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CANONICAL BASES FOR CLUSTER ALGEBRAS

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Introduction

0.1. **Statement of the main results.** Fock and Goncharov conjectured that the algebra of functions on a cluster variety has a canonical vector space basis parameterized by the tropical points of the mirror cluster variety. Unfortunately, as shown in [GHK13] by the first three authors of this paper, this conjecture is usually false: in general the cluster variety may have far too few global functions. One can only expect a power series version of the conjecture, holding in the "large complex structure limit", and honest global functions parameterized by a subset of the mirror tropical points. For the conjecture to hold as stated, one needs further affineness assumptions. Here we apply methods developed in the study of mirror symmetry, in particular *scattering diagrams*, introduced by Kontsevich and Soibelman in [KS06] for two dimensions and by Gross and Siebert in [GS11] for all dimensions, *broken lines*, introduced by Gross in [G09] and developed further by Carl, Pumperla, and Siebert in [CPS], and *theta functions*, introduced by Gross, Hacking, Keel, and Siebert, see [GHK11], [CPS], [GS12], and [GHKS], to prove the conjecture in this corrected form. We give in addition a formula for the structure constants in this basis, nonnegative integers given by counts of broken lines. Definitions of all these objects, essentially combinatorial in nature, in the context of cluster algebras will be given in later sections. Here are more precise statements of our results.

For basic cluster variety notions we follow the notation of [GHK13, §2], for convenience, as we have collected there a number of definitions across the literature; nothing there is original. We recall some of this notation in Appendices A and B. The various flavors of cluster varieties are all varieties of the form $V = \bigcup_{\mathbf{s}} T_{L,\mathbf{s}}$, where $T_{L,\mathbf{s}}$ is a copy of the algebraic torus

$$T_L := L \otimes_{\mathbb{Z}} \mathbb{G}_m = \operatorname{Hom}(L^*, \mathbb{G}_m) = \operatorname{Spec} \Bbbk[L^*]$$

over a field $\Bbbk$ of characteristic 0, and $L = \mathbb{Z}^n$ is a lattice, indexed by $\mathbf{s}$ running over a set of *seeds* (a seed being roughly an ordered basis for L). The birational transformations induced by the inclusions of two different copies of the torus are compositions of *mutations*. Fock and Goncharov introduced a simple way to dualize

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the mutations, and using this define the *Fock–Goncharov dual*,[1] $V^\vee = \bigcup_{\mathbf{s}} T_{L^*,\mathbf{s}}$. We write $\mathbb{Z}^T$ for the tropical semifield of integers under max, +. There is a notion of the set of $\mathbb{Z}^T$-valued points of V, written as $V(\mathbb{Z}^T)$. This can also be viewed as being canonically in bijection with $V^{\mathrm{trop}}(\mathbb{Z})$, the set of divisorial discrete valuations on the field of rational functions of V where the canonical volume form has a pole; see §2. Each choice of seed $\mathbf{s}$ determines an identification $V(\mathbb{Z}^T) = L$.

Our main object of study is the $\mathcal{A}$ *cluster variety with principal coefficients*, $\mathcal{A}_{\mathrm{prin}} = \bigcup_{\mathbf{s}} T_{\widetilde{N}^\circ,\mathbf{s}}$; see Appendices A and B for notation. This comes with a canonical fibration over a torus $\pi : \mathcal{A}_{\mathrm{prin}} \to T_M$ and a canonical free action by a torus T_{N°. We let $\mathcal{A}_t := \pi^{-1}(t)$. The fiber $\mathcal{A}_e \subset \mathcal{A}_{\mathrm{prin}}$ ($e \in T_M$ the identity) is the Fock–Goncharov $\mathcal{A}$ variety (whose algebra of regular functions is the Fomin–Zelevinsky upper cluster algebra). The quotient $\mathcal{A}_{\mathrm{prin}}/T_{N^\circ}$ is the Fock–Goncharov $\mathcal{X}$ variety.

Definition 0.1. A *global monomial* on a cluster variety $V = \bigcup_{\mathbf{s}\in S} T_{L,\mathbf{s}}$ is a regular function on V which restricts to a character on some torus $T_{L,\mathbf{s}}$ in the atlas. For V an $\mathcal{A}$-type cluster variety, a global monomial is the same as a cluster monomial. One defines the *upper cluster algebra* $\mathrm{up}(V)$ associated to V by $\mathrm{up}(V) := \Gamma(V, \mathcal{O}_V)$ and the *ordinary cluster algebra* $\mathrm{ord}(V)$ to be the subalgebra of $\mathrm{up}(V)$ generated by global monomials.

For example, $\mathrm{ord}(\mathcal{A})$ is the original cluster algebra defined by Fomin and Zelevinsky in [FZ02a], and $\mathrm{up}(\mathcal{A})$ is the corresponding upper cluster algebra as defined in [BFZ05].

Given a global monomial f on V, there is a seed $\mathbf{s}$ such that $f|_{T_{L,\mathbf{s}}}$ is a character z^m, $m \in L^*$. Because the seed $\mathbf{s}$ gives an identification of $V^\vee(\mathbb{Z}^T)$ with L^*, we obtain an element $\mathbf{g}(m) \in V^\vee(\mathbb{Z}^T)$, which we show is well-defined (independent of the open set $T_{L,\mathbf{s}}$); see Lemma 7.10. This is the *g-vector* of the global monomial f. We show this notion of g-vector coincides with the notion of g-vector from [FZ07] in the $\mathcal{A}$ case; see Corollary 5.9. Let $\Delta^+(\mathbb{Z}) \subset V^\vee(\mathbb{Z}^T)$ be the set of g-vectors of all global monomials on V. Finally, we write $\mathrm{can}(V)$ for the $\Bbbk$-vector space with basis $V^\vee(\mathbb{Z}^T)$, i.e.,

$$\mathrm{can}(V) := \bigoplus_{q \in V^\vee(\mathbb{Z}^T)} \Bbbk \cdot \vartheta_q \tag{0.2}$$

(where ϑ_q for the moment indicates the abstract basis element corresponding to $q \in V^\vee(\mathbb{Z}^T)$).

Fock and Goncharov's dual basis conjecture says that $\mathrm{can}(V)$ is canonically identified with the vector space $\mathrm{up}(V)$, and so in particular $\mathrm{can}(V)$ should have a canonical $\Bbbk$-algebra structure. Note that such an algebra structure is determined by its structure constants, a function

$$\alpha : V^\vee(\mathbb{Z}^T) \times V^\vee(\mathbb{Z}^T) \times V^\vee(\mathbb{Z}^T) \to \Bbbk$$

such that for fixed p, q, $\alpha(p,q,r) = 0$ for all but finitely many r and

$$\vartheta_p \cdot \vartheta_q = \sum_r \alpha(p,q,r)\vartheta_r.$$

[1] Roughly one can view the Fock–Goncharov dual as the mirror variety, but this is not always precisely the case. With some additional effort, one can make this precise "at the boundary", but we shall not do so here.

With this in mind, we have:

Theorem 0.3. *Let V be one of $\mathcal{A}, \mathcal{X}, \mathcal{A}_{\mathrm{prin}}$. The following hold:*

(1) *There are canonically defined nonnegative structure constants*
$$\alpha : V^\vee(\mathbb{Z}^T) \times V^\vee(\mathbb{Z}^T) \times V^\vee(\mathbb{Z}^T) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$
These are given by counts of broken lines*, certain combinatorial objects which we will define. The value ∞ is not taken in the $\mathcal{X}$ or $\mathcal{A}_{\mathrm{prin}}$ case.*

(2) *There is a canonically defined subset $\Theta \subset V^\vee(\mathbb{Z}^T)$ with $\alpha(\Theta \times \Theta \times \Theta) \subseteq \mathbb{Z}_{\geq 0}$ such that the restriction of α gives the vector subspace* $\mathrm{mid}(V) \subset \mathrm{can}(V)$ *with basis indexed by Θ the structure of an associative commutative $\Bbbk$-algebra.*

(3) *$\Delta^+(\mathbb{Z}) \subset \Theta$, i.e., Θ contains the g-vector of each global monomial.*

(4) *For the lattice structure on $V^\vee(\mathbb{Z}^T)$ determined by any choice of seed, $\Theta \subset V^\vee(\mathbb{Z}^T)$ is closed under addition. Furthermore, $\Theta \subset V^\vee(\mathbb{Z}^T)$ is saturated: for $k > 0$ and $x \in V^\vee(\mathbb{Z}^T)$, $k \cdot x \in \Theta$ if and only if $x \in \Theta$.*

(5) *There is a canonical $\Bbbk$-algebra map $\nu : \mathrm{mid}(V) \to \mathrm{up}(V)$ which sends ϑ_q for $q \in \Delta^+(\mathbb{Z})$ to the corresponding global monomial.*

(6) *The image $\nu(\vartheta_q) \in \mathrm{up}(V)$ is a universal positive Laurent polynomial* (*i.e., a Laurent polynomial with nonnegative integral coefficients in the cluster variables for each seed*)*.*

(7) *ν is injective for $V = \mathcal{A}_{\mathrm{prin}}$ or $V = \mathcal{X}$. Furthermore, ν is injective for $V = \mathcal{A}$ under the additional assumption that there is a seed $\mathbf{s} = (e_1, \ldots, e_n)$ for which all the covectors $\{e_i, \cdot\}$, $i \in I_{\mathrm{uf}}$, lie in a strictly convex cone. When ν is injective, we have canonical inclusions*
$$\mathrm{ord}(V) \subset \mathrm{mid}(V) \subset \mathrm{up}(V).$$

There is an analogue to Theorem 0.3 for $\mathcal{A}_t$ (the main difference is that the *theta functions*, i.e., the canonical basis for $\mathrm{mid}(\mathcal{A}_t)$, are only defined up to scaling each individual element, and the structure constants will not in general be integers). Injectivity in (7) holds for very general $\mathcal{A}_t$; see Theorem 7.16.

