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QUANTUM UNIQUE ERGODICITY AND THE NUMBER OF NODAL DOMAINS OF EIGENFUNCTIONS

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1. Introduction

1.1. **Nodal domains of eigenfunctions on a surface.** Let (M, g) be a smooth compact Riemannian surface without boundary, and let $\{u_n\}$ be an orthonormal Laplacian eigenbasis ordered by the eigenvalue, i.e.,

$$\begin{aligned} -\Delta_g u_n &= \lambda_n^2 u_n, \\ \langle u_n, u_m \rangle_M &= \delta_{nm}, \\ 0 = \lambda_0 &< \lambda_1 \leq \lambda_2 \leq \dots, \end{aligned}$$

where Δ_g is the Laplace-Beltrami operator on M. Here $\langle f, h \rangle_M = \int_M f\bar{h} dV_g$, where dV_g is the volume form of the metric g. We assume throughout the paper that every eigenfunction is real valued. We denote by Z_{u_n} the nodal set $\{x \in M \ : \ u_n(x) = 0\}$ of u_n and by $\mathcal{N}(u_n)$ the number of nodal domains of u_n, where nodal domains are the connected components of $M \backslash Z_{u_n}$.

The purpose of this paper is to understand the growth of $\mathcal{N}(u_n)$ as n tends to $+\infty$. Note that Courant's nodal domain theorem [CH53] and Weyl law imply that $\mathcal{N}(u_n) = O(\lambda_n^2)$. However, it is not true in general that the number of nodal domains necessarily grows with the eigenvalue. For instance, when $M = S^2$ (the standard sphere) or $M = T^2$ (the flat torus), there exists a sequence of eigenfunctions $\{u_{n_k}\}$ with $\lambda_{n_k} \to \infty$ that satisfy $\mathcal{N}(\phi_{n_k}) \leq 3$ [Ste25, Lew77, JN99].

We first state the main result of the paper.

Theorem 1.1. *Let ϕ be a Hecke-Maass eigenform for an arithmetic triangle group with eigenvalue λ. Then we have* $\lim_{\lambda \to +\infty} \mathcal{N}(\phi) = +\infty$.

Note that there are 76 arithmetic triangle groups [Tak77a] which are divided into 18 commensurable classes [Tak77b].

Remark 1.2. This result in the stronger form of a lower bound of $\gg_\epsilon \lambda^{\frac{2}{27}-\epsilon}$ for the number of nodal domains is obtained in [GRS15], however, assuming the generalized Lindelöf hypothesis for a certain family of L-functions.

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Remark 1.3. We refer the readers to Section 6 of [GRS15] for further examples; Section 2.5 of [GRS15] might be required in order to apply Theorem 1.6 to those examples.

Theorem 1.1 is a consequence of Theorem 1.6 given below which considers the number of nodal domains when we have quantum unique ergodicity (QUE). Note that the arithmetic quantum unique ergodicity theorem by Lindenstrauss [Lin06] asserts that QUE holds for Maass-Hecke eigenforms on these triangles. In order to state Theorem 1.6, we first fix $a \mapsto a(x, hD)$, a quantization of a symbol $a(x,\xi) \in C^\infty(T^*M)$, to a pseudo-differential operator. (We refer the readers to [Zwo12] for detailed discussion on the subject.) We say QUE holds for the sequence of eigenfunctions $\{u_n\}_{n\geq 1}$ if we have

$$\lim_{n\to\infty} \left\langle a\left(x, \lambda_n^{-1} D\right) u_n, u_n \right\rangle_M = \int_{S^*M} a(x,\xi) d\mu \tag{1.1}$$

for any fixed symbol $a \in C^\infty(T^*M)$ of finite order. Here $d\mu$ is a normalized Liouville measure on the unit cotangent bundle S^*M. We often write $\mathrm{Op}(a)$ for an operator that acts on an eigenfunction u with the eigenvalue λ as $a(x, \lambda^{-1}D)$.

Remark 1.4. The classical notions of equidistribution of these "Wigner measures" [Šni74, CdV85, Zel87] are concerned with (1.1) for degree zero homogeneous symbols. One can prove that if (1.1) holds with degree zero homogeneous symbols, then (1.1) holds with finite order symbols, by observing that any weak limit of these measures is supported in $S^*M \subset T^*M$.

Remark 1.5. For a compact smooth negatively curved Riemannian manifold, it is conjectured by Rudnick and Sarnak [RS94] that QUE holds for any given orthonormal eigenbasis $\{u_n\}$.

Theorem 1.6. *Let M be a smooth compact Riemannian surface without boundary. Assume that there exists an orientation-reversing isometric involution $\tau : M \to M$ such that* $\mathrm{Fix}(\tau)$ *is separating. Let $\{u_n\}$ be an orthonormal basis of $L^2(M)$ such that each u_n is a joint eigenfunction of the Laplacian and τ. Assume that QUE holds for the sequence $\{u_n\}$. Then*

$$\lim_{n\to\infty} \mathcal{N}(u_n) = +\infty.$$

We say a function f on M is even (resp. odd) if $\tau f = f$ (resp. $\tau f = -f$). In order to prove Theorem 1.6, we first use a topological argument to bound the number of nodal domains of an even (resp. odd) eigenfunction from below by the number of sign changes (resp. the number of singular points) of the eigenfunction along $\mathrm{Fix}(\tau)$. Such an argument is first developed in [GRS13], and we review in Section 4.1 in terms of the nodal graphs and Euler's inequality as in [JZ16]. We then use Bochner's theorem and a Rellich type identity to deduce from QUE that even (resp. odd) eigenfunctions $\{u_n\}$ have a growing number of sign changes (resp. singular points) along $\mathrm{Fix}(\tau)$ as n tends to $+\infty$. This is the main contribution of the paper, and we sketch the argument in the following section.

Remark 1.7. In [JZ16], the same assertion has been obtained when M is a negatively curved surface, but for a density one subsequence of $\{u_n\}$. The argument of [JZ16] to detect a sign change of an eigenfunction u_n on a curve β is to compare

$$\left|\int_\beta u_n(s) ds\right| \quad \text{and} \quad \int_\beta |u_n(s)| ds.$$

(See [GRS13, Jun16, BR15, JZ16, Mag15, GRS15], where such an idea is used to prove a lower bound for the number of sign changes in various contexts.) In order to bound $\|u_n\|_{L^1(\beta)}$ from below using Hölder's inequality, the authors use the quantum ergodic restriction (QER) theorem [TZ13, DZ13] for the lower bound of $\|u_n\|_{L^2(\beta)}$ and the point-wise Weyl law with an improved error term [Bér77] for the upper bound of $\|u_n\|_{L^\infty(\beta)}$. For the upper bound of the integral of u_n over β, the authors use the Kuznecov sum formulas [Zel92]. Note that the result of [Bér77] requires a global assumption on the geometry of M that it does not have conjugate points, which is satisfied if M is negatively curved. Also note that in order to bound such quantities using QER theorem and Kuznecov sum formulas, it is necessary to remove a density 0 subsequence.

