{\underline{H}}||_{L^2_u L^\infty_{\underline{u}} L^2(S)}\right)\\ &\quad\leq C(\mathcal{O}_{\text{ini}}),\end{aligned}$$

for ϵ sufficiently small. □

Using instead the equation for $\nabla_3\psi_H$, we obtain the following estimates in a completely analogous manner:

Proposition 14. *Assume*

$$\mathcal{R}<\infty,\qquad \tilde{\mathcal{O}}_{4,2}<\infty.$$

Then there exists $\epsilon_0=\epsilon_0(\Delta_1,\mathcal{R},\tilde{\mathcal{O}}_{4,2})$ *such that whenever* $\epsilon\leq\epsilon_0$*,*

$$\sum_{i\leq 3}\mathcal{O}_{i,2}[\psi_H]\leq C(\mathcal{O}_{\text{ini}}).$$

In particular this estimate is independent of Δ_1*.*

By the Sobolev embedding theorems given by Proposition 7, we have thus closed our bootstrap assumption (A1) after choosing Δ_1 to be sufficiently large depending on the initial data norm $\mathcal{O}_{\text{ini}}$. We have therefore proved the desired estimates for the Ricci coefficients and their first three angular covariant derivatives. We summarize this in the following proposition.

Proposition 15. *Assume*

$$\mathcal{R}<\infty,\qquad \tilde{\mathcal{O}}_{4,2}<\infty.$$

Then there exists $\epsilon_0=\epsilon_0(\mathcal{O}_{\text{ini}},\mathcal{R},\tilde{\mathcal{O}}_{4,2})$ *such that whenever* $\epsilon\leq\epsilon_0$*,*

$$\sum_{i\leq 3}\mathcal{O}_{i,2}[\psi,\psi_{\underline{H}},\psi_H]\leq C(\mathcal{O}_{\text{ini}}).$$

7. Elliptic estimates for fourth derivatives of the Ricci coefficients

We now estimate the fourth derivative of the Ricci coefficients. We introduce the following bootstrap assumption:

$$\text{(A2)}\qquad \tilde{\mathcal{O}}_{4,2}\leq\Delta_2,$$

where Δ_2 is a constant to be chosen later.

The estimates for the fourth derivative of the Ricci coefficients cannot be achieved only by the transport equations since there would be a loss in derivatives. We can however use the transport equation—the Hodge system type estimates as in [5, 15, 16]. We will first derive estimates for some chosen combination of $\nabla^4(\psi,\psi_H,\psi_{\underline{H}})+\nabla^3(\beta,K,\check{\sigma},\underline{\beta})$ by using transport equations. We will then show that the estimates for all the fourth derivatives of the Ricci coefficients can be proved via elliptic estimates.

In order to apply the elliptic estimates in section 5.5, we need to first control the Gauss curvature and its first and second derivatives in $L^2(S)$.

Proposition 16. *Assume*

$$\mathcal{R} < \infty.$$

Then there exists $\epsilon_0 = \epsilon_0(\Delta_2, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$\sum_{i\leq 2} ||\nabla^i K||_{L^\infty_u L^\infty_{\underline{u}} L^2(S)} \leq C(\mathcal{O}_{\text{ini}}).$$

Proof. K obeys the following Bianchi equation:

$$\nabla_4 K = \nabla\beta + \psi_H K + \sum_{i_1+i_2+i_3\leq 1} \psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_H.$$

Commuting with angular derivatives, we have, for $i \leq 2$,

$$\begin{aligned}
&\nabla_4\nabla^i K \\
&\quad= \sum_{i_1+i_2+i_3\leq 2} \nabla^{i_1}\psi^{i_2}\nabla^{i_3+1}\beta + \sum_{i_1+i_2+i_3+i_4\leq 2} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}K \\
&\qquad+ \sum_{i_1+i_2+i_3+i_4\leq 3} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_H.
\end{aligned}$$

By Proposition 5, in order to control $\nabla^i K$ in $L^\infty_u L^\infty_{\underline{u}} L^2(S)$, we need to bound the right hand side in $L^\infty_u L^1_{\underline{u}} L^2(S)$. We first control the term containing β:

$$\begin{aligned}
&\left\|\sum_{i_1+i_2+i_3\leq 2} \nabla^{i_1}\psi^{i_2}\nabla^{i_3+1}\beta\right\|_{L^\infty_u L^1_{\underline{u}} L^2(S)} \\
&\leq C\left(\sum_{i_1\leq 1, i_2\leq 2} ||\nabla^{i_1}\psi||^{i_2}_{L^\infty_u L^\infty_{\underline{u}} L^\infty(S)}\right) ||f(\underline{u})^{-1}||_{L^2_{\underline{u}}} \left(\sum_{i_3\leq 2} ||f(\underline{u})\nabla^{i_2+1}\beta||_{L^\infty_u L^2_{\underline{u}} L^2(S)}\right) \\
&\leq C(\mathcal{O}_{\text{ini}})\epsilon\mathcal{R},
\end{aligned}$$

where we have used the estimates for ψ given by Proposition 15. The term containing K can be controlled by

$$\begin{aligned}
&\left\|\sum_{i_1+i_2+i_3+i_4\leq 2} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}K\right\|_{L^\infty_u L^1_{\underline{u}} L^2(S)} \\
&\leq C\left(\sum_{i_1\leq 1, i_2\leq 2} ||\nabla^{i_1}\psi||^{i_2}_{L^\infty_u L^\infty_{\underline{u}} L^\infty(S)}\right) \int_0^{\underline{u}} \left(\sum_{i_3+i_4\leq 2} ||\nabla^{i_3}\psi_H\nabla^{i_4}K||_{L^\infty_u L^2(S_{u,\underline{u}'})}\right) d\underline{u}' \\
&\leq C(\mathcal{O}_{\text{ini}}) \int_0^{\underline{u}} \left(\sum_{i_1+i_2\leq 2} ||\nabla^{i_1}\psi_H\nabla^{i_2}K||_{L^\infty_u L^2(S_{u,\underline{u}'})}\right) d\underline{u}' \\
&\leq C(\mathcal{O}_{\text{ini}}) \int_0^{\underline{u}} \left(\sum_{i_1\leq 2} ||\nabla^{i_1}\psi_H||_{L^\infty_u L^2(S_{u,\underline{u}'})}\right) \left(\sum_{i_2\leq 2} ||\nabla^{i_2}K||_{L^\infty_u L^2(S_{u,\underline{u}'})}\right) d\underline{u}'.
\end{aligned}$$

The remaining term has been bounded in the previous section. By (42) and Proposition 15,

$$\left\| \sum_{i_1+i_2+i_3\leq 2} \psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_H \right\|_{L^\infty_u L^1_{\underline{u}} L^2(S)} \leq C(\mathcal{O}_{\text{ini}})\epsilon.$$

Therefore, by Proposition 5,

$$\sum_{i\leq 2}||\nabla^i K||_{L^\infty_u L^2(S_{u,\underline{u}})} \leq C(\mathcal{O}_{\text{ini}}) \left(1+\epsilon+\epsilon\mathcal{R}+\int_0^{\underline{u}}\left(\sum_{i_1\leq 2}||\nabla^{i_1}\psi_H||_{L^\infty_u L^2(S_{u,\underline{u}'})}\right) \times \left(\sum_{i_2\leq 2}||\nabla^{i_2}K||_{L^\infty_u L^2(S_{u,\underline{u}'})}\right) d\underline{u}'\right).$$

Gronwall's inequality implies

$$\sum_{i\leq 2}||\nabla^i K||_{L^\infty_u L^2(S_{u,\underline{u}})} \leq C(\mathcal{O}_{\text{ini}})\exp\left(\sum_{i\leq 2}||\nabla^i\psi_H||_{L^1_{\underline{u}}L^\infty_u L^2(S)}\right) \leq C(\mathcal{O}_{\text{ini}})$$

since by Proposition 15, $\sum_{i\leq 1}||\nabla^i\psi_H||_{L^1_{\underline{u}}L^\infty_u L^2(S)} \leq C(\mathcal{O}_{\text{ini}})$ for ϵ sufficiently small. □

It is easy to see that since $\check{\sigma}$ satisfies a similar schematic Bianchi equation as K, we also have the following estimates for $\check{\sigma}$ and its derivative.

Proposition 17. *Assume*

$$\mathcal{R} < \infty.$$

Then there exists $\epsilon_0 = \epsilon_0(\Delta_2, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$\sum_{i\leq 2}||\nabla^i\check{\sigma}||_{L^\infty_u L^\infty_{\underline{u}} L^2(S)} \leq C(\mathcal{O}_{\text{ini}}).$$

Using Proposition 16, we now control the fourth derivatives of the Ricci coefficients. We first bound $\nabla^4\text{tr}\,\chi$ using the transport equation.

Proposition 18. *There exists* $\epsilon_0 = \epsilon_0(\mathcal{O}_{\text{ini}}, \Delta_2)$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$||f(\underline{u})\nabla^4\text{tr}\,\chi||_{L^2_{\underline{u}}L^\infty_u L^2(S)} \leq C(\mathcal{O}_{\text{ini}}).$$

Proof. Consider the following equation:

$$\nabla_4\text{tr}\,\chi = \psi_H\psi_H,$$

After commuting with angular derivatives, we have

$$\nabla_4\nabla^4\text{tr}\,\chi = \sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_H.$$

By Proposition 5, in order to control $\nabla^4\text{tr}\,\chi$ in $L^2(S_{u,\underline{u}})$, we need to bound the right-hand side in $L^1_{\underline{u}}L^2(S)$. Using the fact that $f(\underline{u})$ is decreasing, this can be achieved using Sobolev embedding (Proposition 7) by

$$\begin{aligned}
&\sum_{i_1+i_2+i_3+i_4\leq 4}\int_0^{\underline{u}}||\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_H||_{L^2(S_{u,\underline{u}'})}d\underline{u}'\\
&\leq Cf(\underline{u})^{-2}\left(\sum_{i_1\leq 3}\sum_{i_2\leq 4}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^2(S)}\right)\left(\sum_{i_3\leq 2}||f(\underline{u})\nabla^{i_3}\psi_H||_{L^2_{\underline{u}}L^2(S)}\right)\\
&\quad\times\left(\sum_{i_4\leq 4}||f(\underline{u})\nabla^{i_4}\psi_H||_{L^2_{\underline{u}}L^2(S)}\right)\\
&\leq Cf(\underline{u})^{-2}\Delta_2.
\end{aligned}$$

By Proposition 5, we have

$$||\nabla^4\mathrm{tr}\,\chi||_{L^2(S_{u,\underline{u}})}\leq C(\mathcal{O}_{\mathrm{ini}})+C(\mathcal{O}_{\mathrm{ini}})f(\underline{u})^{-2}\Delta_2. \tag{47}$$

Multiplying (47) by $f(\underline{u})$ and taking first the L^∞ norm in u and then the L^2 norm in $\underline{u}$, we have

$$||f(\underline{u})\nabla^4\mathrm{tr}\,\chi||_{L^2_{\underline{u}}L^\infty_uL^2(S)}\leq C(\mathcal{O}_{\mathrm{ini}})+C(\mathcal{O}_{\mathrm{ini}})||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}\Delta_2\leq C(\mathcal{O}_{\mathrm{ini}})+C\epsilon\Delta_2,$$

where we have used

$$||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}\leq C\epsilon.$$

Thus, the conclusion follows by choosing ϵ to be sufficiently small depending on Δ_2. □