Note that (5)–(6) immediately imply:

Corollary 0.4 (Positivity of the Laurent phenomenon)**.** *Each cluster variable of an $\mathcal{A}$-cluster algebra is a Laurent polynomial with nonnegative integer coefficients in the cluster variables of any given seed.*

This was conjectured by Fomin and Zelevinsky in their original paper [FZ02a]. Positivity was obtained independently in the skew-symmetric case by [LS13] by an entirely different argument. In our proof the positivity in (1) and (6) both come from positivity in the scattering diagram, a powerful tool fundamental to the entire paper; see Theorem 1.13.

We conjecture that injectivity in (7) holds for all $\mathcal{A}_t$ (without the convexity assumption). Note (7) includes the linear independence of cluster monomials, which has already been established (without convexity assumptions) for skew-symmetric cluster algebras in [CKLP] by a very different argument. The linear independence of cluster monomials in the principal case also follows easily from our scattering diagram technology, as pointed out to us by Greg Muller; see Theorem 7.20.

When there are frozen variables, one obtains a partial compactification $V \subset \overline{V}$ (where the frozen variables are allowed to take the value 0) for $V = \mathcal{A}$, $\mathcal{A}_{\mathrm{prin}}$, or $\mathcal{A}_t$. The notions of ord, up, can, and mid extend naturally to $\overline{V}$; see Construction B.9.

Of course if $\mathrm{ord}(V) = \mathrm{up}(V)$, and we have injectivity in (7), $\mathrm{ord}(V) = \mathrm{mid}(V) = \mathrm{up}(V)$ has a canonical basis Θ with the given properties. Also, $\mathrm{ord}(V) = \mathrm{up}(V)$ implies, under certain hypotheses, $\mathrm{ord}(\overline{V}) = \mathrm{up}(\overline{V})$; see Lemma 9.10. Such partial compactifications are essential for representation-theoretic applications:

Example 0.5. Let $G = \mathrm{SL}_r$. Choose a Borel subgroup B of G, $H \subset B$ a maximal torus, and let $N = [B, B]$ be the unipotent radical of B. These choices determine a cluster variety structure (with frozen variables) on $\overline{\mathcal{A}} = G/N$, with $\mathrm{up}(\overline{\mathcal{A}}) = \mathrm{ord}(\overline{\mathcal{A}}) = \mathcal{O}(G/N)$, the ring of regular functions on G/N; see [GLS, §10.4.2].

Theorem 0.3 implies that these choices canonically determine a vector space basis $\Theta \subset \mathcal{O}(G/N)$. Each basis element is an H-eigenfunction for the natural (right) action of H on G/N. For each character $\lambda \in \chi^*(H)$, $\Theta \cap \mathcal{O}(G/N)^\lambda$ is a basis of the weight space $\mathcal{O}(G/N)^\lambda =: V_\lambda$. The V_λ are the collection of irreducible representations of G, each of which thus inherits a basis, canonically determined by the choice of $H \subset B \subset G$.

We give, combining our results with results of T. Magee, much more precise results; see Corollary 0.20.

Canonical bases for $\mathcal{O}(G/N)$ have been constructed by Lusztig. Here we will obtain bases by a procedure very different from Lusztig's, as a special case of the more general [GHK11, Conjecture 0.6], which applies in theory to any variety with the right sort of volume form. See Remark 0.16 for further commentary on this. □

The tools necessary for the proof of Theorem 0.3 are developed in the first six sections of the paper, with the proof given in §7. This material is summarized in more detail in §0.2.

The second part of the paper turns to criteria for the full Fock–Goncharov conjecture to hold. Precisely:

Definition 0.6. We say the *full Fock–Goncharov conjecture holds* for a cluster variety V if the map $\nu : \mathrm{mid}(V) \to \mathrm{up}(V)$ of Theorem 0.3 is injective,

$$\mathrm{up}(V) = \mathrm{can}(V) \quad \text{and} \quad \Theta = V^\vee(\mathbb{Z}^T).$$

Note this implies $\mathrm{mid}(V) = \mathrm{up}(V) = \mathrm{can}(V)$.

We prove a number of criteria which guarantee the full Fock–Goncharov conjecture holds. One such condition, which seems to be very natural in our setup and is implied, say, by the existence of a maximal green sequence, is:

Proposition 0.7 (Proposition 8.25). *If the set $\Delta^+(\mathbb{Z})$ of all g-vectors of global monomials of $\mathcal{A}$ in $\mathcal{A}^\vee(\mathbb{Z}^T)$ is not contained in a half-space under the identification of $\mathcal{A}^\vee(\mathbb{Z}^T)$ with M° induced by some choice of seed, then the full Fock–Goncharov conjecture holds for $\mathcal{A}_{\mathrm{prin}}$, $\mathcal{X}$, very general $\mathcal{A}_t$ and, if the convexity condition* (7) *of Theorem* 0.3 *holds, for $\mathcal{A}$.*

Many of the results in the second part of the paper are proved using a generalized notion of convex function or convex polytope; see §§0.3 and 0.4 for more details.

In §8.5, we turn to results on partial compactifications. We first explain how convex polytopes in our sense give rise, under suitable hypotheses, to compactifications of $\mathcal{A}$-type cluster varieties and toric degenerations of such. This connects our constructions to the mirror symmetry picture described in [GHK11], and in particular describes a partial compactification of $\mathcal{A}_{\mathrm{prin}}$ as giving a degeneration of a family of log Calabi–Yau varieties to a toric variety. Partial compactifications via frozen variables are also important in representation theoretic applications, as already indicated in Example 0.5. We prove results for such partial compactifications which, combined with recent results of T. Magee [Ma15], [Ma17], yield strong representation-theoretic results; see §0.4 for more details.

We now turn to a more detailed summary of the contents of the paper.

0.2. **Toward the main theorem.** Section 1 is devoted to the construction of the fundamental tool of the paper, *scattering diagrams*. While [GS11] defined these in much greater generality, here they are collections of walls living in a vector space with attached functions constructed canonically from a choice of seed data. A precise definition can be found in §1.1. Here we simply highlight the main new result, Theorem 1.13, whose proof, being fairly technical, is deferred to Appendix C. This says that the functions attached to walls of a scattering diagram associated to seed data have positive coefficients. All positivity results in this paper flow from this fundamental observation, and indeed many of our arguments use this in an essential way. For the reader's convenience, we give in §1.2 an elementary construction of the relevant scattering diagrams, drawing on the method given in [KS13]. Since a scattering diagram depends on a choice of seed, §1.3 shows how scattering diagrams associated to mutation equivalent seeds are related. This shows that a scattering diagram has a chamber structure indexed by seeds mutation equivalent to the initial choice of seed.

In §2 we review some notions of tropicalizations of cluster varieties, showing that scattering diagrams naturally live in such tropicalizations. Indeed, the scattering diagram which is associated to a cluster variety V lives naturally in the tropical space of the Fock–Goncharov dual $V^\vee(\mathbb{R}^T)$. These tropicalizations, crucially, can only be viewed as piecewise linear, rather than linear, spaces, with a choice of seed giving an identification of the tropicalization with a linear space. Already the mutation combinatorics becomes apparent:

Theorem 0.8 (Lemma 2.10 and Theorem 2.13)**.** *For each seed* $\mathbf{s} = (e_1, \dots, e_n)$ *of an $\mathcal{A}$-cluster variety, the* (Fock–Goncharov) cluster chamber *associated to* $\mathbf{s}$ *is*

$$\mathcal{C}^+_{\mathbf{s}} := \{x \in \mathcal{A}^\vee(\mathbb{R}^T) \,|\, (z^{e_i})^T(x) \le 0 \text{ for all } 1 \le i \le n\},$$

where $(z^{e_i})^T$ denotes the tropicalization of the monomial z^{e_i}; see §2. The collection Δ^+ of such subsets of $\mathcal{A}^\vee(\mathbb{R}^T)$ over all mutation equivalent seeds form the maximal cones of a simplicial fan, the (Fock–Goncharov) cluster complex*. The Fomin–Zelevinsky exchange graph is the dual graph of this fan.*

The collection of cones Δ^+ was introduced by Fock and Goncharov, who conjectured they formed a fan. It is not at all obvious from the definition that the interiors of the cones cannot overlap. Our description of the chamber structure induced by a scattering diagram in fact shows that part of the chamber structure coincides with the collection of cones Δ^+. This shows the fact that they form a fan directly. In addition, the set $\Delta^+(\mathbb{Z})$ of Theorem 0.3 consists of the integral points of the union of cones in Δ^+.

Section 3 gives the definition of broken line, the second principal combinatorial tool of the paper. These were originally introduced in [G09] and developed further in [CPS] as tropical replacements for Maslov index two disks. In [GHK11], they were used to define *theta functions*, which are, in principle, formal sums over all broken lines with fixed boundary conditions. The relevance of theta functions for us comes in §4. Here we show the direct relationship between scattering diagrams and the $\mathcal{A}$ cluster algebra. We show that if we associate a suitable torus T_L to each chamber of the scattering diagram associated to a mutation of the initial seed, then the walls separating the chambers can be interpreted as giving birational maps between these tori. Gluing together these copies of T_L gives the $\mathcal{A}$ cluster variety; see Theorem 4.4. Further, a theta function ϑ_p depends on a point $p \in \mathcal{A}^\vee(\mathbb{Z}^T)$. If for a given choice of p, ϑ_p is in fact a finite sum, then ϑ_p is a global function on $\mathcal{A}$. We show that this holds in particular when p lies in the cluster complex Δ^+, and in this case ϑ_p agrees with the cluster monomial with g-vector given by p. Because of the positivity result Theorem 1.13, ϑ_p is in any event always a power series with positive coefficients. Thus we get positivity of the Laurent phenomenon, Theorem 4.10, as an easy consequence of our formalism.

In §5 we begin with what is another essential observation for our approach. A choice of initial seed $\mathbf{s}$ provides a partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ of $\mathcal{A}_{\mathrm{prin}}$ by allowing the variables $X_1, \ldots, X_n$ (the principal coefficients) to be zero. These variables induce a flat map $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$, with $\mathcal{A}$ being the fiber over $(1, \ldots, 1)$. Our methods easily show:

Theorem 0.9 (Corollary 5.3(1)). *The central fiber $\pi^{-1}(0) \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ is the algebraic torus $T_{N^\circ,\mathbf{s}}$.*

Though immediate from our scattering diagram methods, the result is not obvious from the original definitions; indeed, it is equivalent to the sign-coherence of c-vectors (see Corollary 5.5).