1.2. **Sketch of the proof: sign changes of even eigenfunctions.** The main step in the proof of Theorem 1.6 is to show that all but finitely many u_n have at least one sign change on any given fixed segment β of $\mathrm{Fix}(\tau)$.

To simplify the discussion, let $\{\psi_n\}$ be a sequence of functions in $C_0^\infty([0,1])$. Assume that for any fixed integer $m \geq 0$ we have

$$\lim_{n\to\infty}\int_0^1 \left|\frac{\partial^m \psi_n}{\partial s^m}(s)\right|^2 ds = a_{2m}, \tag{1.2}$$

for some positive real number a_{2m}. Let $h_n(\xi) = |g_n(\xi)|^2/\|g_n\|_2^2$, where $g_n(\xi)$ is the Fourier transform of ψ_n,

$$g_n(\xi) = (2\pi)^{-\frac{1}{2}}\int_0^1 e^{i\xi s}\psi_n(s)ds.$$

Assume that there exists a unique probability measure $d\mu(\xi)$ whose $2m$th moment is a_{2m}/a_0 and whose $(2m+1)$th moment is zero for any $m \geq 0$. Then (1.2) implies that a sequence of probability measures $h_n(\xi)d\xi$ converges to $d\mu(\xi)$ in moments.

We claim that all but finitely many ψ_n have at least one sign change on $(0,1)$ under the assumption that $d\mu(\xi)$ is not positive-definite, i.e., not a Fourier transform of a positive measure (Lemma 4.6). Assume for contradiction that there exists a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ such that ψ_{n_k} does not change sign on $(0,1)$ for all k. Then by Bochner's theorem, $\{h_{n_k}(\xi)\}$ is a sequence of positive-definite functions, and it cannot converge in moments to a measure that is not positive-definite, contradicting the assumption that $d\mu(\xi)$ is not positive-definite.

Now let $f \in C_0^\infty(\beta)$ be a non-negative function. Our aim is to apply the above argument to $\psi_n(s) = f(s)u_n|_\beta(s)$, when QUE holds for the sequence of eigenfunctions $\{u_n\}$. Note that it is not known whether the limit

$$\lim_{n\to\infty}\int_\beta |\psi_n(s)|^2 ds$$

should exist. However, under the assumption that QUE holds for $\{u_n\}$, we may instead compute the limit (Theorem 3.1)

$$\lim_{n\to\infty}\int_\beta |\psi_n(s)|^2 - \left|\frac{1}{\lambda_n^m}\frac{\partial^m\psi_n}{\partial s^m}(s)\right|^2 ds = 2(1-b_{2m})\int_\beta f^2(s)ds \tag{1.3}$$

for each fixed $m \geq 0$ with an explicit constant $0 < b_{2m} \leq 1$ using the Rellich identity, as in the proof of the quantum uniquely ergodic restriction theorem of [CTZ13].

We first deduce from (1.3) that (Corollary 3.2)

$$\liminf_{n\to\infty} \int_\beta |\psi_n(s)|^2\, ds \geq 2 \int_\beta f^2(s) ds,$$

and so

$$\limsup_{n\to\infty} 2 \int_\beta f^2(s) ds \left(\int_\beta |\psi_n(s)|^2\, ds \right)^{-1} \leq 1.$$

Assume for simplicity that, for some $0 \leq a \leq 1$, we have

$$\lim_{n\to\infty} 2 \int_\beta f^2(s) ds \left(\int_\beta |\psi_n(s)|^2\, ds \right)^{-1} = a.$$

Then (1.3) implies that

$$\lim_{n\to\infty} \int_\beta \left| \frac{1}{\lambda_n^m} \frac{\partial^m \psi_n}{\partial s^m}(s) \right|^2 ds \left(\int_\beta |\psi_n(s)|^2\, ds \right)^{-1} = (1-a) + ab_{2m},$$

and we may apply the argument to

$$h_n(\xi) = \lambda_n \left| \widehat{\psi_n}(\lambda_n \xi) \right|^2 \left(\int_\beta |\psi_n(s)|^2\, ds \right)^{-1}$$

to conclude that all but finitely many u_n have at least one sign change on β, by verifying that the unique measure having $(1-a)+ab_{2m}$ as the $2m$th moment and 0 as the $(2m+1)$th moment is not positive-definite for any given $0 \leq a \leq 1$. This implies that the number of sign changes of u_n along $\mathrm{Fix}(\sigma)$ tends to $+\infty$ as $n \to \infty$ (Theorem 4.3).

2. L^p ESTIMATES FOR THE RESTRICTION TO A CURVE OF DERIVATIVES OF EIGENFUNCTIONS

Let u be a Laplacian eigenfunction with the eigenvalue λ. Let L be a degree m linear differential operator on M; i.e., for any coordinate patch (U,p) there exist smooth functions $a_\alpha \in C^\infty(\mathbb{R}^n)$ (in which $a_\alpha \not\equiv 0$ for some α with $|\alpha| = m$) such that for any $\phi, \psi \in C_0^\infty(U)$ and for each $f \in C^\infty(M)$,

$$\phi L(\psi f) = \phi p^* \sum_{|\alpha|\leq m} a_\alpha(x) \partial^\alpha \left(p^{-1}\right)^* (\psi f).$$

Recall that

$$\sup_{x\in M} |Lu(x)| = O\left(\lambda^{m+\frac{1}{2}}\right), \tag{2.1}$$

which is a consequence of the generalization of remainder estimate for spectral function by Avakumovic-Levitan-Hörmander to that for the derivatives of spectral function [Bin04]. Denoting by $\langle f,g \rangle_\beta = \int_\beta f(s)\overline{g(s)} ds$, (2.1) implies that

$$|\langle Lu, u \rangle_\beta| \ll_\beta \sup_{x\in M} |Lu(x)| \sup_{x\in M} |u(x)| = O\left(\lambda^{m+1}\right). \tag{2.2}$$

In the proof of Theorem 3.1, we need an improvement over (2.2), and we achieve an improvement by combining the L^2 eigenfunction restrictions estimates along curves due to Burq, Gérard, and Tzvetkov [BGT07] and (2.1).