Once we have the estimates for $\nabla^4\mathrm{tr}\,\chi$, we can control $\nabla^4\hat{\chi}$ using elliptic estimates:

Proposition 19. *Assume*

$$\mathcal{R}<\infty.$$

Then there exists $\epsilon_0=\epsilon_0(\mathcal{O}_{\mathrm{ini}},\Delta_2,\mathcal{R})$ *such that whenever* $\epsilon\leq\epsilon_0$,

$$||f(\underline{u})\nabla^4\hat{\chi}||_{L^\infty_uL^2_{\underline{u}}L^2(S)}\leq C(\mathcal{O}_{\mathrm{ini}})+C\mathcal{R}.$$

Proof. We now use the Codazzi equation

$$\mathrm{div}\,\hat{\chi}=\frac{1}{2}\nabla\mathrm{tr}\,\chi-\beta+\psi\psi_H$$

and apply elliptic estimates from Proposition 11 to get

$$\begin{aligned}
||\nabla^4\hat{\chi}||_{L^2(S)}\leq C&\left(\sum_{i\leq 4}||\nabla^i\mathrm{tr}\,\chi||_{L^2(S)}+\sum_{i\leq 3}||\nabla^i\beta||_{L^2(S)}\right.\\
&\left.+\sum_{i_1+i_2\leq 3}||\nabla^{i_1}\psi\nabla^{i_2}\psi_H||_{L^2(S)}+\sum_{i\leq 3}||\nabla^i\hat{\chi}||_{L^2(S)}\right).
\end{aligned} \tag{48}$$

Notice that we can apply elliptic estimates using Proposition 11, since we have the estimates for the Gauss curvature from Proposition 16. Multiply (48) by $f(\underline{u})$ and take the $L^\infty_u L^2_{\underline{u}}$ norm to get

$$\begin{aligned}
&||f(\underline{u})\nabla^4\hat{\chi}||_{L^\infty_u L^2_{\underline{u}} L^2(S)} \\
&\quad \leq C\left(\sum_{i\leq 4}||f(\underline{u})\nabla^i \mathrm{tr}\,\chi||_{L^\infty_u L^2_{\underline{u}} L^2(S)} + \sum_{i\leq 3}||f(\underline{u})\nabla^i\beta||_{L^\infty_u L^2_{\underline{u}} L^2(S)}\right. \\
&\qquad \left. + \sum_{i_1+i_2\leq 3}||f(\underline{u})\nabla^{i_1}\psi\nabla^{i_2}\psi_H||_{L^\infty_u L^2_{\underline{u}} L^2(S)} + \sum_{i\leq 3}||f(\underline{u})\nabla^i\hat{\chi}||_{L^\infty_u L^2_{\underline{u}} L^2(S)}\right) \\
&\quad \leq C(\mathcal{O}_{\mathrm{ini}}) + C\mathcal{R} + C\sum_{i_1+i_2\leq 3}||f(\underline{u})\nabla^{i_1}\psi\nabla^{i_2}\psi_H||_{L^\infty_u L^2_{\underline{u}} L^2(S)}.
\end{aligned}$$

By Proposition 15 and Sobolev embedding theorem in Proposition 7, we have

$$\begin{aligned}
&\sum_{i_1+i_2\leq 3}||f(\underline{u})\nabla^{i_1}\psi\nabla^{i_2}\psi_H||_{L^\infty_u L^2_{\underline{u}} L^2(S)} \\
&\quad \leq C\left(\sum_{i_1\leq 3}||\nabla^{i_1}\psi||_{L^\infty_u L^\infty_{\underline{u}} L^2(S)}\right)\left(\sum_{i_2\leq 3}||f(\underline{u})\nabla^{i_2}\psi_H||_{L^\infty_u L^2_{\underline{u}} L^2(S)}\right) \leq C(\mathcal{O}_{\mathrm{ini}}).
\end{aligned}$$

Therefore,

$$||f(\underline{u})\nabla^4\hat{\chi}||_{L^\infty_u L^2_{\underline{u}} L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}}) + C\mathcal{R}. \qquad \square$$

The $\tilde{\mathcal{O}}_{4,2}$ estimates for $\nabla^4\mathrm{tr}\,\underline{\chi}$ and $\nabla^4\hat{\underline{\chi}}$ follow identically as that for $\nabla^4\mathrm{tr}\,\chi$ and $\nabla^4\hat{\chi}$:

Proposition 20. *Assume*

$$\mathcal{R} < \infty.$$

Then there exists $\epsilon_0 = \epsilon_0(\mathcal{O}_{\mathrm{ini}}, \Delta_2, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$||f(u)\nabla^4\mathrm{tr}\,\underline{\chi}||_{L^\infty_{\underline{u}} L^2_u L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}})$$

and

$$||f(u)\nabla^4\hat{\underline{\chi}}||_{L^\infty_{\underline{u}} L^2_u L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}}) + C\mathcal{R}.$$

We then prove estimates for $\nabla^4\eta$. To do so, we first prove estimates for third derivatives of $\mu = -\mathrm{div}\,\eta + K$ and recover the control for $\nabla^4\eta$ via elliptic estimates.

Proposition 21. *Assume*

$$\mathcal{R} < \infty.$$

Then there exists $\epsilon_0 = \epsilon_0(\mathcal{O}_{\mathrm{ini}}, \Delta_2, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$||f(u)\nabla^4\eta||_{L^\infty_u L^2_{\underline{u}} L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}})(\epsilon^{\frac{1}{2}} + \mathcal{R})$$

and

$$||f(\underline{u})\nabla^4\eta||_{L^\infty_{\underline{u}} L^2_u L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}})(\epsilon^{\frac{1}{2}} + \mathcal{R}).$$

Proof. Recall that

$$\mu = -\mathrm{div}\,\eta + K.$$

Then μ satisfies the following equation:[26]

$$\nabla_4\mu = \psi_H(K,\check{\sigma}) + \sum_{i_1+i_2+i_3=1} \psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_H.$$

After commuting with angular derivatives, we get

$$\nabla_4\nabla^3\mu = \sum_{i_1+i_2+i_3+i_4=3} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}(K,\check{\sigma}) + \sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_H.$$

We now control each of the terms on the right-hand side in $L^1_{\underline{u}}L^2(S)$. The first term, which contains curvature components, can be estimated by

$$\begin{aligned}
&\sum_{i_1+i_2+i_3+i_4=3} \int_0^{\underline{u}} ||\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}(K,\check{\sigma})||_{L^2(S_{u,\underline{u}'})}d\underline{u}' \\
&\quad\leq Cf(u)^{-1}f(\underline{u})^{-1}\left(\sum_{i_1\leq 3}\sum_{i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^2(S)}\right)\left(\sum_{i_3\leq 3}||f(\underline{u})\nabla^{i_3}\psi_H||_{L^2_{\underline{u}}L^2(S)}\right) \\
&\qquad\times\left(\sum_{i_4\leq 3}||f(u)\nabla^{i_4}(K,\check{\sigma})||_{L^2_{\underline{u}}L^2(S)}\right) \\
&\quad\leq C(\mathcal{O}_{\text{ini}})f(u)^{-1}f(\underline{u})^{-1}\mathcal{R},
\end{aligned}$$

using the bounds obtained in Proposition 15. The second term can be controlled using Sobolev embedding in Proposition 7 by

$$\begin{aligned}
&\sum_{i_1+i_2+i_3+i_4=4} \int_0^{\underline{u}} ||\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_H||_{L^2(S_{u,\underline{u}'})}d\underline{u}' \\
&\quad\leq Cf(u)^{-1}f(\underline{u})^{-1}\left(\sum_{i_1\leq 4}\sum_{i_2\leq 5}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^2(S)}\right)\left(\sum_{i_3\leq 4}||f(\underline{u})\nabla^{i_3}\psi_H||_{L^2_{\underline{u}}L^2(S)}\right) \\
&\qquad\times\left(\sum_{i_4\leq 4}||f(u)\nabla^{i_4}\psi||_{L^2_{\underline{u}}L^2(S)}\right) \\
&\quad\leq C(\mathcal{O}_{\text{ini}})f(u)^{-1}f(\underline{u})^{-1}(1+\Delta_2)^2
\end{aligned}$$

using the estimates in Proposition 15. Therefore, by Proposition 5, we have

$$||\nabla^3\mu||_{L^2(S_{u,\underline{u}})} \leq C(\mathcal{O}_{\text{ini}})(1+f(u)^{-1}f(\underline{u})^{-1}(\mathcal{R}+(1+\Delta_2)^2)). \tag{49}$$

Recall that the $L^2_{\underline{u}}$ norm of $f(\underline{u})^{-1}$ is bounded by ϵ. Thus, multiplying (49) by $f(u)$ and taking the L^2 norm in $\underline{u}$, we get

$$||f(u)\nabla^3\mu||_{L^2_{\underline{u}}L^2(S)} \leq C(\mathcal{O}_{\text{ini}})(\epsilon^{\frac{1}{2}}+\epsilon(\mathcal{R}+(1+\Delta_2)^2)) \leq C(\mathcal{O}_{\text{ini}})\epsilon^{\frac{1}{2}}$$

for ϵ sufficiently small. Similarly, multiplying (49) by $f(\underline{u})$ and taking the L^2 norm in u, we get

$$||f(\underline{u})\nabla^3\mu||_{L^2_u L^2(S)} \leq C(\mathcal{O}_{\text{ini}})\epsilon^{\frac{1}{2}}.$$

[26] It is important to note that the potentially harmful term $\psi_{\underline{H}}\psi_H\psi_H$ is absent in this equation. This required structure is the reason that we perform this renormalization instead of using $\mu = -\operatorname{div}\eta - \rho + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}$ as in [18, 19].