The last major ingredient in the proof of Theorem 0.3 is a formal version of the Fock–Goncharov conjecture. As mentioned above, this conjecture does not hold in general, but in §6, we show that the Fock–Goncharov conjecture holds in a formal neighborhood of the torus fiber of $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n$. We show the structure constants given in Theorem 0.3(1) have a tropical interpretation and determine an associative product on $\mathrm{can}(\mathcal{A}_{\mathrm{prin}})$, except that $\vartheta_p \cdot \vartheta_q$ will in general be an infinite sum of theta functions. Further, canonically associated to each universal Laurent polynomial $g \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ is a formal power series $\sum_{q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)} \alpha_q \vartheta_q$ which converges to g in a formal neighborhood of the central fiber. For the precise statement see Theorem 6.8, which we interpret as saying that the Fock–Goncharov dual basis conjecture always holds in the *large complex structure limit.* This is all one should expect from log Calabi–Yau mirror symmetry in the absence of further affineness assumptions. A crucial point, shown in the proof of Theorem 6.8, is that the expansion of $g \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ is independent of the choice of seed $\mathbf{s}$ determining the compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$; i.e., it is independent of which degeneration is used to perform the expansion.

In §7 we introduce the middle cluster algebra $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$. The idea is that while we do not know that every regular function on $\mathcal{A}_{\mathrm{prin}}$ can be written as a linear combination of theta functions, there is a set $\Theta \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ indexing those p for which ϑ_p is a regular function on $\mathcal{A}_{\mathrm{prin}}$. These in fact yield a vector space basis for a subalgebra of $\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ which necessarily includes all cluster monomials, hence includes the ordinary cluster algebra. With this in hand, Theorem 0.3 becomes a summary of the results proved up to this point. We then deduce the result for $\mathcal{X}$ and $\mathcal{A}$-type cluster varieties from the $\mathcal{A}_{\mathrm{prin}}$ case.

0.3. **Convexity conditions.** We now turn to the use of convexity conditions to prove the Fock–Goncharov conjecture in a number of different situations, as covered in §8. To motivate the concepts, let us define a *partial minimal model* of a log Calabi–Yau variety V. This is an inclusion $V \subset Y$ as an open subset such that the canonical volume form on V has a simple pole along each irreducible divisor of the boundary $Y \setminus V$. For example, a partial minimal model for an algebraic torus is the same as a toric compactification. We wish to extend elementary constructions of toric geometry to the cluster case. For example, the partial compactification $\mathcal{A} \subset \overline{\mathcal{A}}$ determined by frozen variables is a partial minimal model.

The generalization of the cocharacter lattice $N \subset N_{\mathbb{R}}$ of the algebraic torus $T_N := N \otimes \mathbb{G}_m$ is the tropical set $V(\mathbb{Z}^T) \subset V(\mathbb{R}^T)$ of V. The main difference between the torus and the general case is that $V(\mathbb{R}^T)$ is not in general a vector space. Indeed, the identification of $V(\mathbb{Z}^T)$ with the cocharacter lattices of various charts of V induce piecewise linear (but not linear) identifications between the cocharacter lattices. As a result, a piecewise straight path in $V(\mathbb{R}^T)$ which is straight under one identification $V(\mathbb{R}^T) = N_{\mathbb{R}}$ will be bent under another. Thus the usual notions of straight lines, convex functions, or convex sets do not make sense on $V(\mathbb{R}^T)$.

The idea for generalizing the notion of convexity is to instead make use of broken lines, which are piecewise linear paths in $V(\mathbb{R}^T)$. Using broken lines in place of straight lines, we can say which piecewise linear functions, and thus which polytopes, are convex; see Definition 8.2. Each regular function $W : V \to \mathbb{A}^1$ has a canonical piecewise linear tropicalization $w := W^T : V(\mathbb{R}^T) \to \mathbb{R}$, which we conjecture is convex in the sense of Definition 8.2; see Conjecture 8.11. The conjecture is easy for $W \in \operatorname{ord}(V) \subset \operatorname{up}(V)$; see Proposition 8.13. Each *convex* piecewise linear w gives a *convex* polytope $\Xi_w = \{x \,|\, w(x) \geq -1\}$ and a *convex* cone $\{x \in V^\vee(\mathbb{R}^T) \,|\, w(x) \geq 0\}$, where italics indicates convexity in our broken line sense. We believe the existence of a bounded polytope is equivalent to the full Fock–Goncharov conjecture:

Conjecture 0.10. *The full Fock–Goncharov conjecture holds for $\mathcal{A}_{\mathrm{prin}}$ if and only if the tropical space $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ contains a full-dimensional bounded polytope, convex in our sense.*

The examples of [GHK13, §7], show that for the full Fock–Goncharov conjecture to hold, we need to assume V has enough global functions. In that case tropicalizing a general function gives (conjecturally) a bounded *convex* polytope. As we are unable to prove Conjecture 8.11 except in the monomial case, we use a restricted version (which happily still has wide application):

Definition 0.11. A cluster variety V has *Enough Global Monomials* (EGM) if for each valuation $0 \neq v \in V^{\mathrm{trop}}(\mathbb{Z})$ there is a global monomial f with $v(f) < 0$.

The condition that V has EGM is equivalent to the existence of $W \in \operatorname{ord}(V)$ whose associated convex polytope Ξ_{W^T} is bounded; see Lemma 8.15.

The following theorem demonstrates the value of the EGM condition:

Theorem 0.12. *Let V be a cluster variety. Then:*

(1) (*Corollaries* 8.18 *and* 8.21) *If $V^\vee$ satisfies the EGM condition, then the multiplication rule on* $\operatorname{can}(V)$ *is polynomial, i.e., for given $p, q \in V^\vee(\mathbb{Z}^T)$, $\alpha(p,q,r) = 0$ for all but finitely many $r \in V^\vee(\mathbb{Z}^T)$. This gives* $\operatorname{can}(V)$ *the structure of a finitely generated commutative associative* $\mathbb{k}$*-algebra.*

(2) (*Proposition* 8.22) *If* $V = \mathcal{A}_{\mathrm{prin}}$ *and* V *satisfies the EGM condition, then there are canonical inclusions*

$$\mathrm{ord}(V) \subset \mathrm{mid}(V) \subset \mathrm{up}(V) \subset \mathrm{can}(V).$$

Remark 0.13. We believe, based on calculations in [M13, §7.1], that the conditions of the theorem ($\mathcal{A}_{\mathrm{prin}}$ has EGM, and $\Theta = \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$) hold for the cluster variety associated with the once-punctured torus; see some details in Examples 2.14 and 7.18. However, the equality $\mathrm{up}(\mathcal{A}) = \mathrm{can}(\mathcal{A})$ is expected to fail, and in particular in this case we expect the full Fock–Goncharov conjecture holds for $\mathcal{A}_{\mathrm{prin}}, \mathcal{X}$, and very general $\mathcal{A}_t$, but not for $\mathcal{A}$.

We note that $\mathcal{A}_{\mathrm{prin}}$ has EGM in many cases:

Proposition 0.14. *Consider the following conditions on a cluster algebra* $\mathcal{A}$*:*

(1) *The exchange matrix has full rank,* $\mathrm{up}(\mathcal{A})$ *is generated by finitely many cluster variables, and* $\mathrm{Spec}(\mathrm{up}(\mathcal{A}))$ *is a smooth affine variety.*
(2) $\mathcal{A}$ *has an acyclic seed.*
(3) $\mathcal{A}$ *has a seed with a maximal green sequence.*
(4) *For some seed, the cluster complex* $\Delta^+(\mathbb{Z}) \subset \mathcal{A}^\vee(\mathbb{R}^T)$ *is not contained in a half-space.*
(5) $\mathcal{A}_{\mathrm{prin}}$ *has EGM.*

Then (1) *implies* (5) (*Proposition* 8.27). *Furthermore,* (2) *implies* (3) *implies* (4) *implies* (5) (*Propositions* 8.24 *and* 8.25). *Finally,* (4) *implies the full Fock–Goncharov conjecture, for* $V = \mathcal{A}_{\mathrm{prin}}$*,* $\mathcal{X}$*, or very general* $\mathcal{A}_t$*, or, under the convexity assumption* (7) *of Theorem* 0.3*, for* $\mathcal{A}$ (*Proposition* 8.25).

Example 0.15. A recent paper [GY13] of Goodearl and Yakimov announces the equality up = ord for all double Bruhat cells in semisimple groups. In this case, Yakimov has furthermore announced the existence of a maximal green sequence. Many cluster varieties $\mathcal{A}$ associated to a marked bordered surface with at least two punctures also have a maximal green sequence; see [CLS, §1.3] for a summary of known results on this. The recent [GS16, Theorems 1.12 and 1.17], shows that (4) holds for the Fock–Goncharov cluster varieties of PGL_m local systems on *most* decorated surfaces. Together with Proposition 0.14, these results imply the full Fock–Goncharov theorem in any of these cases.

We note that for the cluster algebra associated to a marked bordered surface, a canonical basis of $\mathrm{up}(\mathcal{X})$ parameterized by $\mathcal{A}(\mathbb{Z}^T)$ has been previously obtained by Fock–Goncharov [FG06, Theorem 12.3]. They show that the $\mathcal{A}$ and $\mathcal{X}$ varieties have natural modular meaning as moduli spaces of local systems. They identify $\mathcal{A}(\mathbb{Z}^T)$ with a space of integer laminations (isotopy classes of disjoint loops with integer weights) and their associated basis element is a natural function given by trace of monodromy around a loop. We checked, together with A. Neitzke, that our basis agrees with the Fock–Goncharov basis of trace functions in the case of a sphere with four punctures, for primitive elements of the tropical set. Our theta function basis comes canonically from the cluster structure (it does not depend on any modular interpretation).