Lemma 2.1. *For any fixed degree m differential operator L, we have*

$$|\langle Lu, u \rangle_\beta| = O\left(\lambda^{m+\frac{3}{4}}\right).$$

Proof. By Hölder's inequality,

$$|\langle Lu, u\rangle_\beta| \le \sup_{x\in M} |Lu(x)| \, \|u\|_{L^1(\beta)}.$$

From [BGT07], we have $\|u\|_{L^2(\beta)} = O\left(\lambda^{\frac{1}{4}}\right)$; hence

$$\|u\|_{L^1(\beta)} \le l(\beta)^{\frac{1}{2}} \|u\|_{L^2(\beta)} = O_\beta\left(\lambda^{\frac{1}{4}}\right).$$

Therefore by (2.1), we conclude

$$|\langle Lu, u\rangle_\beta| = O\left(\lambda^{m+\frac{3}{4}}\right).$$

□

Since we only need any power saving over $O\left(\lambda^{m+1}\right)$ in (2.2) in our proof, it is unnecessary to optimize our bound in Lemma 2.1. The optimal upper bound for $\|Lu\|_{L^2(\beta)}$ is $O\left(\lambda^{m+\frac{1}{4}}\right)$ which is sharp when $L = 1$ and M is the standard sphere S^2. Note that when L corresponds to a normal derivative along β, the bound can be improved to $O(1)$ using second-microlocalization techniques, due to [CHT15].

3. Rellich type analysis when QUE holds: even eigenfunctions

In this section, we prove (1.3) with explicit constants $\{b_{2m}\}$. The main idea is to follow the computation involving the Rellich identity in [CTZ13], with a specific choice of symbols.

Theorem 3.1. *Assume that QUE holds for the sequence of even eigenfunctions $\{u_n\}$. Fix a segment $\beta \subset \mathrm{Fix}(\tau)$. For any fixed real valued function $f \in C_0^\infty(\beta)$ and for any fixed non-negative integer m, we have*

$$\lim_{n\to\infty} \int_\beta |f(t)u_n(t)|^2 dt - \lambda_n^{-2m} \int_\beta \left|\partial_t^m (f(t)u_n(t))\right|^2 dt$$
$$= 2\left(1 - \frac{1}{\pi}\int_{-1}^{1} \xi^{2m} \frac{d\xi}{\sqrt{1-\xi^2}}\right) \int_\beta f^2(t)dt.$$

Proof. We drop the subscript n in u_n and λ_n for simplicity.

Let (t, n) be Fermi normal coordinates in a small tubular neighborhood U_ϵ of β near a point $x_0 \in \beta$. Let $p : U_\epsilon \to \mathbb{R}^2$ be the coordinate chart. We may assume that

$$U = U_\epsilon = p^{-1}\left(\{(t, n) \mid t \in V,\ |n| < \epsilon\}\right)$$

in these coordinates, where $V \subset \mathbb{R}$ is a coordinate chart that contains x_0. Let (t, n, ξ_t, ξ_n) be the local coordinates of $T^*(U)$ under the identification

$$\mathbb{R}^2 \to T^*_{x=(t,n)}(U),$$
$$(\xi_t, \xi_n) \mapsto \xi_t dt + \xi_n dn.$$

We consider the standard quantization in these coordinates; i.e., for any given symbol $a(t, n, \xi_t, \xi_n)$ of finite order, we let

$$\mathrm{Op}(a)u(t_0, n_0)$$
$$= \frac{\lambda^n}{(2\pi)^n} \int_{p(U)\times\mathbb{R}^2} e^{\lambda i((t_0-t)\xi_t + (n_0-n)\xi_n)} a(t_0, n_0, \xi_t, \xi_n) u(t, n) dt dn d\xi_t d\xi_n.$$

For example, if $a(t,n,\xi_t,\xi_n)=\sum_{|\alpha|\le N} a_\alpha(t,n)\xi_t^{\alpha_1}\xi_n^{\alpha_2}$, then

$$\operatorname{Op}(a)u=\sum_{|\alpha|\le N} a_\alpha(t,n)\left(\frac{\partial_t}{i\lambda}\right)^{\alpha_1}\left(\frac{\partial_n}{i\lambda}\right)^{\alpha_2} u.$$

Let $U_- \subset U$ be given by

$$U_-=\{(t,n)\in U \mid n<0\}.$$

For any pseudo-differential operator T on M, from Green's formula, we have

$$\langle \Delta_g Tu, u\rangle_{U_-}-\langle Tu,\Delta_g u\rangle_{U_-}=\langle \partial_n Tu|_\beta, u|_\beta\rangle_\beta-\langle Tu|_\beta,\partial_n u|_\beta\rangle_\beta. \tag{3.1}$$

Since u is an eigenfunction, $\langle Tu,\Delta_g u\rangle_{U_-}=\langle T\Delta_g u,u\rangle_{U_-}$. Also since we are assuming that u is even, $\langle Tu|_\beta,\partial_n u|_\beta\rangle_\beta=0$. Therefore we have the Rellich identity,

$$\frac{1}{\lambda}\langle[-\Delta_g,T]u,u\rangle_{U_-}=-\frac{1}{\lambda}\langle\partial_n Tu|_\beta,u|_\beta\rangle_\beta, \tag{3.2}$$

where λ^{-1} is the normalizing factor.

Now fix $\chi\in C_0^\infty(\mathbb{R})$ such that

$$\chi(x)=\begin{cases}0 & \text{if } |x|\ge 1,\\ 1 & \text{if } |x|\le \frac{1}{2}.\end{cases}$$

We define a symbol supported near β for $0<\delta<\epsilon$,

$$a_{\delta,m}(x,\xi)=\chi\left(\frac{n}{\delta}\right)f^2(t)i\xi_n\sum_{k=0}^{m-1}\xi_t^{2k},$$

and let $T=\operatorname{Op}(a_{\delta,m})$. Observing that $-(\partial_n^2+\partial_t^2)u=\lambda^2u$ along β, we may rewrite the right-hand side (RHS) of (3.2) as

$$\left\langle f^2\left(1+(-1)^{m-1}\lambda^{-2m}\partial_t^{2m}\right)u|_\beta,u|_\beta\right\rangle_\beta.$$

We integrate by parts to further simplify the second term as follows:

$$\begin{aligned}&\left\langle(-1)^{m-1}f^2\lambda^{-2m}\partial_t^{2m}u|_\beta,u|_\beta\right\rangle_\beta+\lambda^{-2m}\left\langle\partial_t^m\left(fu|_\beta\right),\partial_t^m\left(fu|_\beta\right)\right\rangle_\beta\\ =&\left\langle(-1)^{m-1}\lambda^{-2m}f[f,\partial_t^{2m}]u|_\beta,u|_\beta\right\rangle_\beta\\ =&\,O_{m,f}(\lambda^{-\frac14}),\end{aligned}$$

where we used Lemma 2.1 with $L=f[f,\partial_t^{2m}]$ in the last estimate. So we have

$$-\frac{1}{\lambda^2}\langle\partial_n Tu|_\beta,u|_\beta\rangle_\beta=\int_\beta|f(t)u(t)|^2dt-\lambda^{-2m}\int_\beta\left|\partial_t^m(f(t)u(t))\right|^2dt+O_{m,f}\left(\lambda^{-\frac14}\right). \tag{3.3}$$