We can obtain bounds for $\nabla^4\eta$ from the control of $\nabla^3\mu$ using elliptic estimates as follows. By the div-curl systems

$$\mathrm{div}\,\eta = -\mu + K, \qquad \mathrm{curl}\,\eta = \check{\sigma},$$

and the elliptic estimates given by Propositions 10 and 16, we have

$$||\nabla^4\eta||_{L^2(S)} \leq C\left(\sum_{i\leq 3}||\nabla^i\mu||_{L^2(S)} + \sum_{i\leq 3}||\nabla^i(K,\check{\sigma})||_{L^2(S)} + \sum_{i\leq 3}||\nabla^i\eta||_{L^2(S)}\right).$$

Therefore,

$$\begin{aligned}&||f(u)\nabla^4\eta||_{L^2_{\underline{u}}L^2(S)}\\ &\leq C\left(\sum_{i\leq 3}||f(u)\nabla^i\mu||_{L^2_{\underline{u}}L^2(S)}+\sum_{i\leq 3}||f(u)\nabla^i(K,\check{\sigma})||_{L^2_{\underline{u}}L^2(S)}+\sum_{i\leq 3}||f(u)\nabla^i\eta||_{L^2_{\underline{u}}L^2(S)}\right)\\ &\leq C(\mathcal{O}_{\mathrm{ini}})(\epsilon^{\frac{1}{2}}+\mathcal{R}).\end{aligned}$$

Similarly,

$$||f(\underline{u})\nabla^4\eta||_{L^2_u L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}})(\epsilon^{\frac{1}{2}}+\mathcal{R}). \qquad \square$$

A similar proof shows that the conclusion of Proposition 21 holds also for $\nabla^3\underline{\eta}$:

Proposition 22. *Assume*

$$\mathcal{R} < \infty.$$

Then there exists $\epsilon_0 = \epsilon_0(\mathcal{O}_{\mathrm{ini}}, \Delta_2, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$||f(u)\nabla^4\underline{\eta}||_{L^\infty_u L^2_{\underline{u}}L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}})(\epsilon^{\frac{1}{2}}+\mathcal{R})$$

and

$$||f(\underline{u})\nabla^4\underline{\eta}||_{L^\infty_{\underline{u}}L^2_u L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}})(\epsilon^{\frac{1}{2}}+\mathcal{R}).$$

We now move to the estimates for $\nabla^4\underline{\omega}$:

Proposition 23. *Assume*

$$\mathcal{R} < \infty.$$

Then there exists $\epsilon_0 = \epsilon_0(\mathcal{O}_{\mathrm{ini}}, \Delta_2, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$||f(u)\nabla^4\underline{\omega}||_{L^\infty_{\underline{u}}L^2_u L^2(S)} \leq C(\mathcal{O}_{\mathrm{ini}})(1+\mathcal{R}).$$

Proof. Let $\underline{\omega}^\dagger$ be defined as the solution to

$$\nabla_4\underline{\omega}^\dagger = \frac{1}{2}\check{\sigma}$$

with zero data on $\underline{H}_0$ and

$$\underline{\kappa} := -\nabla\underline{\omega} +^* \nabla\underline{\omega}^\dagger - \frac{1}{2}\underline{\beta}.$$

By the definition of $\underline{\omega}^\dagger$, it is easy to see that using Proposition 5,

$$\sum_{i\leq 3}||\nabla^i\underline{\omega}^\dagger||_{L^2_u L^\infty_{\underline{u}}L^2(S)} \leq C\epsilon\mathcal{R} \leq C(\mathcal{O}_{\mathrm{ini}}).$$

In other words, $\nabla^i\underline{\omega}^\dagger$ satisfies much better estimates[27] than $\nabla^i\psi_{\underline{H}}$ for $i\leq 3$. With this in mind, in the proof of this proposition, we will also use $\psi_{\underline{H}}$ to denote $\underline{\omega}^\dagger$ (in addition to $\mathrm{tr}\,\underline{\chi}$, $\hat{\underline{\chi}}$, and $\underline{\omega}$).

With this convention, $\underline{\kappa}$ then obeys the schematic equation

$$\nabla_4\underline{\kappa}=\psi(K,\check{\sigma})+\sum_{i_1+i_2+i_3=1}\psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi+\sum_{i_1+i_2+i_3=1}\psi^{i_1}\nabla^{i_2}\psi_H\nabla^{i_3}\psi_{\underline{H}}.$$

After commuting with angular derivatives, we get

$$\begin{aligned}\nabla_4\nabla^3\underline{\kappa}=&\sum_{i_1+i_2+i_3+i_4=3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})+\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\\&+\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}.\end{aligned}$$

Therefore,

$$\begin{aligned}||\nabla^3\underline{\kappa}||_{L^2(S_{u,\underline{u}})}\leq& C||\nabla^3\underline{\kappa}||_{L^2(S_{u,0})}+C||\sum_{i_1+i_2+i_3+i_4=3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})||_{L^1_{\underline{u}}L^2(S)}\\&+C||\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi||_{L^1_{\underline{u}}L^2(S)}\\&+C||\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}||_{L^1_{\underline{u}}L^2(S)}.\end{aligned}$$

Multiplying by $f(u)$ and taking the L^2 norm in u, we get

$$\begin{aligned}&||f(u)\nabla^3\underline{\kappa}||_{L^2_uL^2(S)}\\&\leq C||f(u)\nabla^3\underline{\kappa}||_{L^2_uL^2(S_{u,0})}+C||f(u)\sum_{i_1+i_2+i_3+i_4=3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})||_{L^2_uL^1_{\underline{u}}L^2(S)}\\&\quad+C||f(u)\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi||_{L^2_uL^1_{\underline{u}}L^2(S)}\\&\quad+C||f(u)\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}||_{L^2_uL^1_{\underline{u}}L^2(S)}.\end{aligned}$$

The first term is an initial data term and it is bounded by a constant depending only on $\mathcal{O}_{\mathrm{ini}}$. We estimate each of the nonlinear terms. The second term can be controlled by

$$\begin{aligned}&||f(u)\sum_{i_1+i_2+i_3+i_4=3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})||_{L^2_uL^1_{\underline{u}}L^2(S)}\\&\leq C\epsilon\left(\sum_{i_1\leq 3}\sum_{i_2\leq 4}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty_{\underline{u}}L^2(S)}\right)\left(\sum_{i_3\leq 3}||f(u)\nabla^{i_3}(K,\check{\sigma})||_{L^\infty_uL^2_{\underline{u}}L^2(S)}\right)\\&\leq C(\mathcal{O}_{\mathrm{ini}})\epsilon\mathcal{R}.\end{aligned}$$

[27]We recall that for $\psi_{\underline{H}}$ we only have the degenerate estimate

$$\sum_{i\leq 3}||f(u)\nabla^i\psi_{\underline{H}}||_{L^2_uL^\infty_{\underline{u}}L^2(S)}\leq C(\mathcal{O}_{\mathrm{ini}}).$$

The third term can be bounded by

$$\begin{aligned}
&||f(u)\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi||_{L^2_uL^1_{\underline{u}}L^2(S)}\\
&\leq C\epsilon\left(\sum_{i_1\leq 3}\sum_{i_2\leq 4}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty_uL^2(S)}\right)\left(\sum_{i_3\leq 3}||\nabla^{i_3}\psi||_{L^\infty_{\underline{u}}L^\infty_uL^2(S)}\right)\\
&\quad\times\left(\sum_{i_4\leq 4}||f(u)\nabla^{i_4}\psi||_{L^\infty_uL^2_{\underline{u}}L^2(S)}\right)\\
&\leq C(\mathcal{O}_{\text{ini}})\epsilon\Delta_2.
\end{aligned}$$

The fourth term can be estimated by

$$\begin{aligned}
&||f(u)\sum_{i_1+i_2+i_3+i_4=4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}||_{L^2_uL^1_{\underline{u}}L^2(S)}\\
&\leq C\epsilon^{\frac{1}{2}}\left(\sum_{i_1\leq 3}\sum_{i_2\leq 4}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_{\underline{u}}L^\infty_uL^2(S)}\right)\left(\sum_{i_3\leq 3}||\nabla^{i_3}\psi_H||_{L^1_{\underline{u}}L^\infty_uL^2(S)}\right)\\
&\quad\times\left(\sum_{i_4\leq 4}||f(u)\nabla^{i_4}\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2_uL^2(S)}\right)\\
&\quad+C||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}||f(\underline{u})\nabla^4\psi_H||_{L^\infty_uL^2_{\underline{u}}L^2(S)}||f(u)\psi_{\underline{H}}||_{L^2_uL^\infty_{\underline{u}}L^\infty(S)}\\
&\leq C(\mathcal{O}_{\text{ini}})\epsilon(1+\Delta_2).
\end{aligned}$$

Therefore,

$$||f(u)\nabla^3\underline{\kappa}||_{L^2_uL^\infty_{\underline{u}}L^2(S)}\leq C(\mathcal{O}_{\text{ini}})(1+\epsilon(1+\Delta_2+\mathcal{R}))\leq C(\mathcal{O}_{\text{ini}}), \tag{50}$$

after choosing ϵ to be sufficiently small. Finally, we retrieve the estimates for $\nabla^4\underline{\omega}$ and $\nabla^4\underline{\omega}^\dagger$ from the bounds for $\nabla^3\underline{\kappa}$. To this end, consider the div-curl system

$$\text{div}\,\nabla\underline{\omega}=-\text{div}\,\underline{\kappa}-\frac{1}{2}\text{div}\,\underline{\beta},$$

$$\text{curl}\,\nabla\underline{\omega}=0,$$

$$\text{div}\,\nabla\underline{\omega}^\dagger=-\text{curl}\,\underline{\kappa}-\frac{1}{2}\text{curl}\,\underline{\beta},$$

$$\text{curl}\,\nabla\underline{\omega}^\dagger=0.$$

By elliptic estimates given by Propositions 10 and 16, we have

$$\begin{aligned}
&||\nabla^4(\underline{\omega},\underline{\omega}^\dagger)||_{L^2(S_{u,\underline{u}})}\\
&\leq C\left(\sum_{i\leq 3}||\nabla^i\underline{\kappa}||_{L^2(S_{u,\underline{u}})}+\sum_{i\leq 3}||\nabla^i\underline{\beta}||_{L^2(S_{u,\underline{u}})}+\sum_{i\leq 3}||\nabla^i(\underline{\omega},\underline{\omega}^\dagger)||_{L^2(S_{u,\underline{u}})}\right).
\end{aligned}$$

Therefore, using Proposition 12, (50), and the curvature norm,

$$||f(u)\nabla^4(\underline{\omega},\underline{\omega}^\dagger)||_{L^\infty_{\underline{u}}L^2_uL^2(S)}\leq C(\mathcal{O}_{\text{ini}})(1+\mathcal{R}).$$

□

By switching $\underline{\omega}$ and ω as well as e_3 and e_4, we also have the following estimates for $\nabla^4\omega$:

Proposition 24. *Assume*

$$\mathcal{R} < \infty.$$

Then there exists $\epsilon_0 = \epsilon_0(\mathcal{O}_{\text{ini}}, \Delta_2, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$||f(\underline{u})\nabla^4\omega||_{L^\infty_u L^2_{\underline{u}} L^2(S)} \leq C(\mathcal{O}_{\text{ini}})(1+\mathcal{R}).$$

We have thus controlled the fourth angular derivatives of all Ricci coefficients and have closed the bootstrap assumption (A2) after choosing Δ_2 to be sufficiently large depending on $\mathcal{O}_{\text{ini}}$ and $\mathcal{R}$. We summarize this in the following proposition:

Proposition 25. *Assume*

$$\mathcal{R} < \infty.$$

There exists $\epsilon_0 = \epsilon_0(\mathcal{O}_{\text{ini}}, \mathcal{R})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$\tilde{\mathcal{O}}_{4,2} \leq C(\mathcal{O}_{\text{ini}})(1+\mathcal{R}).$$

8. Estimates for curvature

In this section, we derive and prove the energy estimates. To this end, we introduce the following bootstrap assumptions:

$$\mathcal{R} \leq \Delta_3, \tag{A3}$$

where Δ_3 is a constant to be chosen later.