Remark 0.16. In general, we conjecture the bases we construct for rings of global functions on cluster varieties V or partial compactifications $\overline{V}$ are intrinsic to the underlying log Calabi–Yau variety V and do not depend on the particular cluster structure on V. This is a nontrivial statement: there exist varieties with multiple cluster structures (in particular different atlases of tori for the same variety). Yan Zhou will show in her PhD thesis that the (principal coefficient version of) the cluster variety associated to the once-punctured torus is an example.

This conjecture is suggested by [GHK11, Conjecture 0.6], and the results of [GHK11], [GHK12], and [GHKII] prove this in the case of the $\mathcal{X}$ cluster varieties where the skew-symmetric form has rank 2, which includes the case of the sphere with four punctures. Thus we have the (at least to us) remarkable conclusion that in many cases where bases occur because of some extrinsic interpretation of the spaces, in fact this extrinsic interpretation is irrelevant. For example, the theta functions given by trace functions above, which would appear to depend on the realization of the cluster variety as a moduli space of local systems, are actually intrinsic to the underlying variety. In the case of Example 0.5, where bases may arise from representation theory, our basis does not use the group-theoretic aspects of the spaces. The suggestion that the canonical basis is independent of the cluster structure may surprise some, as understanding the canonical basis was the initial motivation for the Fomin–Zelevinsky definition of cluster algebras.

Returning to the role of convexity notions, we note that our formula for the structure constants α of Theorem 0.3(1) is given by counting broken lines. As a result, our notion of convexity interacts nicely with the multiplication rule. This allows us to generalize basic polyhedral constructions from toric geometry in a straightforward way.

A polytope $\Xi \subset V^\vee(\mathbb{R}^T)$ convex in our sense determines (by familiar Rees-type constructions for graded rings) a compactification of V. Furthermore, for any choice of seed, $V^\vee(\mathbb{R}^T)$ is identified with a linear space $\mathbb{R}^n$ and Ξ with an ordinary convex polytope. Our construction also gives a flat degeneration of this compactification of V to the ordinary polarized toric variety for $\Xi \subset \mathbb{R}^n$; see §8.5. We expect this specializes to a uniform construction of many degenerations of representation theoretic objects to toric varieties; see, e.g., [C02], [AB], and [KM05]. Applied to the Fock–Goncharov moduli spaces of G-local systems, this will give for the first time compactifications of character varieties with nice (e.g., toroidal anticanonical) boundary; see Remark 8.34. The polytope can be chosen so that the boundary of the compactification is very simple, a union of toric varieties. For example, let $\mathrm{Gr}^o(k,n) \subset \mathrm{Gr}(k,n)$ be the open subset where the frozen variables for the standard cluster structure are nonvanishing. Then the boundary $\mathrm{Gr}(k,n) \setminus \mathrm{Gr}^o(k,n)$ consists of a union of certain Schubert cells. Using a polytope, we obtain an alternative compactification where the Schubert cells (which are highly nontoric) are replaced by toric varieties; see Theorem 8.35.

The Fock–Goncharov conjecture is the cluster special case of [GHK11, Conjecture 0.6], which says (roughly) that affine log Calabi–Yau varieties with maximal boundary come in canonical dual pairs with the tropical set of one parameterizing a canonical basis of functions on the other. We can view the conjecture as having two parts: First, the vector space, can, with this basis $V^{\mathrm{trop}}(\mathbb{Z})$ is naturally an algebra in a such a way that $V^\vee := \mathrm{Spec}(\mathrm{can})$ is an affine log CY. And then furthermore, this log CY is the mirror—in the cluster case the Fock–Goncharov dual (it is natural to further ask if this is the mirror in the sense of homological mirror symmetry but we do not consider this question here). Our deepest mirror theoretic result is the following weakening of the first part:

Theorem 0.17. *Assume $\mathcal{A}^\vee_{\mathrm{prin}}$ has EGM. Let $V = \mathcal{X}$, $\mathcal{A}_{\mathrm{prin}}$, or $\mathcal{A}_t$ for very general t, or let $V = \mathcal{A}$ and assume the convexity condition* (7) *of Theorem* 0.3 *holds. Then the structure constants of Theorem* 0.3 *define an algebra structure on* $\mathrm{can}(V)$ *such that it is a finitely generated $\Bbbk$-algebra and* $\mathrm{Spec}(\mathrm{can}(V))$ *is a log canonical Gorenstein K-trivial affine variety of dimension* $\dim(V)$.

For the proof see Theorem 8.32.

0.4. **Representation-theoretic applications.** We turn to §9. Here we study features of partial compactifications coming from frozen variables. As explained in Example 0.5, these partial compactifications are often the relevant ones in representation-theoretic examples. In particular, for a partial minimal model $\mathcal{A} \subset \overline{\mathcal{A}}$, often the vector subspace $\mathrm{up}(\overline{\mathcal{A}}) \subset \mathrm{up}(\mathcal{A})$ is more important than $\mathrm{up}(\mathcal{A})$ itself. For example there is a cluster structure with frozen variables for the open double Bruhat cell U in a semisimple group G. Then $\mathrm{up}(\mathcal{A})$ is the ring of functions on the open double Bruhat cell and $\mathrm{up}(\overline{\mathcal{A}}) = H^0(G, \mathcal{O}_G)$. Of course $H^0(G, \mathcal{O}_G)$ is the most important representation of G. However, one cannot expect a canonical basis of $\mathrm{up}(\overline{\mathcal{A}})$, i.e., one determined by the intrinsic geometry of $\overline{\mathcal{A}}$. For example, G has no nonconstant global functions which are eigenfunctions for the action of G on itself. But we expect, and in the myriad cases above can prove, that the affine log Calabi–Yau open subset $\mathcal{A} \subset \overline{\mathcal{A}}$ has a canonical basis Θ, and we believe that $\Theta \cap \mathrm{up}(\overline{\mathcal{A}})$, the set of theta functions on $\mathcal{A}$ that extends regularly to all of $\overline{\mathcal{A}}$, is a basis for $\mathrm{up}(\overline{\mathcal{A}})$, canonically associated to the choice of log Calabi–Yau open subset $\mathcal{A} \subset \overline{\mathcal{A}}$; see [GHK13, Remark 1.10]. This is not a basis of G-eigenfunctions, but they are eigenfunctions for the associated maximal torus, which is the subgroup of G that preserves U. This is exactly what one should expect: the basis is not intrinsic to G, instead it is (we conjecture) intrinsic to the pair $U \subset G$; see Remark 0.16.

We shall now describe in more detail what can be proved for partial compactifications of cluster varieties coming from frozen variables. A key point is a technical but combinatorial hypothesis that *each variable has an optimized seed*; see Definition 9.1 and Lemmas 9.2 and 9.3. The main need for this hypothesis is Proposition 9.7, which states that if a linear combination of theta functions extends across a boundary divisor, then each theta function in the sum extends across the divisor. Thus the middle cluster algebra, in this case, behaves well with respect to boundary divisors. Happily, this condition holds for the cluster structures on the Grassmannian, and, for $G = \mathrm{SL}_r$, for the cluster structure on a maximal unipotent subgroup $N \subset G$, the basic affine space $\mathcal{A} = G/N$, and the Fock–Goncharov cluster structure on $(\mathcal{A} \times \mathcal{A} \times \mathcal{A})/G$; see Remark 9.5.

Let us now work with the principal cluster variety $\mathcal{A}_{\mathrm{prin}}$. Consider the partial compactification $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}_{\mathrm{prin}}$ by allowing the frozen variables to be zero. Each boundary divisor $E \subset \overline{\mathcal{A}}_{\mathrm{prin}}$ gives a point $E \in \mathcal{A}_{\mathrm{prin}}(\mathbb{Z}^T)$, and thus (in general conjecturally) a canonical theta function ϑ_E on $\mathcal{A}^\vee_{\mathrm{prin}}$. We then define the *potential* $W = \sum_{E \subset \partial \overline{\mathcal{A}}_{\mathrm{prin}}} \vartheta_E \in \mathrm{up}(\mathcal{A}^\vee_{\mathrm{prin}})$ as the sum of these theta functions. We have its piecewise linear tropicalization $W^T : \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) \to \mathbb{R}$. This defines a cone

$$\Xi := \{x \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) \mid W^T(x) \geq 0\} \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T). \tag{0.18}$$

Theorem 0.19 (Corollaries 9.17 and 9.18)**.** *Assume that each frozen index i has an optimized seed. Then:*

(1) W^T *and* Ξ *are convex in our sense.*
(2) *The set* $\Xi \cap \Theta$ *parameterizes a canonical basis of an algebra* $\mathrm{mid}(\overline{\mathcal{A}}_{\mathrm{prin}})$, *and*
$$\mathrm{mid}(\overline{\mathcal{A}}_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}_{\mathrm{prin}}) \cap \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}}).$$
(3) *Now assume further that we have EGM on* $\mathcal{A}^\vee_{\mathrm{prin}}$. *If for some seed* $\mathbf{s}$, Ξ *is contained in the convex hull of* Θ (*which itself contains the convex hull of* $\Delta^+(\mathbb{Z})$), *then* $\Theta = \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$, $\mathrm{mid}(\overline{\mathcal{A}}_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}_{\mathrm{prin}})$ *is finitely generated, and the integer points* $\Xi \cap \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ *parameterize a canonical basis.*

Each choice of seed identifies $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ with a lattice and the cone $\Xi \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ with a rational polyhedral cone, described by canonical linear inequalities given by the tropicalization of the potential. Note that Ξ is convex in our generalized sense.

We show, making use of recent results of Magee [Ma15], [Ma17] and Goncharov and Shen [GS16] that in the representation-theoretic examples, which were the original motivation for the definition of cluster algebras, our polyhedral cones Ξ specialize to the piecewise linear parameterizations of canonical bases of Berenstein and Zelevinsky [BZ01], Knutson and Tao [KT99], and Goncharov and Shen [GS13]:

Corollary 0.20. *Let* $G = \mathrm{SL}_{r+1}$, *and let* $\mathcal{A} \subset \overline{\mathcal{A}}$ *be the Fomin–Zelevinsky cluster variety for the basic affine space* G/N.