Now let $-\Delta_g=\sum_{|\alpha|=1,2}b_\alpha(n,t)\left(\frac{\partial_n}{i}\right)^{\alpha_1}\left(\frac{\partial_t}{i}\right)^{\alpha_2}$. Observe from (2.1) that if $\alpha_1+\alpha_2=1$,

$$\frac{1}{\lambda}\left[b_\alpha(n,t)\left(\frac{\partial_n}{i}\right)^{\alpha_1}\left(\frac{\partial_t}{i}\right)^{\alpha_2},\chi\left(\frac{n}{\delta}\right)f^2(t)\frac{(-1)^k\partial_n\partial_t^{2k}}{\lambda^{2k+1}}\right]u=O_{\delta,m,f}(\lambda^{-\frac12}),$$

and that if $\alpha_1+\alpha_2=2$,

$$\begin{aligned}&\frac{1}{\lambda}\left[b_\alpha(n,t)\left(\frac{\partial_n}{i}\right)^{\alpha_1}\left(\frac{\partial_t}{i}\right)^{\alpha_2},\chi\left(\frac{n}{\delta}\right)f^2(t)\frac{\partial_n(-1)^k\partial_t^{2k}}{\lambda^{2k+1}}\right]u\\&=\mathrm{Op}\left(\frac{\alpha_1}{\delta}b_\alpha(n,t)\chi'\left(\frac{n}{\delta}\right)f^2(t)\xi_n^{\alpha_1}\xi_t^{2k+\alpha_2}\right)u\\&+\mathrm{Op}\left(\chi\left(\frac{n}{\delta}\right)R_{m,f,\alpha}(n,t,\xi_n,\xi_t)\right)u+O_{\delta,m,f}(\lambda^{-\frac{1}{2}})\end{aligned}$$

for some symbol $R_{m,f,\alpha}$ of finite order depending only on m,f,α. Therefore we may reexpress the left-hand side (LHS) of (3.2) as

$$\begin{aligned}&\left\langle\mathrm{Op}\left(\sum_{|\alpha|=2}\frac{\alpha_1\xi_n^{\alpha_1}\xi_t^{\alpha_2}}{\delta}b_\alpha(n,t)\chi'\left(\frac{n}{\delta}\right)f^2(t)\sum_{k=0}^{m-1}\xi_t^{2k}\right)u,u\right\rangle_{U_-}\\&+\left\langle\mathrm{Op}\left(\chi\left(\frac{n}{\delta}\right)R_{m,f}(n,t,\xi_n,\xi_t)\right)u,u\right\rangle_{U_-}\\&+O_{\delta,m,f}(\lambda^{-\frac{1}{2}})\end{aligned}$$

for some finite order symbol $R_{m,f}$.

We bound the second inner product using Cauchy-Schwartz inequality by

$$\begin{aligned}&\left|\left\langle\mathrm{Op}\left(\chi\left(\frac{n}{\delta}\right)R_{m,f}(n,t,\xi_n,\xi_t)\right)u,u\right\rangle_{U_-}\right|\\&\le\left\|\mathrm{Op}\left(\chi\left(\frac{n}{\delta}\right)R_{m,f}(n,t,\xi_n,\xi_t)\right)u\right\|^2_{L^2(U_-)}\\&\le\left\|\mathrm{Op}\left(\chi\left(\frac{n}{\delta}\right)R_{m,f}(n,t,\xi_n,\xi_t)\right)u\right\|^2_{L^2(U)},\end{aligned}$$

and from the assumption that the QUE holds, we may estimate the last quantity as $O_{m,f}(\delta)+o_{\delta,m,f}(1)$ as λ tends to $+\infty$.

Now let $\chi_0\in C_0^\infty(\mathbb{R})$ be given by $\chi_0(x)=\chi'(x)$ if $x<0$, and $\chi_0(x)=0$ otherwise. We then have

$$\begin{aligned}&\left\langle\mathrm{Op}\left(\sum_{|\alpha|=2}\frac{\alpha_1\xi_n^{\alpha_1}\xi_t^{\alpha_2}}{\delta}b_\alpha(n,t)\chi'\left(\frac{n}{\delta}\right)f^2(t)\sum_{k=0}^{m-1}\xi_t^{2k}\right)u,u\right\rangle_{U_-}\\&=\left\langle\mathrm{Op}\left(\sum_{|\alpha|=2}\frac{\alpha_1\xi_n^{\alpha_1}\xi_t^{\alpha_2}}{\delta}b_\alpha(n,t)\chi_0\left(\frac{n}{\delta}\right)f^2(t)\sum_{k=0}^{m-1}\xi_t^{2k}\right)u,u\right\rangle_{U}\\&=\int_{S^*U}\sum_{|\alpha|=2}\frac{\alpha_1\xi_n^{\alpha_1}\xi_t^{\alpha_2}}{\delta}b_\alpha(n,t)\chi_0\left(\frac{n}{\delta}\right)f^2(t)\sum_{k=0}^{m-1}\xi_t^{2k}d\mu+o_{\delta,m,f}(1)\end{aligned}\tag{3.4}$$

as λ tends to $+\infty$ from the assumption that QUE holds.

We therefore conclude from (3.3) and (3.4) that

$$\begin{aligned}\int_\beta |f(t)u(t)|^2 dt - \lambda^{-2m} \int_\beta \left|\partial_t^m (f(t)u(t))\right|^2 dt \\ = \int_{S^*U} \sum_{|\alpha|=2} \frac{\alpha_1 \xi_n^{\alpha_1}\xi_t^{\alpha_2}}{\delta} b_\alpha(n,t)\chi_0\left(\frac{n}{\delta}\right) f^2(t) \sum_{k=0}^{m-1} \xi_t^{2k} d\mu \\ + O_{m,f}\left(\lambda^{-\frac{1}{4}}\right) + O_{m,f}(\delta) + o_{\delta,m,f}(1),\end{aligned}$$

and so

$$\begin{aligned}\lim_{\lambda\to\infty} \int_\beta |f(t)u(t)|^2 dt - \lambda^{-2m} \int_\beta \left|\partial_t^m (f(t)u(t))\right|^2 dt \\ = \int_{S^*U} \sum_{|\alpha|=2} \frac{\alpha_1 \xi_n^{\alpha_1}\xi_t^{\alpha_2}}{\delta} b_\alpha(n,t)\chi_0\left(\frac{n}{\delta}\right) f^2(t) \sum_{k=0}^{m-1} \xi_t^{2k} d\mu + O_{m,f}(\delta).\end{aligned}$$