In order to derive the energy estimates, we need the following integration by parts formula, which can be proved by direct computation:

Proposition 26. *Let $D_{u,\underline{u}}$ be defined as the spacetime region whose coordinates $(u', \underline{u}')$ satisfy $0 \leq u' \leq u$ and $0 \leq \underline{u}' \leq \underline{u}$. Suppose ϕ_1 and ϕ_2 are tensors of rank r, then*

$$\int_{D_{u,\underline{u}}} \phi_1 \nabla_4 \phi_2 + \int_{D_{u,\underline{u}}} \phi_2 \nabla_4 \phi_1 = \int_{\underline{H}_{\underline{u}}(0,u)} \phi_1\phi_2 - \int_{\underline{H}_0(0,u)} \phi_1\phi_2 + \int_{D_{u,\underline{u}}} (2\omega - \mathrm{tr}\,\chi)\phi_1\phi_2,$$

$$\int_{D_{u,\underline{u}}} \phi_1 \nabla_3 \phi_2 + \int_{D_{u,\underline{u}}} \phi_2 \nabla_3 \phi_1 = \int_{H_u(0,\underline{u})} \phi_1\phi_2 - \int_{H_0(0,\underline{u})} \phi_1\phi_2 + \int_{D_{u,\underline{u}}} (2\underline{\omega} - \mathrm{tr}\,\underline{\chi})\phi_1\phi_2.$$

Proposition 27. *Suppose we have a tensor ${}^{(1)}\phi$ of rank r and a tensor ${}^{(2)}\phi$ of rank $r-1$. Then*

$$\int_{D_{u,\underline{u}}} {}^{(1)}\phi^{A_1A_2\cdots A_r} \nabla_{A_r} {}^{(2)}\phi_{A_1\cdots A_{r-1}} + \int_{D_{u,\underline{u}}} \nabla^{A_r}\, {}^{(1)}\phi_{A_1A_2\cdots A_r} {}^{(2)}\phi^{A_1\cdots A_{r-1}}$$
$$= -\int_{D_{u,\underline{u}}} (\eta + \underline{\eta})\, {}^{(1)}\phi\, {}^{(2)}\phi.$$

With these we are now ready to derive energy estimates for $\nabla^i(K, \check{\sigma})$ in $L^2(H_u)$ and for $\nabla^i\underline{\beta}$ in $L^2(\underline{H}_{\underline{u}})$. The most important observation is that the two uncontrollable terms have favorable signs. This in turn is due to the choice of $f(u)$ which is decreasing toward the future.

Proposition 28. *The following L^2 estimates for the curvature hold:*

$$
\begin{aligned}
&\sum_{i\leq 3}(||f(u)\nabla^i(K,\check{\sigma})||^2_{L^\infty_u L^2_{\underline{u}} L^2(S)}+||f(u)\nabla^i\underline{\beta}||^2_{L^\infty_{\underline{u}} L^2_u L^2(S)})\\
&\quad\leq \sum_{i\leq 3}(||f(u)\nabla^i(K,\check{\sigma})||^2_{L^2_{\underline{u}} L^2(S_{0,\underline{u}})}+||f(u)\nabla^i\underline{\beta}||^2_{L^2_u L^2(S_{u,0})})\\
&\quad\quad+\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_{\underline{H}}\right)\right\|_{L^1_u L^1_{\underline{u}} L^1(S)}\\
&\quad\quad+\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_u L^1_{\underline{u}} L^1(S)}\\
&\quad\quad+\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_u L^1_{\underline{u}} L^1(S)}\\
&\quad\quad+\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 2}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}K\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_u L^1_{\underline{u}} L^1(S)}\\
&\quad\quad+\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}\psi_H\right)\right\|_{L^1_u L^1_{\underline{u}} L^1(S)}.
\end{aligned}
$$

Proof. Consider the following schematic Bianchi equations:

$$
\begin{aligned}
\nabla_3\check{\sigma}+\operatorname{div}{}^*\underline{\beta}&=\psi_{\underline{H}}\check{\sigma}+\sum_{i_1+i_2+i_3=1}\psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_{\underline{H}},\\
\nabla_3 K-\operatorname{div}\underline{\beta}&=\psi_{\underline{H}}K+\sum_{i_1+i_2+i_3=1}\psi^{i_1}\nabla^{i_2}\psi\nabla^{i_3}\psi_{\underline{H}},\\
\nabla_4\underline{\beta}-\nabla K-{}^*\nabla\check{\sigma}&=\psi(K,\check{\sigma})+\sum_{i_1+i_2+i_3=1}\psi^{i_1}\nabla^{i_2}\psi_H\nabla^{i_3}\psi_{\underline{H}}.
\end{aligned}
$$

Commute the first equation with i angular derivatives for $i\leq 3$. We get the equation for $\nabla_3\nabla^i\check{\sigma}$,

$$
\begin{aligned}
&\nabla_3\nabla^i\check{\sigma}+\operatorname{div}{}^*\nabla^i\underline{\beta}\\
&\quad=\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}(K,\check{\sigma})\\
&\quad\quad+\sum_{i_1+i_2+i_3+i_4=i+1}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_{\underline{H}}.
\end{aligned}
\tag{51}
$$

Notice that in the above equation, there are terms arising from the commutator $[\nabla^i,\operatorname{div}]\underline{\beta}$, which can be expressed in terms of the Gauss curvature. After substituting also the Codazzi equations for $\underline{\beta}$, we get that these terms have the form of the first term in the above expression. The equation for $\nabla_3\nabla^i K$ has a similar structure:

$$
\begin{aligned}
&\nabla_3\nabla^i K - \operatorname{div}\nabla^i\underline{\beta} \\
&\quad= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}(K,\check{\sigma}) \\
&\quad\quad+ \sum_{i_1+i_2+i_3+i_4=i+1} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_{\underline{H}}.
\end{aligned} \tag{52}
$$

Finally, we have the following structure for $\nabla_4\nabla^i\underline{\beta}$:

$$
\begin{aligned}
&\nabla_4\nabla^i\underline{\beta} - \nabla\nabla^i K -^* \nabla\nabla^i\check{\sigma} \\
&\quad= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma}) + \sum_{i_1+i_2+i_3=i-1} \psi^{i_1}\nabla^{i_2}K\nabla^{i_3}(K,\check{\sigma}) \\
&\quad\quad+ \sum_{i_1+i_2+i_3+i_4=i+1} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}.
\end{aligned} \tag{53}
$$

As a shorthand, we denote by $F_{i,1}$ the terms of the form

$$
F_{i,1} := \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}(K,\check{\sigma}) + \sum_{i_1+i_2+i_3+i_4=i+1} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_{\underline{H}}
$$

and by $F_{i,2}$ the terms of the form

$$
\begin{aligned}
F_{i,2} :=& \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma}) + \sum_{i_1+i_2+i_3=i-1} \psi^{i_1}\nabla^{i_2}K\nabla^{i_3}(K,\check{\sigma}) \\
&+ \sum_{i_1+i_2+i_3+i_4=i+1} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}.
\end{aligned}
$$

Contracting (53) with $\nabla^i\underline{\beta}$, integrating in the region $D_{u,\underline{u}}$, applying Proposition 27 and using equations (51) and (52) yield the following identity on the derivatives of the curvature:

$$
\begin{aligned}
&\int_{D_{u,\underline{u}}} f(u)^2\langle\nabla^i\underline{\beta},\nabla_4\nabla^i\underline{\beta}\rangle_\gamma \\
&\quad= \int_{D_{u,\underline{u}}} f(u)^2\langle\underline{\beta},\nabla\nabla^i K +^* \nabla\nabla^i\sigma\rangle_\gamma + f(u)^2\langle\nabla^i\underline{\beta},F_{i,2}\rangle_\gamma \\
&\quad= \int_{D_{u,\underline{u}}} -f(u)^2\langle\operatorname{div}\nabla^i\underline{\beta},\nabla^i K\rangle_\gamma + f(u)^2\langle\operatorname{div}{}^*\nabla^i\underline{\beta},\nabla^i\check{\sigma}\rangle_\gamma + f(u)^2\langle\nabla^i\underline{\beta},F_{i,2}\rangle_\gamma \\
&\quad= \int_{D_{u,\underline{u}}} -f(u)^2\langle\nabla_3\nabla^i K,\nabla^i K\rangle_\gamma - f(u)^2\langle\nabla_3\nabla^i\check{\sigma},\nabla^i\check{\sigma}\rangle_\gamma \\
&\quad\quad+ \int_{D_{u,\underline{u}}} f(u)^2\langle\nabla^i\underline{\beta},F_{i,2}\rangle_\gamma + f(u)^2\langle\nabla^i(K,\check{\sigma}),F_{i,1}\rangle_\gamma.
\end{aligned} \tag{54}
$$

Using Proposition 26, since $\nabla_4 f(u) = 0$, we have

$$\begin{aligned}
&\int f(u)^2 \langle \nabla^i \underline{\beta}, \nabla_4 \nabla^i \underline{\beta} \rangle_\gamma \\
&= \frac{1}{2}\left(\int_{\underline{H}_{\underline{u}}} f(u)^2 |\nabla^i \underline{\beta}|^2 - \int_{\underline{H}_0} f(u)^2 |\nabla^i \underline{\beta}|^2 \right) \\
&\quad + \int_{D_{u,\underline{u}}} f(u)^2 \left(\omega - \frac{1}{2}\operatorname{tr}\chi \right) |\nabla^i \underline{\beta}|^2.
\end{aligned} \tag{55}$$

For the terms with $\nabla_3 \nabla^i K$ and $\nabla_3 \nabla^i \check{\sigma}$, we similarly apply Proposition 26, but noting that there is an extra contribution coming from $\nabla_3 f(u)$:

$$\begin{aligned}
&\int_{D_{u,\underline{u}}} f(u)^2 \langle \nabla^i K, \nabla_3 \nabla^i K \rangle_\gamma \\
&= -\int_{D_{u,\underline{u}}} f(u) \nabla_3 f(u) |\nabla^i K|^2 \\
&\quad + \frac{1}{2}\left(\int_{H_u} f(u)^2 |\nabla^i K|^2 - \int_{H_0} f(u)^2 |\nabla^i K|^2 \right) \\
&\quad + \int_{D_{u,\underline{u}}} f(u)^2 (\underline{\omega} - \frac{1}{2}\operatorname{tr}\underline{\chi}) |\nabla^i K|^2.
\end{aligned} \tag{56}$$

Similarly,

$$\begin{aligned}
&\int_{D_{u,\underline{u}}} f(u)^2 \langle \nabla^i \check{\sigma}, \nabla_3 \nabla^i \check{\sigma} \rangle_\gamma \\
&= -\int_{D_{u,\underline{u}}} f(u) \nabla_3 f(u) |\nabla^i \check{\sigma}|^2 \\
&\quad + \frac{1}{2}\left(\int_{H_u} f(u)^2 |\nabla^i \check{\sigma}|^2 - \int_{H_0} f(u)^2 |\nabla^i \check{\sigma}|^2 \right) \\
&\quad + \int_{D_{u,\underline{u}}} f(u)^2 \left(\underline{\omega} - \frac{1}{2}\operatorname{tr}\underline{\chi} \right) |\nabla^i \check{\sigma}|^2.
\end{aligned} \tag{57}$$

Combining (54)–(57), we thus have the identity

$$\begin{aligned}
&\int_{\underline{H}_{\underline{u}}} f(u)^2 |\nabla^i \underline{\beta}|^2 + \int_{H_u} f(u)^2 |\nabla^i K|^2 + \int_{H_u} f(u)^2 |\nabla^i \check{\sigma}| \\
&\quad - 2\int_{D_{u,\underline{u}}} f(u) \nabla_3 f(u) |\nabla^i K|^2 - 2\int_{D_{u,\underline{u}}} f(u) \nabla_3 f(u) |\nabla^i \check{\sigma}|^2 \\
&= \int_{\underline{H}_{\underline{u}'}} f(u)^2 |\nabla^i \beta|^2 + \int_{H_{u'}} f(u)^2 |\nabla^i K|^2 + \int_{H_{u'}} f(u)^2 |\nabla^i \check{\sigma}|^2 \\
&\quad - 2\int_{D_{u,\underline{u}}} f(u)^2 \left(\omega - \frac{1}{2}\operatorname{tr}\chi \right) |\nabla^i \underline{\beta}|^2 - 2\int_{D_{u,\underline{u}}} f(u)^2 \left(\underline{\omega} - \frac{1}{2}\operatorname{tr}\underline{\chi} \right) (|\nabla^i K|^2 + |\nabla^i \check{\sigma}|^2) \\
&\quad + \int_{D_{u,\underline{u}}} f(u)^2 \langle \nabla^i \underline{\beta}, F_{i,2} \rangle_\gamma + \int_{D_{u,\underline{u}}} f(u)^2 \langle \nabla^i (K, \check{\sigma}), F_{i,1} \rangle_\gamma.
\end{aligned}$$