(1) *All the hypotheses, and thus the conclusions, of Theorem* 0.19 *hold. In particular* $\Xi \cap \mathcal{A}^\vee(\mathbb{Z}^T) \subset \mathcal{A}^\vee(\mathbb{Z}^T)$ *parameterizes a canonical theta function basis of* $\mathcal{O}(G/N)$.
(2) *Our potential* W *agrees with the (representation theoretically defined) potential function of Berenstein and Kazhdan* [BK07].
(3) *The maximal torus* H *acts canonically on* $\overline{\mathcal{A}}$, *preserving the open set* $\mathcal{A} \subset \overline{\mathcal{A}}$.
(4) *Each theta function is an* H*-eigenfunction, and there is a canonical map*
$$w : \mathcal{A}^\vee(\mathbb{Z}^T) \to \chi^*(H)$$
(*the target is the character lattice of* H), *linear for the linear structure given by any seed, which sends an integer point to the* H*-weight of the corresponding theta function. The slice*
$$\Xi(\mathbb{Z}^T) \cap w^{-1}(\lambda)$$
parameterizes a canonical theta function basis of the eigenspace $\mathcal{O}(G/N)^\lambda = V_\lambda$, *the corresponding irreducible representation of* G.
(5) *For a natural choice of seed, the cone* Ξ *is canonically identified with the Gelfand–Tsetlin cone.*

Corollary 0.21. *Let* $\mathcal{A} \subset \overline{\mathcal{A}}$ *be the Fock–Goncharov cluster variety for*
$$\mathrm{Conf}_3(G/N) := ((G/N)^{\times 3})/G.$$

(1) *All the hypotheses, and thus the conclusions, of Theorem* 0.19 *hold. In particular the cone* $\Xi \cap \mathcal{A}^\vee(\mathbb{Z}^T) \subset \mathcal{A}^\vee(\mathbb{Z}^T)$ *parameterizes a canonical theta function basis of* $\mathcal{O}(\mathrm{Conf}_3(G/N))$.
(2) *Our potential function* W *agrees with the (representation theoretically defined) potential function of Goncharov and Shen* [GS13].

(3) $H^{\times 3}$ *acts canonically on* $\overline{\mathcal{A}}$*, preserving the open subset* $\mathcal{A} \subset \overline{\mathcal{A}}$.

(4) *Each theta function is an* $H^{\times 3}$*-eigenfunction, and there is a canonical map*

$$w : \mathcal{A}^\vee(\mathbb{Z}^T) \to \chi(H^{\times 3})$$

linear for the linear structure given by any seed, which sends an integer point to the $H^{\times 3}$*-weight of the corresponding theta function. The slice*

$$\Xi(\mathbb{Z}^T) \cap w^{-1}(\alpha, \beta, \gamma)$$

parameterizes a canonical theta function basis of the eigenspace

$$\mathcal{O}(G/N)^{(\alpha,\beta,\gamma)} = (V_\alpha \otimes V_\beta \otimes V_\gamma)^G.$$

In particular, the number of integral points in $\Xi(\mathbb{Z}^T) \cap w^{-1}(\alpha, \beta, \gamma)$ *is the corresponding Littlewood–Richardson coefficient.*

(5) *For a natural choice of seed, the cone* Ξ *is canonically identified with the Knutson–Tao hive cone.*

These corollaries are proven at the end of §9.2.

We stress here that the above representation-theoretic results come *for free* from general properties of our mirror symmetry construction: any partial minimal model $V \subset Y$ of an affine log Calabi–Yau variety with maximal boundary determines (in general conjecturally) a cone $\Xi \subset V^\vee(\mathbb{R}^T)$ with the analogous meaning. We are getting these basic representation-theoretic results without representation theory!

We recover the remarkable Gelfand–Tsetlin and hive polytopes for a particular choice of seed. Different (among the infinitely many possible) choices of seed give in general combinatorially different cones, whose integer points parameterize the same theta function basis. The canonical object is the *convex cone* $\Xi \subset \mathcal{A}^\vee(\mathbb{R}^T)$ cut out by W^T, different (by piecewise linear mutation) identifications of $\mathcal{A}^\vee(\mathbb{R}^T)$ with a vector space give different incarnations of Ξ as convex cones in the usual sense.

Potentials were considered in the work of Goncharov and Shen [GS13], which in turn built on work of Berenstein and Zelevinsky [BZ01] and Berenstein and Kazhdan [BK00], [BK07]. The potential constructed by Goncharov and Shen has a beautiful representation-theoretic definition and was found in many situations to coincide with known constructions of Landau–Ginzburg potentials. On the other hand, the construction of the potential in terms of theta functions coincides precisely with the construction of the mirror Landau–Ginzburg potential as carried out in [G09], [CPS]. The latter work can be viewed as a tropicalization of the descriptions of the potential in terms of holomorphic disks in [CO06], [A07]. Thus our construction explains the emergence of the Landau–Ginzburg potentials in [GS13]. Our potentials are determined by the cluster structure (and conjecturally, just the underlying log Calabi–Yau variety), and in particular are independent of any modular or representation-theoretic interpretation of the cluster variety. This gives, as in Remark 0.16, the remarkable suggestion that, e.g., the representation theoretically defined Goncharov–Shen potential, which would seem to depend heavily on the modular interpretation of $\overline{\mathcal{A}} = \mathrm{Conf}_3(G/N)$, is actually intrinsic to the partial minimal model $\mathcal{A} \subset \overline{\mathcal{A}}$.

1. Scattering diagrams and chamber structures

1.1. **Definition and constructions.** Here we recall the basic properties of scattering diagrams, the main technical tool in this paper. Scattering diagrams appeared first in [KS06] in two dimensions, and then in all dimensions in [GS11], with another approach in a more specific case in [KS13]. Here we give a self-contained treatment restricted to the specific case needed in this paper.

We start with a choice of *fixed data* Γ as defined in [GHK13], which for the reader's convenience is described at the beginning of Appendix A. In brief, this entails a lattice N with dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$, a skew-symmetric form

$$\{\cdot,\cdot\} : N \times N \to \mathbb{Q},$$

sublattices $N_{\mathrm{uf}}, N^\circ \subseteq N$ with N_{uf} a saturated sublattice and N° a sublattice of finite index with dual lattice $M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$, an index set $I = \{1, \ldots, n\}$ with $|I| = \operatorname{rank} N$ and a subset $I_{\mathrm{uf}} \subseteq I$ with $|I_{\mathrm{uf}}| = \operatorname{rank} N_{\mathrm{uf}}$, as well as positive integers d_i, $i \in I$. Finally, we also choose an initial seed $\mathbf{s}$, i.e., a basis $e_1, \ldots, e_n$ of N. See Appendix A for the precise properties that all this data must satisfy.

For the construction of the scattering diagram associated to this data, we will require

The injectivity assumption. The map $p_1^* : N_{\mathrm{uf}} \to M^\circ$ given by $n \mapsto \{n, \cdot\}$ is injective.

While this does not hold for a general choice of fixed data, it does hold in the principal coefficient case (see Appendix B) and results in this paper about arbitrary cluster varieties and algebras will be proved via the principal case.

Set

$$N^+ := N^+_{\mathbf{s}} := \left\{ \sum_{i \in I_{\mathrm{uf}}} a_i e_i \,\middle|\, a_i \geq 0, \sum a_i > 0 \right\}.$$

Choose a linear function $d : N \to \mathbb{Z}$ such $d(n) > 0$ for $n \in N^+$.

Under the injectivity assumption, one can choose a strictly convex top-dimensional cone $\sigma \subseteq M_{\mathbb{R}}$, with associated monoid $P := \sigma \cap M^\circ$, such that $p_1^*(e_i) \in J := P \setminus P^\times$ for all $i \in I_{\mathrm{uf}}$. Here $P^\times = \{0\}$ is the group of units of the monoid P. This gives the monomial ideal $J \subseteq \Bbbk[P]$ in the monoid ring $\Bbbk[P]$ over a field $\Bbbk$ of characteristic 0, and we write $\widehat{\Bbbk[P]}$ for the completion with respect to J.

We define the *module of log derivations* of $\Bbbk[P]$ as

$$\Theta(\Bbbk[P]) := \Bbbk[P] \otimes_{\mathbb{Z}} N^\circ,$$

with the action of $f \otimes n$ on $\Bbbk[P]$ being given by

$$(f \otimes n)(z^m) = f\langle n, m\rangle z^m,$$

so we write $f \otimes n$ as $f\partial_n$. Let $\widehat{\Theta(\Bbbk[P])}$ denote the completion of $\Theta(\Bbbk[P])$ with respect to the ideal J.

Using this action, if $\xi \in J\widehat{\Theta(\Bbbk[P])}$, then

$$\exp(\xi) \in \mathrm{Aut}(\widehat{\Bbbk[P]})$$

makes sense using the Taylor series for the exponential. We have the Lie bracket

$$[z^m \partial_n, z^{m'} \partial_{n'}] = z^{m+m'} \partial_{\langle n, m'\rangle n' - \langle n', m\rangle n}.$$

Then $\exp(J\Theta(\Bbbk[P]))$ can be viewed as a subgroup of the group of continuous automorphisms of $\widehat{\Bbbk[P]}$ which are the identity modulo J, with the group law of composition coinciding with the group law coming from the Baker–Campbell–Hausdorff formula.