Note that no terms in the left-hand side depend on δ. Also note that $b_{20}(0,t) = b_{02}(0,t) = 1$ and $b_{11}(0,t) = 0$ since we are taking the Fermi normal coordinate. Therefore by taking $\delta \to 0$, we have

$$\begin{aligned}&\lim_{\delta\to 0} \int_{S^*U} \sum_{|\alpha|=2} \frac{\alpha_1 \xi_n^{\alpha_1}\xi_t^{\alpha_2}}{\delta} b_\alpha(n,t)\chi_0\left(\frac{n}{\delta}\right) f^2(t) \sum_{k=0}^{m-1} \xi_t^{2k} d\mu + O_{m,f}(\delta) \\ &= \int_{S^*_\beta U} \sum_{|\alpha|=2} \alpha_1 b_\alpha(0,t) f^2(t) \xi_n^{\alpha_1}\xi_t^{\alpha_2} \sum_{k=0}^{m-1} \xi_t^{2k} d\mu \\ &= \frac{1}{\pi} \int_\beta f^2(t)dt \int_{\xi_t^2+\xi_n^2=1} (1-\xi_t^{2m}) d\xi \\ &= 2\left(1 - \frac{1}{\pi}\int_{-1}^1 \xi^{2m} \frac{d\xi}{\sqrt{1-\xi^2}}\right) \int_\beta f^2(t)dt.\end{aligned}$$

This implies that

$$\begin{aligned}\lim_{\lambda\to\infty} \int_\beta |f(t)u(t)|^2 dt - \lambda^{-2m} \int_\beta \left|\partial_t^m (f(t)u(t))\right|^2 dt \\ = 2\left(1 - \frac{1}{\pi}\int_{-1}^1 \xi^{2m} \frac{d\xi}{\sqrt{1-\xi^2}}\right) \int_\beta f^2(t)dt + o_{m,f}(1)\end{aligned}$$

as $\delta \to 0$, and since δ can be chosen arbitrarily small, we conclude that

$$\begin{aligned}\lim_{\lambda\to\infty} \int_\beta |f(t)u(t)|^2 dt - \lambda^{-2m} \int_\beta \left|\partial_t^m (f(t)u(t))\right|^2 dt \\ = 2\left(1 - \frac{1}{\pi}\int_{-1}^1 \xi^{2m} \frac{d\xi}{\sqrt{1-\xi^2}}\right) \int_\beta f^2(t)dt.\end{aligned}$$

□

As an immediate application of Theorem 3.1, we give a sharp lower bound for the L^2 estimate of the restriction of eigenfunctions.

Corollary 3.2. *Assume that QUE holds for the sequence of even eigenfunctions $\{u_n\}$. Then for any fixed real valued function $f \in C_0^\infty(\beta)$, we have*

$$\liminf_{n\to\infty} \int_\beta f^2(t)|u_n(t)|^2 dt \geq 2\int_\beta f^2(t)dt.$$

Proof. By the positivity of $\lambda^{-2m}\int_\beta \left|\partial_t^m(f(t)u(t))\right|^2 dt$,

$$\begin{aligned}\liminf_{n\to\infty} \int_\beta f^2(t)|u_n(t)|^2 dt &\geq 2\left(1-\frac{1}{\pi}\int_{-1}^{1}\xi^{2m}\frac{d\xi}{\sqrt{1-\xi^2}}\right)\int_\beta f^2(t)dt \\ &= 2\int_\beta f^2(t)dt + O(1/m^{1/2}).\end{aligned}$$

Since the limit does not depend on m, we conclude that

$$\liminf_{n\to\infty} \int_\beta f^2(t)|u_n(t)|^2 dt \geq 2\int_\beta f^2(t)dt. \qquad \square$$

Remark 3.3. A constant lower bound for the L^2 norm of the restriction of an eigenfunction to a geodesic segment is first proven in [GRS13], when the geodesic segment is sufficiently long, from the arithmetic QUE theorem [Lin06, Sou10].

Remark 3.4. If the geodesic flow on M is ergodic, it is known that there exists a density 1 subsequence $\{u_n\}$ of even eigenfunctions that satisfies

$$\lim_{n\to\infty} \|u_n\|^2_{L^2(\beta)} = 2l(\beta); \tag{3.5}$$

hence the lower bound in Corollary 3.2 is sharp. The existence of such a subsequence is a consequence of results which are studied in [Bur05, TZ13, DZ13, CTZ13].

Remark 3.5. If β is not a part of $\mathrm{Fix}(\tau)$ and satisfies a certain asymmetry condition (see, for instance, [TZ13, Definition 1]), then

$$\lim_{k\to\infty} \|u_{n_k}\|^2_{L^2(\beta)} = l(\beta)$$

along a density 1 subsequence $\{u_{n_k}\}$ of $\{u_n\}$. When β is a segment of $\mathrm{Fix}(\tau)$, then every odd eigenfunction vanishes identically on β, hence explaining why we expect the factor 2 in (3.5).

4. The number of nodal domains of even eigenfunctions

4.1. **Graph structure of the nodal set and Euler's inequality.** In this section we briefly review the topological argument in [GRS13, JZ16] on bounding the number of nodal domains from below by the number of zeros on $\mathrm{Fix}(\tau)$. We refer the readers to [JZ16] for details.

First note that if there exists a segment of $\eta \subset \mathrm{Fix}(\tau)$ such that $\eta \subset Z_u$, then because the normal derivative of u vanishes along $\mathrm{Fix}(\tau)$, any point on η is a singular point, contradicting the upper bound on the number of singular points in [Don92]. Therefore together with the following lemma on the local structure of the nodal set, we conclude that $Z_u \cap \mathrm{Fix}(\tau)$ is a finite set of points.

Lemma 4.1 (Section 6.1, [JZ16])**.** *Assume that u vanishes to order N at x_0. Then there exists a small neighborhood U of x_0 such that the nodal set in U is C^1 equivalent to $2N$ equi-angular rays emanating from x_0.*

From Lemma 4.1, we may view the nodal set as a graph (a *nodal graph*) embedded on a surface as follows:

(1) For each connected component of Z_u that is homeomorphic to a circle and that does not intersect $\mathrm{Fix}(\tau)$, we add a vertex.
(2) Each singular point is a vertex.
(3) Each intersection point in $\mathrm{Fix}(\tau) \cap Z_u$ is a vertex.
(4) Edges are the arcs of $Z_u \cup \mathrm{Fix}(\tau)$ that join the vertices listed above.