The terms

$$-2\int_{D_{u,\underline{u}}} f(u) \nabla_3 f(u) |\nabla^i K|^2 - 2\int_{D_{u,\underline{u}}} f(u) \nabla_3 f(u) |\nabla^i \check{\sigma}|^2$$

on the left-hand side, which cannot be controlled[28] by the curvature flux (i.e., the integrals of ∇^i of the curvature components along H_u or $\underline{H}_{\underline{u}}$), have a favorable sign! This is because the weight function f satisfies $f(u)\nabla_3 f(u) < 0$. Therefore, we get an *inequality* for every i:

$$\begin{aligned}
&\int_{\underline{H}_{\underline{u}}} f(u)^2|\nabla^i\underline{\beta}|^2 + \int_{H_u} f(u)^2|\nabla^i K|^2 + \int_{H_u} f(u)^2|\nabla^i\check{\sigma}|^2 \\
&\quad\leq \int_{\underline{H}_{\underline{u}'}} f(u)^2|\nabla^i\beta|^2 + \int_{H_{u'}} f(u)^2|\nabla^i K|^2 + \int_{H_{u'}} f(u)^2|\nabla^i\check{\sigma}|^2 \\
&\quad\quad + C\left\|f(u)^2\left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla^i\underline{\beta}\nabla^i\underline{\beta}\right\|_{L^1_u L^1_{\underline{u}} L^1(S)} \\
&\quad\quad + C\left\|f(u)^2\left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla^i(K,\check{\sigma})\nabla^i(K,\check{\sigma})\right\|_{L^1_u L^1_{\underline{u}} L^1(S)} \\
&\quad\quad + C||f(u)^2\nabla^i\underline{\beta}F_{i,2}||_{L^1_u L^1_{\underline{u}} L^1(S)} + C||f(u)^2\nabla^i(K,\check{\sigma})F_{i,1}||_{L^1_u L^1_{\underline{u}} L^1(S)}.
\end{aligned}$$

We now add the above inequalities for $i \leq 3$. One can easily check that the terms

$$\sum_{i\leq 3}\left\|f(u)^2\left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla^i(K,\check{\sigma})\nabla^i(K,\check{\sigma})\right\|_{L^1_u L^1_{\underline{u}} L^1(S)},$$

$$\sum_{i\leq 3}||f(u)^2\nabla^i\underline{\beta}F_{i,2}||_{L^1_u L^1_{\underline{u}} L^1(S)},$$

and

$$\sum_{i\leq 3}||f(u)^2\nabla^i(K,\check{\sigma})F_{i,1}||_{L^1_u L^1_{\underline{u}} L^1(S)}$$

have the form of one of the terms in the statement of the proposition. After applying the Codazzi equation

$$\underline{\beta} = \nabla\psi_{\underline{H}} + \psi(\psi + \psi_{\underline{H}})$$

to one of the $\underline{\beta}$'s, we note that the term

$$\sum_{i\leq 3}\left\|f(u)^2\left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla^i\underline{\beta}\nabla^i\underline{\beta}\right\|_{L^1_u L^1_{\underline{u}} L^1(S)}$$

is also one of the terms in the statement of the proposition. □

To close the energy estimates, we also need to control $\nabla^i\beta$ in $L^2(H)$ and $\nabla^i(K,\check{\sigma})$ in $L^2(\underline{H})$. It is not difficult to see, by virtue of the structure of the Einstein equations, that Proposition 28 also holds when all the barred and unbarred quantities are exchanged. The proof is exactly analogous to that of Proposition 28.

---

[28]In fact, if we do not drop this term, we can control the spacetime integral

$$\int_{D_{u,\underline{u}}}(-f(u)\nabla_3 f(u))|\nabla^i(K,\check{\sigma})|^2,$$

where the weight $(-f(u)\nabla_3 f(u))$ can be singular. For weights such as $f(u) = (u-u_*)^\alpha$ for $\alpha < \frac{1}{2}$ or $f(u) = (u-u_*)^{\frac{1}{2}}\log^\beta(\frac{1}{u-u_*})$ for $\beta > \frac{1}{2}$, this bound is *logarithmically* stronger than simply taking the bound for $\int_{H_u} f(u)^2|\nabla^i(K,\check{\sigma})|^2$ and integrating in u.

Proposition 29. *The following L^2 estimates for the curvature components hold:*

$$
\begin{aligned}
&\sum_{i\leq 3}\Big(||f(\underline{u})\nabla^i(K,\check{\sigma})||^2_{L^\infty_{\underline{u}}L^2_uL^2(S)}+||f(\underline{u})\nabla^i\beta||^2_{L^\infty_u L^2_{\underline{u}}L^2(S)}\Big)\\
&\quad\leq \sum_{i\leq 3}\Big(||f(\underline{u})\nabla^i(K,\check{\sigma})||^2_{L^2_uL^2(S_{u,0})}+||f(\underline{u})\nabla^i\beta||^2_{L^2_{\underline{u}}L^2(S_{0,\underline{u}})}\Big)\\
&\qquad+\left\|f(\underline{u})^2\left(\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi_H\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\qquad+\left\|f(\underline{u})^2\left(\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\qquad+\left\|f(\underline{u})^2\left(\sum_{i\leq 3}\nabla^i\beta\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\qquad+\left\|f(\underline{u})^2\left(\sum_{i\leq 3}\nabla^i\beta\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 2}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}K\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\qquad+\left\|f(\underline{u})^2\left(\sum_{i\leq 3}\nabla^i\beta\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}\psi_H\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}.
\end{aligned}
$$

We now show that we can control all the nonlinear error terms in the energy estimates. We show this for K and $\check{\sigma}$ in $L^2(H_u)$ and $\underline{\beta}$ in $L^2(\underline{H}_{\underline{u}})$. The other case can be dealt with in a similar fashion (see Proposition 31).

Proposition 30. *There exists $\epsilon_0=\epsilon_0(\mathcal{O}_{\text{ini}},\mathcal{R}_{\text{ini}},\Delta_3)$ sufficiently small such that whenever $\epsilon\leq\epsilon_0$,*

$$\sum_{i\leq 3}(||f(u)\nabla^i(K,\check{\sigma})||_{L^\infty_uL^2_{\underline{u}}L^2(S)}+||f(u)\nabla^i\underline{\beta}||_{L^\infty_{\underline{u}}L^2_uL^2(S)})\leq C(\mathcal{O}_{\text{ini}},\mathcal{R}_{\text{ini}}).$$

Proof. To prove the curvature estimates, we use Proposition 28. By assumptions of Theorem 2 (see also Remark 11), the two terms corresponding to the initial data are bounded by a constant $C(\mathcal{R}_{\text{ini}})$ depending only on initial data. Therefore, we need to control the remaining five error terms in Proposition 28. We first look at the term

$$\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}\psi\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}.$$

Using Propositions 15 and 25, together with the bootstrap assumption (A3), we have

$$\begin{aligned}
&\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}\psi\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\leq C\left(\sum_{i\leq 3}||f(u)\nabla^i(K,\check{\sigma})||_{L^\infty_uL^2_{\underline{u}}L^2(S)}\right)\left(\sum_{i_1\leq 3}\sum_{i_2\leq 4}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_uL^\infty_{\underline{u}}L^2(S)}\right)\\
&\quad\times\left(\sum_{i_3\leq 4}||f(u)\nabla^{i_3}\psi||_{L^\infty_uL^2_{\underline{u}}L^2(S)}\right)\left(\sum_{i_4\leq 3}||f(u)\nabla^{i_4}\psi_{\underline{H}}||_{L^2_uL^\infty_{\underline{u}}L^2(S)}\right)||f(u)^{-1}||_{L^2_u}\\
&\quad+C\epsilon\left(\sum_{i\leq 2}||f(u)\nabla^i(K,\check{\sigma})||_{L^\infty_uL^2_{\underline{u}}L^2(S)}\right)||\psi||_{L^\infty_uL^\infty_{\underline{u}}L^\infty(S)}||f(u)\nabla^4\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2_uL^2(S)}\\
&\leq C(\mathcal{O}_{\text{ini}})\Delta_3(1+\Delta_3)\epsilon.
\end{aligned}$$

The term

$$\begin{aligned}
&||f(u)^2\left(\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}(K,\check{\sigma})\right)||_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\qquad\leq C(\mathcal{O}_{\text{ini}})\Delta_3(1+\Delta_3)\epsilon
\end{aligned}$$

similarly as in the previous estimate since by Propositions 16 and 17, $\nabla^i(K,\check{\sigma})$ satisfies exactly the same estimates as $\nabla^{i+1}\psi$. We then consider the third nonlinear term

$$\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}.$$

Using Propositions 15 and 25 and the bootstrap assumptions (A3), we have

$$\begin{aligned}
&\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\qquad\leq C\epsilon\left(\sum_{i\leq 3}||f(u)\nabla^i\underline{\beta}||_{L^\infty_{\underline{u}}L^2_uL^2(S)}\right)\left(\sum_{i_1\leq 3,1\leq i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_uL^\infty_{\underline{u}}L^2(S)}\right)\\
&\qquad\quad\times\left\|f(u)\sum_{i\leq 3}\nabla^i(K,\check{\sigma})\right\|_{L^\infty_uL^2_{\underline{u}}L^2(S)}\\
&\qquad\leq C(\mathcal{O}_{\text{ini}})\Delta_3(1+\Delta_3)\epsilon.
\end{aligned}$$

The fourth nonlinear term can be estimated analogously as the third nonlinear term by

$$\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 2}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}K\nabla^{i_4}(K,\check{\sigma})\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}$$
$$\leq C(\mathcal{O}_{\text{ini}})\Delta_3(1+\Delta_3)\epsilon.$$

As before, this is because by Propositions 16 and 17, $\nabla^i(K,\check{\sigma})$ satisfies exactly the same estimates as $\nabla^{i+1}\psi$. Thus it remains to control

$$\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}.$$

This term can be bounded as follows:

$$\begin{aligned}
&\left\|f(u)^2\left(\sum_{i\leq 3}\nabla^i\underline{\beta}\right)\left(\sum_{i_1+i_2+i_3+i_4\leq 4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}\right)\right\|_{L^1_uL^1_{\underline{u}}L^1(S)}\\
&\leq C\left(\sum_{i\leq 3}||f(u)\nabla^i\underline{\beta}||_{L^\infty_{\underline{u}}L^2_uL^2(S)}\right)\left(\sum_{i_1\leq 3}\sum_{i_2\leq 3}||\nabla^{i_1}\psi||^{i_2}_{L^\infty_uL^\infty_{\underline{u}}L^2(S)}\right)\\
&\quad\times\left(\sum_{i_3\leq 3}||f(\underline{u})\nabla^{i_3}\psi_H||_{L^2_{\underline{u}}L^\infty_uL^2(S)}\right)\left(\sum_{i_4\leq 4}||f(u)\nabla^{i_4}\psi_{\underline{H}}||_{L^\infty_{\underline{u}}L^2_uL^2(S)}\right)||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}\\
&\quad+C\left(\sum_{i\leq 2}||f(u)\nabla^i\underline{\beta}||_{L^\infty_{\underline{u}}L^2_uL^2(S)}\right)||f(\underline{u})^{-1}||_{L^2_{\underline{u}}}||f(\underline{u})\nabla^4\psi_H||_{L^\infty_uL^2_{\underline{u}}L^\infty(S)}\\
&\quad\times||f(u)\psi_{\underline{H}}||_{L^2_uL^\infty_{\underline{u}}L^\infty(S)}\\
&\leq C(\mathcal{O}_{\text{ini}})\Delta_3(1+\Delta_3)\epsilon.
\end{aligned}$$