Define the sub-Lie algebra of $\Theta(\Bbbk[P]) = \Bbbk[P] \otimes_{\mathbb{Z}} N^\circ$,

$$\mathfrak{g} := \bigoplus_{n \in N^+} \mathfrak{g}_n,$$

where $\mathfrak{g}_n$ is the one-dimensional subspace of $\Theta(\Bbbk[P])$ spanned by $z^{p_1^*(n)}\partial_n$. We calculate that $\mathfrak{g}$ is in fact closed under Lie bracket:

$$\begin{aligned}
[z^{p_1^*(n)}\partial_n, z^{p_1^*(n')}\partial_{n'}] &= z^{p_1^*(n+n')}\left(\langle p_1^*(n'), n\rangle \partial_{n'} - \langle p_1^*(n), n'\rangle\partial_n\right) \\
&= z^{p_1^*(n+n')}\left(\{n', n\}\partial_{n'} - \{n, n'\}\partial_n\right) \qquad (1.1)\\
&= \{n', n\} z^{p_1^*(n+n')}\partial_{n+n'}.
\end{aligned}$$

We have

$$\mathfrak{g}^{>k} := \bigoplus_{d(n)>k} \mathfrak{g}_n \subset \mathfrak{g}$$

a Lie subalgebra, and $\mathfrak{g}^{\leq k} := \mathfrak{g}/\mathfrak{g}^{>k}$ a nilpotent Lie algebra. We let $G^{\leq k} := \exp(\mathfrak{g}^{\leq k})$ be the corresponding nilpotent group. This group, as a set, is just $\mathfrak{g}^{\leq k}$, but multiplication is given by the Baker–Campbell–Hausdorff formula. We set

$$G := \exp(\mathfrak{g}) := \varprojlim G^{\leq k}$$

the corresponding pronilpotent group. We have the canonical set bijections

$$\exp : \mathfrak{g}^{\leq k} \to G^{\leq k} \quad \text{and} \quad \exp : \varprojlim \mathfrak{g}^{\leq k} \to G.$$

For $n_0 \in N^+$ we define

$$\begin{aligned}
\mathfrak{g}^{\parallel}_{n_0} &= \bigoplus_{k>0} \mathfrak{g}_{k\cdot n_0} \subset \mathfrak{g} \quad \text{(note this is a Lie subalgebra)}\\
G^{\parallel}_{n_0} &= \exp(\mathfrak{g}^{\parallel}_{n_0}) \subset G.
\end{aligned}$$

Note that by the commutator formula (1.1), $\mathfrak{g}^{\parallel}_{n_0}$, hence $G^{\parallel}_{n_0}$, is abelian.

In what follows, noting that G is a subgroup of $\mathrm{Aut}_{\Bbbk}(\widehat{\Bbbk[P]})$, we will often describe elements of $G^{\parallel}_{n_0}$ as follows.

Definition 1.2. Let $n_0 \in N^+$, $m_0 := p_1^*(n_0)$, and $f = 1 + \sum_{k=1}^\infty c_k z^{km_0} \in \widehat{\Bbbk[P]}$. Define $\mathfrak{p}_f$ to be the automorphism of $\widehat{\Bbbk[P]}$ given by

$$\mathfrak{p}_f(z^m) = f^{\langle n_0', m\rangle} z^m,$$

where n_0' is the generator of the monoid $\mathbb{R}_{\geq 0} n_0 \cap N^\circ$.

Lemma 1.3. *For $n_0 \in N^+$, $G^{\parallel}_{n_0} \subset \mathrm{Aut}(\widehat{\Bbbk[P]})$ is the subgroup of automorphisms of the form $\mathfrak{p}_f$ for f as in Definition 1.2 with the given $n_0 \in N^+$. More specifically, $\exp(\sum_{k>0} c_k z^{kp_1^*(n_0)}\partial_{kn_0}) \in G^{\parallel}_{n_0}$ acts as the automorphism $\mathfrak{p}_f$ with $f = \exp(\sum_{k>0} d^{-1}kc_k z^{kp_1^*(n_0)})$, where $d \in \mathbb{Q}$ is the smallest positive rational number such that $dn_0 \in N^\circ$.*

Proof. Let $H \subset \mathrm{Aut}(\widehat{\Bbbk[P]})$ be the set of $\mathfrak{p}_f$ of the given form. Then H is a subgroup as $\mathfrak{p}_{f_1} \circ \mathfrak{p}_{f_2} = \mathfrak{p}_{f_1 f_2}$. Note that $\sum_{k>0} c_k z^{kp_1^*(n_0)} \partial_{kn_0} = \left(\sum_{k>0} d^{-1} k c_k z^{kp_1^*(n_0)}\right) \partial_{dn_0}$, where $d \in \mathbb{Q}$ is as described in the statement. The exponential of this vector field is easily seen to act as $\mathfrak{p}_f$ with $f = \exp(\sum_{k>0} d^{-1} k c_k z^{kp_1^*(n_0)})$. Hence $G_{n_0}^{\parallel} \subset H$. From this, we see also that if $\log(f) = \sum_{k>0} c_k z^{kp_1^*(n_0)}$, then $\mathfrak{p}_f = \exp(\sum_{k>0} dk^{-1} c_k z^{kp_1^*(n_0)} \partial_{kn_0})$, and the latter lies in $G_{n_0}^{\parallel}$. $\square$

Definition 1.4. A *wall* in $M_{\mathbb{R}}$ (for N^+ and $\mathfrak{g}$) is a pair $(\mathfrak{d}, g_{\mathfrak{d}})$ such that

(1) $g_{\mathfrak{d}} \in G_{n_0}^{\parallel}$ for some primitive $n_0 \in N^+$.;
(2) $\mathfrak{d} \subset n_0^{\perp} \subset M_{\mathbb{R}}$ is a $(\operatorname{rank} N - 1)$-dimensional convex (but not necessarily strictly convex) rational polyhedral cone.

The set $\mathfrak{d} \subset M_{\mathbb{R}}$ is called the *support* of the wall $(\mathfrak{d}, g_{\mathfrak{d}})$.

Remark 1.5. Using Lemma 1.3, we often write a wall as $(\mathfrak{d}, f_{\mathfrak{d}})$ for $f_{\mathfrak{d}} \in \widehat{\Bbbk[P]}$, necessarily of the form $f_{\mathfrak{d}} = 1 + \sum_{k \geq 1} c_k z^{km_0}$. We shall use this notation interchangeably without comment.

Definition 1.6. A *scattering diagram $\mathfrak{D}$ for N^+ and* $\mathfrak{g}$ is a set of walls such that for every degree $k > 0$, there are only a finite number of $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$ with the image of $g_{\mathfrak{d}}$ in $G^{\leq k}$ not the identity.

If $\mathfrak{D}$ is a scattering diagram, we write

$$\mathrm{Supp}(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \mathfrak{d}, \qquad \mathrm{Sing}(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial\mathfrak{d} \quad \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \in \mathfrak{D} \\ \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = n-2}} \mathfrak{d}_1 \cap \mathfrak{d}_2$$

for the support and singular locus of the scattering diagram. If $\mathfrak{D}$ is a finite scattering diagram, then its support is a finite polyhedral cone complex. A *joint* is an $(n-2)$-dimensional cell of this complex, so that $\mathrm{Sing}(\mathfrak{D})$ is the union of all joints of $\mathfrak{D}$.

Remark 1.7. We will often (especially in Appendix C) want to use a slightly more general notion of scattering diagram, where the elements attached to walls lie in some other choice of group G' arising from an N^+-graded Lie algebra $\mathfrak{g}'$. In this case we talk about a scattering diagram for $\mathfrak{g}'$. For example, any scattering diagram for $\mathfrak{g}$ induces a finite scattering diagram for $\mathfrak{g}^{\leq k}$ by taking the image of the attached group elements under the projection $G \to G^{\leq k}$.

Given a scattering diagram $\mathfrak{D}$, we obtain the *path-ordered product.* Assume given a smooth immersion

$$\gamma : [0,1] \to M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$$

with endpoints not contained in the support of $\mathfrak{D}$. Assume γ is transversal to each wall of $\mathfrak{D}$ that it crosses. For each degree $k > 0$, we can find numbers

$$0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$$

and elements $\mathfrak{d}_i \in \mathfrak{D}$ with the image of $g_{\mathfrak{d}_i}$ in $G^{\leq k}$ nontrivial such that

$$\gamma(t_i) \in \mathfrak{d}_i,$$

$\mathfrak{d}_i \neq \mathfrak{d}_j$ if $t_i = t_j$, and s taken as large as possible. (The t_i are the times at which the path γ hits a wall. We allow $t_i = t_{i+1}$ because we may have two different walls $\mathfrak{d}_i, \mathfrak{d}_{i+1}$ which span the same hyperplane.)

For each i, define

$$\epsilon_i = \begin{cases} +1 & \langle n_0, \gamma'(t_i)\rangle < 0, \\ -1 & \langle n_0, \gamma'(t_i)\rangle > 0, \end{cases}$$

where $n_0 \in N^+$ with $\mathfrak{d} \subseteq n_0^\perp$. We then define

$$\mathfrak{p}^k_{\gamma,\mathfrak{D}} = g^{\epsilon_s}_{\mathfrak{d}_s} \cdots g^{\epsilon_1}_{\mathfrak{d}_1}.$$

If $t_i = t_{i+1}$, then $\mathfrak{d}_i, \mathfrak{d}_{i+1}$ span the same hyperplane $n_0^\perp$, hence $g_{\mathfrak{d}_i}, g_{\mathfrak{d}_{i+1}} \in G^{\parallel}_{n_0}$. Thus, since this latter group is abelian, $g_{\mathfrak{d}_i}$ and $g_{\mathfrak{d}_{i+1}}$ commute, so this product is well-defined. We then take

$$\mathfrak{p}_{\gamma,\mathfrak{D}} = \lim_{k\to\infty} \mathfrak{p}^k_{\gamma,\mathfrak{D}} \in G.$$

We note that $\mathfrak{p}_{\gamma,\mathfrak{D}}$ depends only on its homotopy class (with fixed endpoints) in $M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$. We also note that the definition can easily be extended to piecewise smooth paths γ, provided that the path always crosses a wall if it intersects it.

Definition 1.8. Two scattering diagrams $\mathfrak{D}$, $\mathfrak{D}'$ are *equivalent* if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ for all paths γ for which both are defined.

Call $x \in M_{\mathbb{R}}$ *general* if there is at most one rational hyperplane $n_0^\perp$ with $x \in n_0^\perp$. For x general and $\mathfrak{D}$ a scattering diagram, let $g_x(\mathfrak{D}) := \prod_{\mathfrak{d}\ni x} g_{\mathfrak{d}} \in G^{\parallel}_{n_0}$. One checks easily:

Lemma 1.9. *Two scattering diagrams $\mathfrak{D}, \mathfrak{D}'$ are equivalent if and only if $g_x(\mathfrak{D}) = g_x(\mathfrak{D}')$ for all general x.*

Definition 1.10. A scattering diagram $\mathfrak{D}$ is *consistent* if $\mathfrak{p}_{\gamma,\mathfrak{D}}$ only depends on the endpoints of γ for any path γ for which $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is defined.