Let $V(u)$ and $E(u)$ be the finite set of vertices and the finite set of edges given above, respectively. This way, we obtain a *nodal graph* $V(u), E(u)$ of u embedded into the surface M.

From the assumption that $\mathrm{Fix}(\tau)$ is separating, the nodal domains that intersect $\mathrm{Fix}(\tau)$ are cut in two by $\mathrm{Fix}(\tau)$. Therefore the number of faces divided by two bounds the number of nodal domains $\mathcal{N}(u)$ from below.

Observe from Lemma 4.1 that every vertex of a nodal graph has a degree at least 2. Then by Euler's inequality [JZ16, (6.1)],

$$|V(u)| - |E(u)| + |F(u)| - m(u) \geq 1 - 2\mathfrak{g},$$

we obtain a lower bound for the number of nodal domains by the number of zeros on $\mathrm{Fix}(\tau)$. Here $m(G)$ is the number of connected components of the nodal graph, and $\mathfrak{g}$ is the genus of the surface M.

Lemma 4.2 (Lemma 6.4, [JZ16])**.**

$$\mathcal{N}(u) \geq \frac{1}{2}\#\left(Z_u \cap \mathrm{Fix}(\tau)\right) + 1 - \mathfrak{g}.$$

Therefore in order to prove Theorem 1.6, it is sufficient to prove the following theorem.

Theorem 4.3. *Assume that QUE holds for the sequence of even eigenfunctions $\{u_n\}_{n\geq 1}$. Then*

$$\lim_{n\to\infty} \#\left(Z_{u_n} \cap \mathrm{Fix}(\tau)\right) = +\infty.$$

4.2. **Lemmata from probability theory.** In order to prove Theorem 4.3, we first recall some facts about probability measures. We assume that all random variables in this section are defined on the real line.

Lemma 4.4. *Suppose a random variable X has moments $\mu_k = \mathbb{E}[X^k]$ that satisfies the condition*

$$\limsup_{k\to\infty} \mu_{2k}^{\frac{1}{2k}}/2k = r < \infty.$$

Then, X has the unique distribution with moments $(\mu_k)_{k\geq 1}$.

Proof. See [Dur10, Theorem 3.3.11]. □

Lemma 4.5. *If X_n converges to X in moments and the distribution of X is uniquely determined by its moments, then for each $t \in \mathbb{R}$, $\mathbb{E}[e^{itX_n}]$ converges to $\mathbb{E}[e^{itX}]$.*

Proof. Suppose we have a counterexample of this lemma. That is, we have a sequence (X_n) of random variables and X a random variable, such that $\mathbb{E}[X_n^m] \to \mathbb{E}[X^m]$ for all $m > 0$, but $\mathbb{E}[\exp(it_0 X_n)] \not\to \mathbb{E}[\exp(it_0 X)]$ for some $t_0 \in \mathbb{R}$.

Let $F_n(x) := \Pr[X_n \leq x]$ be the cumulative distribution functions of random variable X_n. By Helly's selection theorem [Dur10, Theorem 3.2.6], together with the tightness of (X_n)'s [Dur10, Theorems 3.2.7 and 3.2.8] there exists a subsequence (F_{n_k}) that converges to a cumulative distribution function G of some random variable Y on the real line.

[Dur10, Theorem 3.2.2] implies that, by appropriately settling the probability space Ω for X_{n_k}'s and Y, we can have $X_{n_k}(\omega) \to Y(\omega)$ almost surely for $\omega \in \Omega$. (For instance, we can set $\Omega = (0,1)$, $\Pr =$ (the Lebesgue measure), and $X_{n_k}(\omega) = \sup\{x \in \mathbb{R} \mid F_{n_k}(x) < \omega\}$, etc.) In particular, as $\exp(it_0 X_{n_k}) \to \exp(it_0 Y)$ almost surely, together with $|\exp(it_0 X_{n_k})| \le 1$ for all k implies that $\mathbb{E}[\exp(it_0 X_{n_k})] \to \mathbb{E}[\exp(it_0 Y)]$ by the bounded convergence theorem.

From the assumption that X is the unique random variable with the sequence of the moments $(\mathbb{E}[X^m])$, we claim $X = Y$ by showing that $\mathbb{E}[X^m] = \mathbb{E}[Y^m]$ for all $m > 0$. Equivalently, $\mathbb{E}[X_{n_k}^m] \to \mathbb{E}[Y^m]$ as $k \to \infty$. Denote by χ_M the indicator function of $[-M, M]$ for $M > 0$. We first estimate

$$\begin{aligned}|\mathbb{E}[X_{n_k}^m] - \mathbb{E}[Y^m]| \le \mathbb{E}[|X_{n_k}|^m(1-\chi_M(X_{n_k}))] + \mathbb{E}[|Y|^m(1-\chi_M(Y))]\\ + |\mathbb{E}[X_{n_k}^m \chi_M(X_{n_k}) - Y^m \chi_M(Y)]|.\end{aligned}$$

We bound the first term by Cauchy-Schwarz inequality and Markov inequality,

$$\begin{aligned}\mathbb{E}[|X_{n_k}|^m(1-\chi_M(X_{n_k}))] &\le \mathbb{E}[|X_{n_k}|^{2m}]^{\frac{1}{2}} \mathbb{E}[(1-\chi_M(X_{n_k}))^2]^{\frac{1}{2}}\\ &= \mathbb{E}[X_{n_k}^{2m}]^{\frac{1}{2}} (\Pr[|X_{n_k}| > M])^{\frac{1}{2}}\\ &\le \mathbb{E}[X_{n_k}^{2m}]^{\frac{1}{2}} \mathbb{E}[X_{n_k}^2]^{\frac{1}{2}} M^{-1} \le K M^{-1},\end{aligned}$$

where $K = \sup\{\mathbb{E}[X_{n_k}^{2m}], \mathbb{E}[X_{n_k}^2] \mid k\} < \infty$. We bound the second term by Fatou's lemma,

$$\mathbb{E}[|Y|^m(1-\chi_M(Y))] \le \liminf_{k\to\infty} \mathbb{E}[|X_{n_k}|^m(1-\chi_M(X_{n_k}))] \le K M^{-1},$$

where we used the estimate of the first term in the last inequality. Finally, observe that the third term converges to 0, i.e., $|\mathbb{E}[X_{n_k}^m \chi_M(X_{n_k}) - Y^m \chi_M(Y)]| \to 0$ as $k \to \infty$, by the bounded convergence theorem.