Therefore, gathering all the above estimates, we have

$$\sum_{i\leq 3}\left(||f(u)\nabla^i(K,\check{\sigma})||^2_{L^\infty_uL^2_{\underline{u}}L^2(S)}+||f(u)\nabla^i\underline{\beta}||^2_{L^\infty_{\underline{u}}L^2_uL^2(S)}\right)$$
$$\leq C(\mathcal{O}_{\text{ini}},\mathcal{R}_{\text{ini}})+C(\mathcal{O}_{\text{ini}})\Delta_3(1+\Delta_3)\epsilon,$$

which implies the conclusion of the proposition after taking ϵ to be sufficiently small. □

Notice that the schematic equations are symmetric under the change $\nabla_3\leftrightarrow\nabla_4$, $u\leftrightarrow\underline{u}$, and $\psi_H\leftrightarrow\psi_{\underline{H}}$. Since the conditions for the initial data are also symmetric, we also have the following analogous energy estimates for $\nabla^i\beta$ on H_u and $\nabla^i(K,\check{\sigma})$ on $\underline{H}_{\underline{u}}$:

Proposition 31. *There exists $\epsilon_0=\epsilon_0(\mathcal{O}_{\text{ini}},\mathcal{R}_{\text{ini}},\Delta_3)$ sufficiently small such that whenever $\epsilon\leq\epsilon_0$,*

$$\sum_{i\leq 3}(||f(\underline{u})\nabla^i\beta||_{L^\infty_uL^2_{\underline{u}}L^2(S)}+||f(\underline{u})\nabla^i(K,\check{\sigma})||_{L^\infty_{\underline{u}}L^2_uL^2(S)})\leq C(\mathcal{O}_{\text{ini}},\mathcal{R}_{\text{ini}}).$$

Propositions 30 and 31 together imply

Proposition 32. *There exists* $\epsilon_0 = \epsilon_0(\mathcal{O}_{\text{ini}}, \mathcal{R}_{\text{ini}})$ *such that whenever* $\epsilon \leq \epsilon_0$,

$$\mathcal{R} \leq C(\mathcal{O}_{\text{ini}}, \mathcal{R}_{\text{ini}}).$$

Proof. Let

$$\Delta_3 \gg C(\mathcal{O}_{\text{ini}}, \mathcal{R}_{\text{ini}}),$$

where $C(\mathcal{O}_{\text{ini}}, \mathcal{R}_{\text{ini}})$ is taken to be the maximum of the upper bounds in Propositions 30 and 31. Hence, the choice of Δ_3 depends only on $\mathcal{O}_{\text{ini}}$ and $\mathcal{R}_{\text{ini}}$. Thus, by Propositions 30 and 31, the bootstrap assumption (A3) can be improved by choosing ϵ sufficiently small depending on $\mathcal{O}_{\text{ini}}$ and $\mathcal{R}_{\text{ini}}$. □

Combining Propositions 15, 25, and 32, we conclude the proof of Theorem 5. As mentioned previously, standard methods then imply Theorem 2.

9. Nature of the singular boundary

As described by Theorems 3 and 4, we will also prove the regularity and singularity of the boundary H_{u_*} and $\underline{H}_{\underline{u}_*}$. We first prove the regularity of the boundary asserted in Theorem 3.

Proof of Theorem 3. The fact that $(\mathcal{M}, g)$ can be extended continuously up to and beyond H_{u_*} and $\underline{H}_{\underline{u}_*}$ simply follows from the continuity of the metric components Ω, γ, and b proved in Propositions 1–4. To obtain the higher regularity for γ, we recall the equations (32), (36), and (37):

$$\frac{\partial}{\partial \underline{u}}\Omega^{-1} = 2\omega, \qquad \frac{\partial}{\partial \underline{u}}\gamma_{AB} = 2\Omega\chi_{AB}, \qquad \frac{\partial}{\partial \underline{u}}b^A = -4\Omega^2\zeta^A.$$

Commuting these equations with $(\frac{\partial}{\partial\theta})^i$ and using the bounds[29] for the Ricci coefficients obtained in the proof of Theorem 5, we conclude that

$$\sum_{i_1+i_2\leq 4}\sup_{0\leq u\leq u_*}\sup_{0\leq \underline{u}\leq \underline{u}_*}\left\|\left(\frac{\partial}{\partial\theta^1}\right)^{i_1}\left(\frac{\partial}{\partial\theta^2}\right)^{i_2}(\gamma, b, \Omega)\right\|_{L^2(U_i(u,\underline{u}))} \leq C.$$

The boundedness of ψ and its angular derivatives

$$\sum_{i\leq 3}\|\nabla^i\psi\|_{L^\infty_u L^\infty_{\underline{u}} L^2(S)} \leq C$$

are already proved in Theorem 5. To control $\psi_{\underline{H}}$ and its angular derivatives on the singular boundary $\underline{H}_{\underline{u}_*}$, we first note that by the smoothness assumption on the *interior* of the initial hypersurface $\underline{H}_0$, we have that for every fixed $U \in [0, u_*)$,

$$\sum_{i\leq 5}\sup_{0\leq u\leq U}\|\nabla^i\psi_{\underline{H}}\|_{L^2(S_{u,0})} \leq C_U$$

for some finite C_U. We now revisit the proof of Proposition 13 to bound $\nabla^i\psi_{\underline{H}}$ up to $i \leq 3$ for $u \in [0, U]$. Restricting to $[0, U]$, $f(u)^{-1}$ is bounded. Therefore, the

[29]Notice that by controlling γ and its coordinate angular derivatives $(\frac{\partial}{\partial\theta})^i\gamma$, we can show also that $\frac{\partial}{\partial\theta}$ and ∇ are comparable up to lower-order terms, which allows us to apply the estimates for $\nabla^i\mathrm{tr}\,\chi$, $\nabla^i\hat{\chi}$, $\nabla^i\eta$, and $\nabla^i\underline{\eta}$ to bound the coordinate angular derivatives of the metric components.

estimates in (43), (44), and (45) are bounded uniformly in u. Finally, (46) can be replaced by the estimate

$$\int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4\leq 3} \|\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi_H\nabla^{i_4}\psi_{\underline{H}}\|_{L^2(S_{u,\underline{u}'})}d\underline{u}'$$
$$\leq C\int_0^{\underline{u}}\left(\sum_{i_1\leq 3}||\nabla^{i_1}\psi_H||_{L^2(S_{u,\underline{u}'})}\right)\left(\sum_{i_2\leq 3}\sup_{0\leq \underline{u}''\leq \underline{u}}||\nabla^{i_2}\psi_{\underline{H}}||_{L^2(S_{u,\underline{u}''})}\right)d\underline{u}'.$$

Putting these bounds together, we have

$$\sum_{i\leq 3}\sup_{\substack{0\leq u\leq U\\ 0\leq \underline{u}'\leq \underline{u}}}\|\nabla^i\psi_{\underline{H}}\|_{L^2(S_{u,\underline{u}'})}$$
$$\leq C_U + C\int_0^{\underline{u}}\left(\sum_{i_1\leq 3}||\nabla^{i_1}\psi_H||_{L^2(S_{u,\underline{u}'})}\right)\left(\sum_{i_2\leq 3}\sup_{\substack{0\leq u'\leq U\\ 0\leq \underline{u}''\leq \underline{u}'}}||\nabla^{i_2}\psi_{\underline{H}}||_{L^2(S_{u,\underline{u}''})}\right)d\underline{u}',$$

which implies

$$\sum_{i\leq 3}\sup_{\substack{0\leq u\leq U\\ 0\leq \underline{u}\leq \underline{u}_*}}\|\nabla^i\psi_{\underline{H}}\|_{L^2(S_{u,\underline{u}})}\leq C_U \tag{58}$$

after applying Gronwall's inequality.

To conclude the proof, it remains to control $\nabla_3\nabla^i\psi$ and $\nabla_3\nabla^i\psi_{\underline{H}}$ for $i\leq 2$. Since $\underline{\eta}$ obeys a ∇_3 equation (see (9)), by directly controlling the right-hand side of the null structure equation (commuted with angular derivatives) and using the bounds in Theorem 5, we get

$$\sum_{i\leq 2}\sup_{\substack{0\leq u\leq U\\ 0\leq \underline{u}\leq \underline{u}_*}}\|\nabla_3\nabla^i\underline{\eta}\|_{L^2(S_{u,\underline{u}})}\leq C_U.$$

To control the term $\nabla_3\nabla^i\eta$, notice that combining the $\nabla_3\underline{\eta}$ equation in (9) and the equations in (7), we have

$$\nabla_3\eta = -\nabla_3\underline{\eta}+2\nabla_3\nabla(\log\Omega) = \underline{\chi}\cdot(\underline{\eta}-\eta)-\underline{\beta}-\nabla\underline{\omega}-4\underline{\omega}\nabla(\log\Omega)-2\underline{\chi}\cdot\nabla(\log\Omega).$$

Upon expressing $\underline{\beta}$ in terms of $\psi_{\underline{H}}$ using the Codazzi equation in (10), commuting the equation with ∇^i, and using the bound (58), we get

$$\sum_{i\leq 2}\sup_{\substack{0\leq u\leq U\\ 0\leq \underline{u}\leq \underline{u}_*}}\|\nabla_3\nabla^i\eta\|_{L^2(S_{u,\underline{u}})}\leq C_U. \tag{59}$$

Finally, we control the terms $\nabla_3\nabla^i\psi_{\underline{H}}$. Commuting the null structure equations for $\nabla_4\psi_{\underline{H}}$ in (8) and (9) with $\nabla_3\nabla^i$, we have

$$\begin{aligned}\nabla_4\nabla_3\nabla^i\psi_{\underline{H}}&\\ =&\sum_{\substack{j_1+j_2+j_3+j_4=1\\ i_1+i_2+i_3+i_4=i}}(\nabla^{i_1}\psi_{\underline{H}}^{j_1}\nabla_3^{j_2}\nabla^{i_2}\psi^{i_3}\nabla_3^{j_3}\nabla^{i_4}K+\nabla^{i_1}\psi_{\underline{H}}^{j_1}\nabla_3^{j_2}\nabla^{i_2}\psi^{i_3}\nabla_3^{j_3}\nabla^{i_4}\nabla\psi)\\ &+\sum_{\substack{j_1+j_2+j_3=1\\ i_1+i_2+i_3+i_4+i_5=i}}\nabla^{i_1}\psi_{\underline{H}}^{j_1}\nabla_3^{j_2}\nabla^{i_2}\psi^{i_3}\nabla_3^{j_3}\nabla^{i_4}\psi\nabla^{i_5}\psi\\ &+\sum_{\substack{j_1+j_2+j_3+j_4=1\\ i_1+i_2+i_3+i_4+i_5=i}}\nabla^{i_1}\psi_{\underline{H}}^{j_1}\nabla_3^{j_2}\nabla^{i_2}\psi^{i_3}\nabla_3^{j_3}\nabla^{i_4}\psi_{\underline{H}}\nabla_3^{j_4}\nabla^{i_5}\psi_H.\end{aligned}$$