Definition 1.11. We say a wall $\mathfrak{d} \subset n_0^\perp$ is *incoming* if

$$p_1^*(n_0) \in \mathfrak{d}.$$

Otherwise, we say the wall is *outgoing* (note in any case $p_1^*(n_0)$ lies in the span of the wall $n_0^\perp$).

We call $-p_1^*(n_0)$ the *direction* of the wall. (This terminology comes from the case $N = \mathbb{Z}^2$, where an outgoing wall is then a ray containing its direction vector, thus one that points outward.)

We need one particular scattering diagram, determined by the fixed data and seed data. Setting $v_i = p_1^*(e_i)$, $i \in I_{\mathrm{uf}}$, we start with the scattering diagram

$$\mathfrak{D}_{\mathrm{in},\mathbf{s}} := \{(e_i^\perp, 1 + z^{v_i}) \,|\, i \in I_{\mathrm{uf}}\}.$$

The main result on scattering diagrams, which follows easily from Theorem 1.21, is the following. A more general version of this was proved in two dimensions in [KS06] and in a much more general context in all dimensions in [GS11]. A simpler argument which applies to the case at hand was given in [KS13], which shall be reviewed in §1.2.

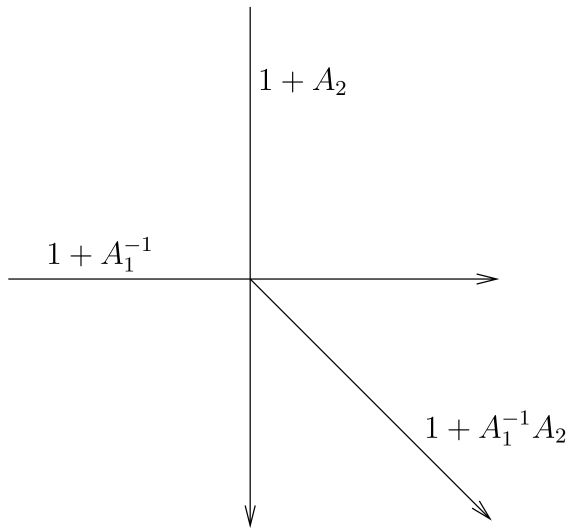

FIGURE 1.1. Scattering diagram for Example 1.14

Theorem 1.12. *There is a scattering diagram $\mathfrak{D}_{\mathbf{s}}$ satisfying:*

(1) $\mathfrak{D}_{\mathbf{s}}$ *is consistent,*
(2) $\mathfrak{D}_{\mathbf{s}} \supset \mathfrak{D}_{\mathrm{in},\mathbf{s}}$*,*
(3) $\mathfrak{D}_{\mathbf{s}} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ *consists only of outgoing walls.*

Moreover, $\mathfrak{D}_{\mathbf{s}}$ satisfying these three properties is unique up to equivalence.

The crucial positivity result satisfied by $\mathfrak{D}_{\mathbf{s}}$ is now easily stated:

Theorem 1.13. *The scattering diagram $\mathfrak{D}_{\mathbf{s}}$ is equivalent to a scattering diagram all of whose walls $(\mathfrak{d}, f_{\mathfrak{d}})$ satisfy $f_{\mathfrak{d}} = (1 + z^m)^c$ for some $m = p^*(n), n \in N^+$ and c a positive integer. In particular, all nonzero coefficients of $f_{\mathfrak{d}}$ are positive integers.*

The proof is given in Appendix C. The basic idea is that the construction of the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ can be reduced to repeated applications of the following example:

Example 1.14. Take $N = N^\circ = N_{\mathrm{uf}} = \mathbb{Z}^2$, $d_1, d_2 = 1$, and the skew-symmetric form $\{\cdot, \cdot\} : N \times N \to \mathbb{Q}$ given by the matrix $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, where $\epsilon_{ij} = \{e_i, e_j\}$. Let f_1, f_2 be the dual basis of e_1, e_2, and write $A_1 = z^{f_1}$, $A_2 = z^{f_2}$. We get

$$\mathfrak{D}_{\mathrm{in},\mathbf{s}} = \{(e_1^\perp, 1 + A_2), (e_2^\perp, 1 + A_1^{-1})\}.$$

Then one checks easily that

$$\mathfrak{D}_{\mathbf{s}} = \mathfrak{D}_{\mathrm{in},\mathbf{s}} \cup \{(\mathbb{R}_{\geq 0}(1, -1), 1 + A_1^{-1}A_2)\}.$$

See Figure 1.1. (See for example [GPS, Example 1.6].)

Example 1.15. Take $N = N_{\mathrm{uf}} = \mathbb{Z}^2$, with basis e_1, e_2, and take N° to be the sublattice generated by be_1, ce_2. Further, take $d_1 = b$, $d_2 = c$, where b, c are two positive integers, and take the skew-symmetric form to be the same as in the

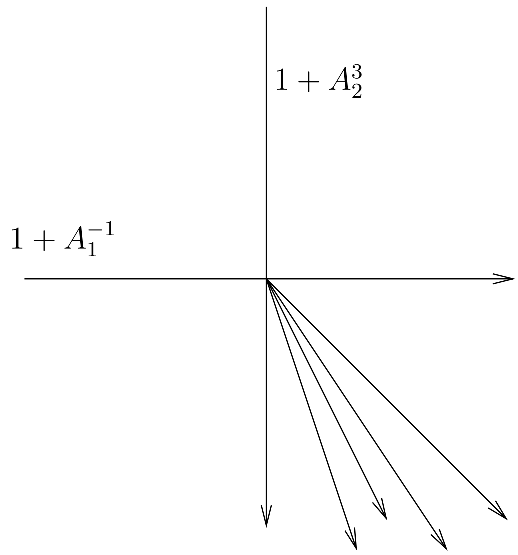

FIGURE 1.2. Scattering diagram for Example 1.15, $b = 1, c = 3$. The unlabeled rays intersecting the interior of the fourth quadrant have attached functions $1 + A_1^{-3}A_2^3$, $1 + A_1^{-2}A_2^3$, $1 + A_1^{-3}A_2^6$, and $1 + A_1^{-1}A_2^3$ in clockwise order.

previous example. Then $f_1 = e_1^*/b$, $f_2 = e_2^*/c$. Taking as before $A_1 = z^{f_1}$, $A_2 = z^{f_2}$, we get

$$\mathfrak{D}_{\mathrm{in},\mathbf{s}} = \{(e_1^\perp, 1 + A_2^c), (e_2^\perp, 1 + A_1^{-b})\}.$$

For most choices of b and c, this is a very complicated scattering diagram. A very similar scattering diagram, with functions $(1+A_2)^b$ and $(1+A_1)^c$, has been analyzed in [GP10], but it is easy to translate this latter diagram to the one considered here by replacing A_1 by A_1^{-1} and using the change of lattice trick, which is given in Step IV of the proof of Proposition C.13. All rays of $\mathfrak{D}_\mathbf{s} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ are contained strictly in the fourth quadrant (i.e., in particular are not contained in an axis). Without giving the details, we summarize the results. There are two linear operators S_1, S_2 given by the matrices in the basis f_1, f_2 as

$$S_1 = \begin{pmatrix} -1 & -b \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ -c & -1 \end{pmatrix}.$$

Then $\mathfrak{D}_\mathbf{s} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ is invariant under S_1 and S_2, in the sense that if $(\mathfrak{d}, f_\mathfrak{d}(z^m)) \in \mathfrak{D}_\mathbf{s} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$, we have $(S_i(\mathfrak{d}), f_\mathfrak{d}(z^{S_i(m)})) \in \mathfrak{D}_\mathbf{s} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ provided $S_i(\mathfrak{d})$ is contained strictly in the fourth quadrant. It is also the case that applying S_2 to $(\mathbb{R}_{\geq 0}(1,0), 1 + A_1^{-b})$ or S_1 to $(\mathbb{R}_{\geq 0}(0,-1), 1 + A_2^c)$ gives an element of $\mathfrak{D}_\mathbf{s} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$. Further, $\mathfrak{D}_\mathbf{s}$ contains a discrete series of rays consisting of those rays in the fourth quadrant obtained by applying S_1 and S_2 alternately to the above rays supported on $\mathbb{R}_{\geq 0}(1,0)$ and $\mathbb{R}_{\geq 0}(0,-1)$. These rays necessarily have functions of the form $1 + A_1^{-b\alpha}A_2^{-b\beta}$ or $1 + A_1^{c\alpha}A_2^{c\beta}$ for various choices of α and β. If $bc < 4$, we obtain a finite diagram. (Moreover, the corresponding $\mathcal{A}$ cluster variety is the cluster variety of finite type [FZ03a] associated to the root system A_2, B_2, or G_2 for $bc = 1$, 2, or 3, respectively.) See Figure 1.2 for the case $bc = 3$. If $bc \geq 4$, these rays converge to the rays

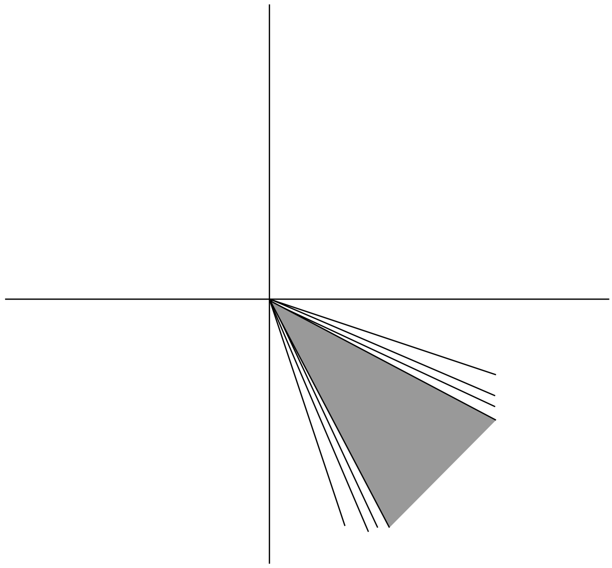

FIGURE 1.3. The general appearance of the scattering diagram of Example 1.15 for $bc > 4$

contained in the two eigenspaces of $S_1 \circ S_2$ and $S_2 \circ S_1$. These are rays of slope $-(bc \pm \sqrt{bc(bc-4)})/2b$. This gives a complete description of the rays outside of the cone spanned by these two rays. The expectation is that every ray of rational slope appears in the interior of this cone, and the attached functions are in general unknown (see Figure 1.3). However, in the $b = c$ case, it is known [R12] that the function attached to the ray of slope -1 is

$$\left(\sum_{k=0}^{\infty} \frac{1}{(b^2-2b)k+1} \binom{(b-1)^2 k}{k} A_1^{-bk} A_2^{bk}\right)^b.$$

The chamber structure one sees outside the quadratic irrational cone is very well-behaved and familiar in cluster algebra theory. In particular, the interiors of the first, second, and third quadrants are all connected components of $M^\circ_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$, and there are for $bc \geq 4$ an infinite number of connected components in the fourth quadrant. We will see in §2 that this chamber structure is precisely the Fock–Goncharov cluster complex.