Therefore

$$\limsup_{k\to\infty} |\mathbb{E}[X_{n_k}^m] - \mathbb{E}[Y^m]| = O(M^{-1}),$$

and since M can be chosen arbitrarily large, we conclude $\mathbb{E}[X_{n_k}^m] \to \mathbb{E}[Y^m]$ which implies that $X = Y$. Therefore $\mathbb{E}[\exp(it_0 X_{n_k})] \to \mathbb{E}[\exp(it_0 Y)] = \mathbb{E}[\exp(it_0 X)]$, contradicting the initial assumption

$$\mathbb{E}[\exp(it_0 X_{n_k})] \not\to \mathbb{E}[\exp(it_0 X)]. \qquad \square$$

We now present a new method for detecting sign changes of functions using Lemma 4.4, Lemma 4.5, and Bochner's theorem.

Lemma 4.6. *Let $\{f_n\}$ be a sequence of real valued functions in $C_0^\infty([0,1])$, and let $\{a_n\}$ be a sequence of positive reals such that for each fixed non-negative integer m we have*

$$\lim_{n\to\infty} a_n^{-2m} \int_0^1 |\partial_x^m f_n(x)|^2 \, dx = b_{2m} \tag{4.1}$$

for some positive real numbers b_{2m}. Assume that $d\mu(\xi)$ is the unique probability distribution whose $2m$th moment is b_{2m}/b_0 and whose $(2m+1)$th moment is zero for any $m \ge 0$. If $d\mu(\xi)$ is not positive-definite, then all but finitely many f_n has at least one sign change on $(0,1)$.

Proof. Assume for contradiction that there exists a subsequence $\{f_{n_k}\}$ such that f_{n_k} does not change sign on $(0,1)$ for all $k \geq 1$. Let h_k be given by

$$h_k(\xi) = \frac{a_{n_k}}{2\pi} \left| \int_0^1 f_{n_k}(x) e^{i a_{n_k} \xi x} dx \right|^2 \left(\int_0^1 |f_{n_k}(x)|^2 dx \right)^{-1}.$$

Then from (4.1), we have for each $m \geq 0$,

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} \xi^{2m} h_k(\xi) d\xi = b_{2m}/b_0,$$

and since $h_k(\xi)$ is an even function in ξ, the sequence of probability distribution $\{h_k(\xi)d\xi\}$ converges in moments to $d\mu(\xi)$. We therefore conclude from Lemma 4.5 that the sequence of characteristic functions of $h_k(\xi)d\xi$ converges point-wise to the characteristic function of $d\mu(\xi)$.

Now observe that since f_{n_k} does not change sign along $(0,1)$, $h_k(\xi)$ is a positive-definite function in ξ for each k by Bochner's theorem. Therefore the characteristic function of $h_k(\xi)d\xi$ is a non-negative function for each k. However, since we assumed $d\mu(\xi)$ is not positive-definite, the characteristic function $\int_{-\infty}^{\infty} e^{it\xi} d\mu(\xi)$ is negative for some $t \in \mathbb{R}$, which contradicts the point-wise convergence of characteristic functions. We therefore conclude that all but finitely many f_n has at least one sign change on $(0,1)$. $\square$

4.3. **Sign changes of even eigenfunctions on fixed segments.**

Lemma 4.7. *Assume that QUE holds for the sequence of even eigenfunctions $\{u_n\}_{n\geq 1}$. For any fixed segment $\beta \subset \mathrm{Fix}(\tau)$, all but finitely many u_n have at least one sign change on β.*

Proof. Assume for contradiction that there exists a subsequence of even eigenfunctions $\{u_{n_k}\}_{k\geq 1}$ such that u_{n_k} does not change sign along β for all $k \geq 1$. Fix a non-negative function $f \in C_0^\infty(\beta)$.

First, by Corollary 3.2, we can find a subsequence $\{u_{j_k}\}_{k\geq 1} \subset \{u_{n_k}\}_{k\geq 1}$ such that

$$\lim_{k \to \infty} 2 \int_\beta f^2(t) dt ||f u_{j_k}||^{-2}_{L^2(\beta)} = a$$

for some $0 \leq a \leq 1$. Then by Theorem 3.1, we have that

$$\lim_{k \to \infty} 1 - \lambda_{j_k}^{-2m} \int_\beta |\partial_t^m (f(t) u_{j_k}(t))|^2 \, ||f u_{j_k}||^{-2}_{L^2(\beta)} dt = a - \frac{a}{\pi} \int_{-1}^1 \xi^{2m} \frac{d\xi}{\sqrt{1-\xi^2}}.$$

We therefore have

$$\begin{aligned} \lim_{k \to \infty} \lambda_{j_k}^{-2m} \int_\beta |\partial_t^m (f(t) u_{j_k}(t))|^2 \, ||f u_{j_k}||^{-2}_{L^2(\beta)} dt &= (1-a) + \frac{a}{\pi} \int_{-1}^1 \xi^{2m} \frac{d\xi}{\sqrt{1-\xi^2}} \\ &= \int_{-\infty}^{\infty} \xi^{2m} d\mu_a(\xi) \\ &= b_{2m}, \end{aligned}$$

where $d\mu_a(\xi)$ is the probability measure given by

$$d\mu_a(\xi) = \frac{(1-a)}{2} (\delta_{-1}(\xi) + \delta_1(\xi)) d\xi + \frac{a}{\pi} I_{[-1,1]}(\xi) \frac{d\xi}{\sqrt{1-\xi^2}}.$$

Here $I_{[-1,1]}(\xi)$ is the indicator function of $[-1,1]$. Observe that

$$\int_{-\infty}^{\infty} \xi^{2m} d\mu_a(\xi) \le (1-a) + \frac{a}{\pi}\int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}} = 1,$$

and

$$\limsup_{m\to\infty} \frac{1}{2m} b_{2m}^{\frac{1}{2m}} = 0 < +\infty,$$

so by Lemma 4.4, $d\mu_a(\xi)$ is the only probability measure on $\mathbb{R}$ whose $2m$th moment is b_{2m} and whose $(2m+1)$th moment is zero for any $m \ge 0$.

Note that the characteristic function of $d\mu_a(\xi)$ is given by

$$\int_{-\infty}^{\infty} e^{it\xi} d\mu_a(\xi) = (1-a)\cos(t) + aJ_0(t),$$

where $J_0(t)$ is the Bessel function of the first kind and that

$$(1-a)\cos(\pi) + aJ_0(\pi) \le J_0(\pi) = -0.3042\ldots < 0$$

which implies that $d\mu_a(\xi)$ is not positive-definite.