Estimating directly the right-hand side of the null structure equations or the Bianchi equations, we can easily show that

$$\sum_{i\leq 2}\sup_{0\leq u\leq U}\|\nabla_3(\nabla^iK,\nabla^i\underline{\eta},\nabla^i\psi_H)\|_{L^1_{\underline{u}}L^2(S)}\leq C_U.$$

Using also (59), we thus have

$$\begin{aligned}&\sum_{i\leq 2}\sup_{0\leq u\leq U}\|\nabla_3\nabla^i\psi_{\underline{H}}\|_{L^2(S_{u,\underline{u}})}\\ &\qquad\leq C_U+C_U\int_0^{\underline{u}}\sum_{i_1+i_2+i_3+i_4\leq 2}\|\nabla^{i_1}\psi^{i_2}\nabla_3\nabla^{i_3}\psi_{\underline{H}}\nabla^{i_4}\psi_H\|_{L^2(S_{u,\underline{u}'})}d\underline{u}'.\end{aligned}$$

Using Gronwall's inequality, we get

$$\sum_{i\leq 2}\sup_{\substack{0\leq u\leq U\\ 0\leq\underline{u}\leq\underline{u}_*}}\|\nabla_3\nabla^i\psi_{\underline{H}}\|_{L^2(S_{u,\underline{u}})}\leq C_U.$$

In particular combining the above estimates, we obtain

$$\sum_{i\leq 3-j,\,j\leq 1}\sup_{0\leq u\leq U}\|\nabla^j\nabla^i(\psi_{\underline{H}},\psi)\|_{L^2(S_{u,\underline{u}_*})}\leq C_U$$

on $\underline{H}_{\underline{u}_*}$, as desired. □

Finally, we move to the proof of Theorem 4. First, we prove

Proposition 33. *Suppose, in addition to the assumptions in Theorem 2, $\hat{\chi}$ initially obeys*

$$\int_0^{\underline{u}_*}|\hat{\chi}\restriction_\gamma(\underline{u}')|^2d\underline{u}'=\infty,$$

along an outgoing null generator γ of H_0. Let $\Phi_u(\gamma)$ be the image of γ under the one-parameter family of diffeomorphisms generated by $\underline{L}$. Then

$$\int_0^{\underline{u}_*}(\mathrm{tr}\,\chi\restriction_{\Phi_u(\gamma)}(\underline{u}'))^2+|\hat{\chi}\restriction_{\Phi_u(\gamma)}(\underline{u}')|^2d\underline{u}'=\infty,$$

holds for every $0\leq u<u_*$.

Similarly suppose, in addition to the assumptions in Theorem 2, $\underline{\hat{\chi}}$ initially obeys

$$\int_0^{u_*}|\underline{\hat{\chi}}\restriction_\gamma(u')|^2du'=\infty,$$

along an outgoing null generator γ of $\underline{H}_0$. Let $\underline{\Phi}_{\underline{u}}(\gamma)$ be the image of γ under the one-parameter family of diffeomorphisms generated by L. Then

$$\int_0^{u_*} (\mathrm{tr}\,\underline{\chi}\restriction_{\Phi_{\underline{u}}(\gamma)}(u'))^2 + |\hat{\underline{\chi}}\restriction_{\Phi_{\underline{u}}(\gamma)}(u')|^2 du' = \infty,$$

holds for every $0 \leq \underline{u} < \underline{u}_*$.

Proof. Fix $U \in (0, u_*)$. Suppose

$$\int_0^{\underline{u}_*} (\mathrm{tr}\,\chi\restriction_{\Phi_U(\gamma)}(\underline{u}'))^2 d\underline{u}' < \infty. \tag{60}$$

We want to show that under the assumption (60), we have

$$\int_0^{\underline{u}_*} |\hat{\chi}\restriction_{\Phi_U(\gamma)}(\underline{u}')|^2 d\underline{u}' = \infty,$$

which will then imply the desired conclusion.

Using (60), define $h : [0, \underline{u}_*) \to \mathbb{R}$ by

$$h(\underline{u}) = |\mathrm{tr}\,\chi\restriction_{\Phi_U(\gamma)}(\underline{u})|$$

such that

$$\int_0^{\underline{u}_*} h(\underline{u}')^2 d\underline{u}' < \infty.$$

Consider the following null structure equation for $\mathrm{tr}\,\chi$:

$$\nabla_3 \mathrm{tr}\,\chi + \mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi = 2\underline{\omega}\mathrm{tr}\,\chi - 2K + 2\mathrm{div}\,\eta + 2|\eta|^2.$$

Along the integral curve of $-e_3$ emanating from $\Phi_u(\gamma)$, we thus have

$$\begin{aligned}&\frac{d}{du}\left(e^{\int_U^u (\Omega\mathrm{tr}\,\underline{\chi} - 2\Omega\underline{\omega})\restriction_{\Phi_{u'}(\gamma)}(\underline{u})du'}\mathrm{tr}\,\chi\restriction_{\Phi_u(\gamma)}(u)\right)\\ &\quad = e^{\int_U^u (\Omega\mathrm{tr}\,\underline{\chi} - 2\Omega\underline{\omega})\restriction_{\Phi_{u'}(\gamma)}(\underline{u})du'}(-2K + 2\mathrm{div}\,\eta + 2|n|^2).\end{aligned}$$

By the estimates derived in the proof of Theorem 5, K, $\nabla\eta$, η are bounded and $\mathrm{tr}\,\underline{\chi}$, $\underline{\omega}$ are in $L^1_u L^\infty(S)$. Therefore,

$$|\mathrm{tr}\,\chi\restriction_{\Phi_u(\gamma)}(\underline{u})| \leq Ch(\underline{u}) \quad \text{for all } u. \tag{61}$$

Consider the following null structure equation for $\hat{\chi}$:

$$\nabla_3\hat{\chi} + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\hat{\chi} = \nabla\widehat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\,\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta.$$

Contract this equation with $\hat{\chi}$ to get

$$\frac{1}{2}\nabla_3|\hat{\chi}|^2 + \frac{1}{2}\mathrm{tr}\,\underline{\chi}|\hat{\chi}|^2 - 2\underline{\omega}|\hat{\chi}|^2 = \left(\nabla\widehat{\otimes}\eta - \frac{1}{2}\mathrm{tr}\,\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta\right)\cdot\hat{\chi},$$

which implies

$$\left|\nabla_3|\hat{\chi}| + \frac{1}{2}\mathrm{tr}\,\underline{\chi}|\hat{\chi}| - 2\underline{\omega}|\hat{\chi}|\right| \leq |\nabla\widehat{\otimes}\eta| + \left|\frac{1}{2}\mathrm{tr}\,\chi\hat{\underline{\chi}}\right| + |\eta\widehat{\otimes}\eta|.$$

This implies that along the integral curve of e_3, we have

$$\begin{aligned}&\left|\frac{d}{du}(e^{\int_U^u (\frac{1}{2}\Omega\mathrm{tr}\,\underline{\chi} - 2\Omega\underline{\omega})\restriction_{\Phi_{u'}(\gamma)}(\underline{u})du'}|\hat{\chi}|\restriction_{\Phi_u(\gamma)}(\underline{u}))\right|\\ &\quad \leq 2e^{\int_U^u (\frac{1}{2}\Omega\mathrm{tr}\,\underline{\chi} - 2\Omega\underline{\omega})\restriction_{\Phi_{u'}(\gamma)}(\underline{u})du'}\left(|\nabla\widehat{\otimes}\eta| + \left|\frac{1}{2}\mathrm{tr}\,\chi\hat{\underline{\chi}}\right| + |\eta\widehat{\otimes}\eta|\right).\end{aligned}$$

Using again the fact that K, $\nabla\eta$, η, $\mathrm{tr}\,\underline{\chi}$, $\hat{\underline{\chi}}$, $\underline{\omega}$ are bounded for $u \le U$, as well as the estimate (61), we have

$$\left|\left(e^{\int_U^u (\frac{1}{2}\Omega\mathrm{tr}\,\underline{\chi}-2\Omega\underline{\omega})\restriction_{\Phi_{u'}(\gamma)}(\underline{u})du'}|\hat{\chi}|\restriction_{\Phi_u(\gamma)}(\underline{u})\right)-\left(e^{\int_U^u (\frac{1}{2}\Omega\mathrm{tr}\,\underline{\chi}-2\Omega\underline{\omega})\restriction_{\gamma}(\underline{u})du'}|\hat{\chi}|\restriction_{\gamma}(\underline{u})\right)\right|$$
$$\le C_U(1+h(\underline{u})).$$

Notice that $e^{\int_U^u (\frac{1}{2}\Omega\mathrm{tr}\,\underline{\chi}-2\Omega\underline{\omega})\restriction_{\Phi_{u'}(\gamma)}(\underline{u})du'}$ is bounded above and below uniformly in $\underline{u}$. Taking the $L^2_{\underline{u}}$ norm implies that for $u \le U$, we have

$$\int_0^{\underline{u}_*} |\hat{\chi}\restriction_{\Phi_u(\gamma)}(\underline{u}')|^2 d\underline{u}' \ge c\int_0^{\underline{u}_*} |\hat{\chi}\restriction_{\gamma}(\underline{u}')|^2 d\underline{u}' - C - C\int_0^{\underline{u}_*} h^2(\underline{u}')d\underline{u}' = \infty$$

by the assumption of the proposition. The blowup for $\underline{\chi}$ can be proved in a similar manner. □

This implies

Proposition 34. *Suppose the assumptions of Theorem 4 hold. Then, in a neighborhood of any point on $\underline{H}_{\underline{u}_*}$, $|\chi|^2$ is not integrable with respect to the spacetime volume form. Similarly, in a neighborhood of any point on H_{u_*}, $|\underline{\chi}|^2$ is not integrable with respect to the spacetime volume form.*

Proof. We begin with $|\underline{\chi}|^2$ near H_{u_*}. By definition, the image of the initial incoming null generator under the map $\underline{\Phi}_{\underline{u}}$ defined in Proposition 33 has constant $\underline{u}$, θ^1, and θ^2 values. Also, by Propositions 1 and 2, the spacetime volume element $2\Omega^2\sqrt{\det\gamma}$ is bounded uniformly above and below. Therefore, for any neighborhood $\mathcal{N}$ of $p = (u, \underline{u}_*, \theta^1, \theta^2) \in \underline{H}_{\underline{u}_*}$, we have

$$\int_{\mathcal{N}}((\mathrm{tr}\,\underline{\chi})^2 + |\hat{\underline{\chi}}|^2)$$
$$\ge c\int_{\theta^2-\delta}^{\theta^2+\delta}\int_{\theta^1-\delta}^{\theta^1+\delta}\int_{u-\delta}^{u+\delta}\int_{\underline{u}_*-\delta}^{\underline{u}_*}((\mathrm{tr}\,\underline{\chi})^2+|\hat{\underline{\chi}}|^2)(u', \underline{u}', (\theta^1)', (\theta^2)')d\underline{u}'\,du'\,d(\theta^1)'\,d(\theta^2)'$$
$$= \infty,$$

by Proposition 33.