It now follows from Lemma 4.6 that $f(t)u_{j_k}$ has at least one sign change along β for all but finitely many k, which contradicts the assumption that u_{n_k} does not change sign on β for all $k \ge 1$. We therefore conclude that all but finitely many u_n have at least one sign change on β. ☐

We complete the proof of Theorem 1.6 by proving Theorem 4.3.

Proof of Theorem 4.3. Fix $N \in \mathbb{N}$. Let $\beta_1, \ldots, \beta_N \subset \mathrm{Fix}(\tau)$ be a set of disjoint segments. Then by Lemma 4.7, for all sufficiently large k, u_n has at least one sign change on each curve β_i for $i = 1, \ldots, N$. Hence we have

$$\liminf_{k\to\infty} \#\left(Z_{u_n} \cap \mathrm{Fix}(\tau)\right) \ge N,$$

and since N can be chosen arbitrarily large, we conclude that

$$\lim_{k\to\infty} \#\left(Z_{u_n} \cap \mathrm{Fix}(\tau)\right) = +\infty. \qquad ☐$$

5. Nodal domains of odd eigenfunctions

In this section we prove an analogy of Theorem 1.6 for sequence odd eigenfunctions assuming QUE. Recall from (3.1) that

$$\langle \Delta_g Tu, u\rangle_{U_-} - \langle Tu, \Delta_g u\rangle_{U_-} = \langle \partial_n Tu|_\beta, u|_\beta\rangle_\beta - \langle Tu|_\beta, \partial_n u|_\beta\rangle_\beta.$$

From the assumption that u is an odd eigenfunction, we have the Rellich identity for odd eigenfunctions

$$\frac{1}{\lambda}\langle [-\Delta_g, T]u, u\rangle_{U_-} = \frac{1}{\lambda}\langle Tu|_\beta, \partial_n u|_\beta\rangle_\beta. \tag{5.1}$$

Let

$$a_{\delta,m}(x,\xi) = \chi\left(\frac{n}{\delta}\right) f^2(t)\xi_t^{2m}\xi_n,$$

and let $T = \mathrm{Op}(a_{\delta,m})$. For simplicity, let $Nu = \lambda^{-1}\partial_n u|_\beta$. Then the RHS of (5.1) is

$$(-1)^m \lambda^{-2m} \int_\beta f^2(t)\left(\partial_t^{2m} Nu(t)\right) Nu(t)dt$$
$$= \lambda^{-2m} \int_\beta \left|\partial_t^m \left(f(t)Nu(t)\right)\right|^2 dt + O_{m,f}\left(\lambda^{-\frac{1}{4}}\right),$$

and the LHS of (5.1) is

$$\frac{2}{\pi}\int_{-1}^{1}\xi^{2m}\sqrt{1-\xi^2}d\xi\int_{\beta}f^2(t)dt+o_{\delta,m,f}(1)+O_{m,f}(\delta).$$

Therefore Theorem 3.1 for odd eigenfunctions assuming QUE is

$$\begin{aligned}\lim_{k\to\infty}\lambda_n^{-2m}\int_{\beta}|\partial_t^m\left(f(t)Nu_n(t)\right)|^2dt &= \frac{2}{\pi}\int_{\beta}f^2(t)dt\int_{-1}^{1}\xi^{2m}\sqrt{1-\xi^2}d\xi\\ &= b_{2m}\int_{\beta}f^2(t)dt.\end{aligned}$$

Let $d\mu(\xi)=\pi^{-1}\sqrt{1-\xi^2}d\xi$. Observe that

$$b_{2m}=\frac{2}{\pi}\int_{-1}^{1}\xi^{2m}\sqrt{1-\xi^2}d\xi\le\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}d\xi=1,$$

and

$$\limsup_{m\to\infty}\frac{1}{2m}b_{2m}^{\frac{1}{2m}}=0<+\infty,$$

so by Lemma 4.4, $d\mu(\xi)$ is the only probability measure on $\mathbb{R}$ whose $2m$th moment is b_{2m} and whose $(2m+1)$th moment is zero for any $m\ge 0$. Now note that

$$\int e^{it\xi}d\mu(\xi)=J_1(t)/t,$$

and since $J_1(5)/5=-0.0655\ldots<0$, $d\mu(\xi)$ is not positive-definite, so we may apply Lemma 4.6 to conclude.

Lemma 5.1. *Assume that QUE holds for a sequence of odd eigenfunctions $\{u_n\}$. For any fixed segment $\beta\subset\mathrm{Fix}(\tau)$, all but finitely many $\partial_n u_n|_\beta$ has at least one sign change on β.*

As in Theorem 4.3, Lemma 5.1 implies the following.

Theorem 5.2. *Assume that QUE holds for a sequence of odd eigenfunctions $\{u_n\}$. Then*

$$\lim_{k\to\infty}\#\{x\in\mathrm{Fix}(\tau)\ :\ (\partial_n u_n)(x)=0\}=+\infty.$$

We now use the topological argument in [GRS13, JZ16] to conclude an analogy of Theorem 1.6 for odd eigenfunctions.

Theorem 5.3. *Assume that QUE holds for a sequence of odd eigenfunctions $\{u_n\}$. Then*

$$\lim_{k\to\infty}\mathcal{N}(u_n)\to+\infty.$$

6. Proof of Theorem 1.1

We now prove Theorem 1.1 using Theorem 1.6. Let Γ be an arithmetic triangle group, and let $\mathbb{X}=\Gamma\backslash\mathbb{H}$. Let $\{\phi_j\}_j$ be the complete sequence of Hecke-Maass eigenforms on $\mathbb{X}$; i.e., it is a joint eigenfunction of $-\Delta_g$ and Hecke operators $\{T_n\}_{n\ge1}$. It is shown in [GRS15] that there exists an orientation-reversing isometric involution $\tau:\mathbb{X}\to\mathbb{X}$ such that $\mathrm{Fix}(\tau)$ is separating and that τ commutes with all T_n. From the multiplicity one theorem for Hecke eigenforms [AL70], the sequence of Hecke eigenvalues $\{\lambda_\phi(n)\}_{n\ge1}$ of T_n (i.e., $T_n\phi=\lambda_\phi(n)\phi$) determines ϕ uniquely. Hence any Hecke-Maass eigenform ϕ_j on $\mathbb{X}$ is an eigenfunction of τ so that we have either

$\tau\phi_j = \phi_j$ or $\tau\phi_j = -\phi_j$ for all j. Now from the arithmetic quantum unique ergodicity theorem by Lindenstrauss [Lin06], QUE holds for $\{\phi_j\}_j$—hence we conclude that $\lim_{j\to+\infty} \mathcal{N}(\phi_j) = +\infty$ by Theorem 1.6.

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