To prove the corresponding statement for $|\chi|^2$ near $\underline{H}_{\underline{u}_*}$, we first change to the coordinate system $(u, \underline{u}, \tilde{\theta}^1(\underline{u}; u, \theta), \tilde{\theta}^2(\underline{u}; u, \theta))$ such that $\underline{L} = \frac{\partial}{\partial u}$. This coordinate system can be constructed by solving the ordinary differential equations

$$\frac{d}{du}\tilde{\theta}^A(u; \underline{u}, \theta) = -b^A(u, \underline{u}, \tilde{\theta}^1, \tilde{\theta}^2),$$

with initial condition[30]

$$\tilde{\theta}^A(0; \underline{u}, \theta) = \theta^A.$$

[30]We note that since we do not have a global coordinate chart on $S_{0,0}$, the above ODE only makes sense in $(\Phi_u \circ \underline{\Phi}_{\underline{u}})(U_i) \cap (\underline{\Phi}_{\underline{u}} \circ \Phi_u)(U_j)$, where U_i, U_j are coordinate charts on $S_{0,0}$ and Φ_u and $\underline{\Phi}_{\underline{u}}$ are as defined in Proposition 33. Nevertheless, since $\Phi_u \circ \underline{\Phi}_{\underline{u}}$ and $\underline{\Phi}_{\underline{u}} \circ \Phi_u$ are both diffeomorphisms between $S_{0,0}$ and $S_{u,\underline{u}}$, for every point $p \in S_{u,\underline{u}}$, there exists i and j such that $p \in (\Phi_u \circ \underline{\Phi}_{\underline{u}})(U_i) \cap (\underline{\Phi}_{\underline{u}} \circ \Phi_u)(U_j)$, where this change of coordinates makes sense.

By (37), as well as the estimates for ζ, Ω and their derivatives, b^A and the following first derivatives of b^A are uniformly bounded:

$$|b^A|, \left|\frac{\partial b^A}{\partial \underline{u}}\right|, \left|\frac{\partial b^A}{\partial \theta^B}\right| \leq C.$$

Therefore,

$$\left|\frac{\partial \tilde{\theta}^A}{\partial u}\right|, \left|\frac{\partial \tilde{\theta}^A}{\partial \underline{u}}\right|, \left|\frac{\partial \tilde{\theta}^A}{\partial \theta^B}\right| \leq C.$$

In the new coordinate system, we apply the same argument as in the case for $|\underline{\chi}|^2$ near H_{u_*} and have the estimate

$$\int_{\mathcal{N}} \left((\mathrm{tr}\,\chi)^2 + |\hat{\chi}|^2\right) = \infty$$

for any neighborhood $\mathcal{N}$ of any point $p \in \underline{H}_{\underline{u}_*}$, as desired. □

Finally, this allows us to conclude that the Christoffel symbols do not belong to L^2:

Proposition 35. *Suppose the assumptions of Theorem 4 hold. Then, the Christoffel symbols in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system are not in L^2 in a neighborhood of any point on H_{u_*} or $\underline{H}_{\underline{u}_*}$.*

Proof. Recall that the metric in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinates takes the form

$$g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du).$$

Note that

$$g^{u\underline{u}} = -\frac{1}{2}\Omega^{-2}, \qquad g^{u\alpha} = 0 \text{ for } \alpha \neq \underline{u}.$$

One computes that

$$\Gamma^u_{AB} = -\frac{1}{2} g^{u\underline{u}} \frac{\partial}{\partial \underline{u}} g_{AB} = \frac{1}{4\Omega^2}\frac{\partial}{\partial \underline{u}}\gamma_{AB} = \frac{1}{2\Omega}\chi_{AB}.$$

Since $\frac{1}{2} \leq \Omega \leq 2$ and γ is uniformly bounded and positive definite, Γ^u_{AB} is not in L^2 in a neighborhood of any point on the singular boundary $\underline{H}_{\underline{u}_*}$ in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system.

To show that the incoming hypersurface H_{u_*} is singular, first notice that

$$g^{\underline{u}u} = -\frac{1}{2}\Omega^{-2}, \qquad g^{\underline{u}A} = -\frac{1}{2}\Omega^{-2}b^A, \qquad g^{\underline{u}\underline{u}} = 0.$$

We then compute

$$\begin{aligned}
\Gamma^{\underline{u}}_{AB} =& \frac{1}{2} g^{\underline{u}u}\left(\frac{\partial}{\partial \theta^A} g_{Bu} + \frac{\partial}{\partial \theta^B} g_{Au} - \frac{\partial}{\partial u} g_{AB}\right) \\
&+ \frac{1}{2} g^{\underline{u}C}\left(\frac{\partial}{\partial \theta^B} g_{AC} + \frac{\partial}{\partial \theta^A} g_{BC} - \frac{\partial}{\partial \theta^C} g_{AB}\right) \\
=& \frac{1}{4\Omega^2}\left(\frac{\partial}{\partial u}\gamma_{AB} - \frac{\partial}{\partial \theta^B}(\gamma_{AC} b^C) - \frac{\partial}{\partial \theta^A}(\gamma_{BC} b^C)\right. \\
&\left. - b^C\left(\frac{\partial}{\partial \theta^B}\gamma_{AC} + \frac{\partial}{\partial \theta^A}\gamma_{BC} - \frac{\partial}{\partial \theta^C}\gamma_{AB}\right)\right) \\
=& \frac{1}{2\Omega}\underline{\chi}_{AB} + \text{regular terms},
\end{aligned}$$

where the regular terms denote metric components and their derivatives that are uniformly bounded by the estimates proved in the previous sections. By the same reasoning as in the case near $\underline{H}_{\underline{u}_*}$, $\Gamma^{\underline{u}}_{AB}$ is not in L^2 in a neighborhood of any point on the singular boundary H_{u_*} in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system. □

This concludes the proof of Theorem 4.

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References

[1] A. Bonanno, S. Droz, W. Israel, and S. M. Morsink, *Structure of the charged spherical black hole interior*, Proc. Roy. Soc. London Ser. A **450** (1995), no. 1940, 553–567, DOI 10.1098/rspa.1995.0100. MR1356176

[2] P. R. Brady and J. D. Smith, *Black hole singularities: a numerical approach*, Phys. Rev. Lett. **75** (1995), no. 7, 1256–1259, DOI 10.1103/PhysRevLett.75.1256. MR1343439

[3] L. M. Burko, *Structure of the black hole's Cauchy-horizon singularity*, Phys. Rev. Lett. **79** (1997), no. 25, 4958–4961, DOI 10.1103/PhysRevLett.79.4958. MR1487881

[4] S. Chandrasekhar and J. B. Hartle, *On crossing the Cauchy horizon of a Reissner–Nordström black-hole*, Proc. Roy. Soc. London Ser. A **384** (1982), no. 1787, 301–315, DOI 10.1098/rspa.1982.0160. MR684313

[5] D. Christodoulou, *The formation of black holes in general relativity*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2009. MR2488976

[6] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series, vol. 41, Princeton University Press, Princeton, NJ, 1993. MR1316662

[7] M. Dafermos, *Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-l,kmjnhbgvfMaxwell-scalar field equations*, Ann. of Math. (2) **158** (2003), no. 3, 875–928, DOI 10.4007/annals.2003.158.875. MR2031855

[8] M. Dafermos, *The interior of charged black holes and the problem of uniqueness in general relativity*, Comm. Pure Appl. Math. **58** (2005), no. 4, 445–504, DOI 10.1002/cpa.20071. MR2119866

[9] M. Dafermos, *Black holes without spacelike singularities*, Comm. Math. Phys. **332** (2014), no. 2, 729–757, DOI 10.1007/s00220-014-2063-4. MR3257661

[10] M. Dafermos and I. Rodnianski, *A proof of Price's law for the collapse of a self-gravitating scalar field*, Invent. Math. **162** (2005), no. 2, 381–457, DOI 10.1007/s00222-005-0450-3. MR2199010

[11] M. Dafermos and I. Rodnianski, *The black hole stability problem for linear scalar perturbations*, in Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity (T. Damour *et al.*, Eds.), World Scientific, Singapore, 2011, pp. 132–189.

[12] W. A. Hiscock, *Evolution of the interior of a charged black hole*, Phys. Lett. A **83** (1981), no. 3, 110–112, DOI 10.1016/0375-9601(81)90508-9. MR617171

[13] S. Hod and T. Piran, *Mass inflation in dynamical gravitational collapse of a charged scalar field*, Phys. Rev. Lett. **81** (1998), 1554–1557.

[14] K. A. Khan and R. Penrose, *Scattering of two impulsive gravitational plane waves*, Nature **229** (1971), 185–186.

[15] S. Klainerman and F. Nicolò, *The evolution problem in general relativity*, Progress in Mathematical Physics, vol. 25, Birkhäuser Boston, Inc., Boston, MA, 2003. MR1946854

[16] S. Klainerman and I. Rodnianski, *On the formation of trapped surfaces*, Acta Math. **208** (2012), no. 2, 211–333, DOI 10.1007/s11511-012-0077-3. MR2931382

[17] J. Luk, *On the local existence for the characteristic initial value problem in general relativity*, Int. Math. Res. Not. IMRN **20** (2012), 4625–4678, DOI 10.1093/imrn/rnr201. MR2989616

[18] J. Luk and I. Rodnianski, *Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations*, Cambridge J. Math., to appear.

[19] J. Luk and I. Rodnianski, *Local propagation of impulsive gravitational waves*, Comm. Pure Appl. Math. **68** (2015), no. 4, 511–624, DOI 10.1002/cpa.21531. MR3318018

[20] J. M. McNamara, *Instability of black hole inner horizons*, Proc. Roy. Soc. London Ser. A **358** (1978), no. 1695, 499–517, DOI 10.1098/rspa.1978.0024. MR0489678

[21] H. Müller zum Hagen, *Characteristic initial value problem for hyperbolic systems of second order differential equations* (English, with French summary), Ann. Inst. H. Poincaré Phys. Théor. **53** (1990), no. 2, 159–216. MR1079777

[22] A. Ori and É. É. Flanagan, *How generic are null spacetime singularities?*, Phys. Rev. D (3) **53** (1996), no. 4, R1754–R1758, DOI 10.1103/PhysRevD.53.R1754. MR1380002

[23] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57–59, DOI 10.1103/PhysRevLett.14.57. MR0172678

[24] R. Penrose, *The geometry of impulsive gravitational waves*, General relativity (papers in honour of J. L. Synge), Clarendon Press, Oxford, 1972, pp. 101–115. MR0503490

[25] E. Poisson and W. Israel, *Inner-horizon instability and mass inflation in black holes*, Phys. Rev. Lett. **63** (1989), no. 16, 1663–1666, DOI 10.1103/PhysRevLett.63.1663. MR1018317

[26] E. Poisson and W. Israel, *Internal structure of black holes*, Phys. Rev. D (3) **41** (1990), no. 6, 1796–1809, DOI 10.1103/PhysRevD.41.1796. MR1048877

[27] M. Simpson and R. Penrose, *Internal instability in a Reissner-Nordström black hole*, Internat. J. Theoret. Phys. **7** (1973), 183–197.

[28] P. Szekeres, *Colliding gravitational waves*, Nature **228** (1970), 1183–1184.

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