On the other hand, it is precisely the rich structure inside the quadratic irrational cone which scattering diagram technology brings into the cluster algebra picture.

1.2. **Construction of consistent scattering diagrams.** In this subsection we give more details about the construction of scattering diagrams, and in particular give results leading to the proof of Theorem 1.12. This material can be skipped on first reading but is recommended before reading the more difficult material on scattering diagrams in Appendix C.

Let $\mathfrak{D}$ be a scattering diagram. If we set

$$\begin{aligned} \mathcal{C}^+ &:= \{m \in M_{\mathbb{R}} \,|\, m|_{N^+} \geq 0\}, \\ \mathcal{C}^- &:= \{m \in M_{\mathbb{R}} \,|\, m|_{N^+} \leq 0\}, \end{aligned} \tag{1.16}$$

then since any wall spans a hyperplane $n_0^\perp$, for some $n_0 \in N^+$,

$$\operatorname{Supp}(\mathfrak{D}) \cap \operatorname{Int}(\mathcal{C}^\pm) = \emptyset.$$

In particular, if $\mathfrak{D}$ is a consistent scattering diagram, then $\mathfrak{p}_{\gamma,\mathfrak{D}}$ for γ a path with initial point in $\mathcal{C}^+$ and final point in $\mathcal{C}^-$ is independent of the particular choice of path (or endpoints in $\mathcal{C}^\pm$). Thus we obtain a well-defined element $\mathfrak{p}_{+,-} \in G$ which only depends on the scattering diagram $\mathfrak{D}$.

Theorem 1.17 (Kontsevich and Soibelman)**.** *The assignment of $\mathfrak{p}_{+,-}$ to $\mathfrak{D}$ gives a one-to-one correspondence between equivalence classes of consistent scattering diagrams and elements $\mathfrak{p}_{+,-} \in G$.*

This is a special case of [KS13, 2.1.6]. For the reader's convenience we include the short proof:

Proof. We need to show how to construct $\mathfrak{D}$ given $\mathfrak{p}_{+,-} \in G$. To do so, choose any $n_0 \in N^+$ primitive and a point $x \in n_0^\perp$ general. Then we can determine $g_x(\mathfrak{D})$ as follows, noting by Lemma 1.9 that this information for all such n_0 and general x determines $\mathfrak{D}$ up to equivalence. We can write

$$\mathfrak{g} = \mathfrak{g}_+^x \oplus \mathfrak{g}_0^x \oplus \mathfrak{g}_-^x \tag{1.18}$$

with

$$\mathfrak{g}_+^x = \bigoplus_{\substack{n \in N^+ \\ \langle n,x\rangle > 0}} \mathfrak{g}_n, \quad \mathfrak{g}_-^x = \bigoplus_{\substack{n \in N^+ \\ \langle n,x\rangle < 0}} \mathfrak{g}_n, \quad \mathfrak{g}_0^x = \bigoplus_{\substack{n \in N^+ \\ \langle n,x\rangle = 0}} \mathfrak{g}_n.$$

Each of these subspaces of $\mathfrak{g}$ are closed under Lie bracket, thus defining subgroups $G_\pm^x, G_0^x$ of G. Note by the generality assumption on x, we in fact have $\mathfrak{g}_0^x = \mathfrak{g}_{n_0}^\parallel$. This splitting induces a unique factorization $g = g_+^x \cdot g_0^x \cdot g_-^x$ for any element $g \in G$. Applying this to $\mathfrak{p}_{+,-}$ gives a well-defined element $g_0^x \in G_0^x$. We need to show that the set of data g_0^x determines a scattering diagram $\mathfrak{D}$ such that $g_x(\mathfrak{D}) = g_0^x$ for all general $x \in M_{\mathbb{R}}$. To do this, one needs to know that to any finite order k, the hyperplane $n_0^\perp$ is subdivided into a finite number of polyhedral cones $\mathfrak{d}_1, \ldots, \mathfrak{d}_p$ such that the image of g_0^x in $G^{\leq k}$ is constant for $x \in \mathfrak{d}_i$. This is clear because the number of $n \in N^+$ with $d(n) \leq k$ is finite, as then the decomposition (1.18) varies discretely with x to order k.

We need to show that $\mathfrak{D}$ satisfies the condition that $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{+,-}$ for any path γ from the positive to the negative chamber and that $\mathfrak{p}_{\gamma,\mathfrak{D}}$ only depends on endpoints of γ. To do so, we work modulo $\mathfrak{g}^{>k}$ for any k, so we can assume $\mathfrak{D}$ has a finite number of walls. Choose a general point $x_0 \in \mathcal{C}^+$. Take a general two-dimensional subspace of $M_{\mathbb{R}}$ containing x_0, and after choosing a metric, let γ be a semicircle in the two-dimensional subspace with endpoints x_0 and $-x_0$ and center 0. Then $\mathfrak{p}_{\gamma,\mathfrak{D}} = g_0^{x_n} \cdots g_0^{x_1}$ for points $x_1, \ldots, x_n$ contained in walls crossed by γ, and $g_0^{x_i}$ is the element of $G_0^{x_i}$ determined by the factorization of $\mathfrak{p}_{+,-}$ above. Note that if x_i lies in the hyperplane $n_i^\perp$, all the wall-crossing automorphisms of walls traversed by γ *before* crossing $n_i^\perp$ lie in $G_-^{x_i}$, and all those from walls traversed by γ *after* crossing $n_i^\perp$ lie in $G_+^{x_i}$. It then follows inductively that the factorization of $\mathfrak{p}_{+,-}$ given by x_i takes the form $(g_+)g_0^{x_i}(g_0^{x_{i-1}} \cdots g_0^{x_1})$ for some $g_+ \in G_+^{x_i}$. Indeed, for

$i=1$, this just follows from the definition of $g_0^{x_1}$, while if true for $i-1$, then we have that $\mathfrak{p}_{+,-}=g'\cdot(g_0^{x_{i-1}}\cdots g_0^{x_1})$ is a decomposition of $\mathfrak{p}_{+,-}$ induced by the splitting $\mathfrak{g}=(\mathfrak{g}_+^{x_i}\oplus\mathfrak{g}_0^{x_i})\oplus\mathfrak{g}_-^{x_i}$, and the claim then follows by the definition of $g_0^{x_i}$. In particular, for $i=n+1$, taking $x_{n+1}=-x_0$ and noting that $G_-^{x_{n+1}}=G$, one sees that $\mathfrak{p}_{+,-}=g_0^{x_n}\cdots g_0^{x_1}=\mathfrak{p}_{\gamma,\mathfrak{D}}$.

Next we show the independence of path for the $\mathfrak{D}$ we have constructed, again modulo $\mathfrak{g}^{>k}$. It is sufficient to check $\mathfrak{p}_{\gamma_{\mathfrak{j}},\mathfrak{D}}=\mathrm{id}$ as an element of $G^{\leq k}$ for any small loop $\gamma_{\mathfrak{j}}$ around any joint $\mathfrak{j}$ of $\mathfrak{D}$. Take x' a general point in $\mathfrak{j}$, $n\in N^+$ such that $n^\perp\supseteq\mathfrak{j}$, and choose x,x'' to be points in $n^\perp$ near x' on either side of the joint $\mathfrak{j}$. Let γ,γ'' be two semicircular paths with endpoints x_0 and $-x_0$ and passing through x,x'', respectively. Then up to orientation $\gamma(\gamma'')^{-1}$ is freely homotopic to $\gamma_{\mathfrak{j}}$ in $M_{\mathbb{R}}\setminus\mathrm{Sing}(\mathfrak{D})$. Thus $\mathfrak{p}_{\gamma_{\mathfrak{j}},\mathfrak{D}}=\mathfrak{p}_{\gamma'',\mathfrak{D}}^{-1}\mathfrak{p}_{\gamma,\mathfrak{D}}=\mathfrak{p}_{+,-}^{-1}\mathfrak{p}_{+,-}=\mathrm{id}$.

Thus we have established the one-to-one correspondence between consistent scattering diagrams $\mathfrak{D}$ and elements of G. $\square$

Following [KS13], we give an alternative parameterization of G, as follows. For any $n_0\in N^+$ primitive, we get the splitting

$$\mathfrak{g}=\mathfrak{g}_+^{n_0}\oplus\mathfrak{g}_0^{n_0}\oplus\mathfrak{g}_-^{n_0}, \tag{1.19}$$

where

$$\mathfrak{g}_+^{n_0}:=\bigoplus_{\{n_0,n\}>0}\mathfrak{g}_n,\quad \mathfrak{g}_-^{n_0}:=\bigoplus_{\{n_0,n\}<0}\mathfrak{g}_n,\quad \mathfrak{g}_0^{n_0}:=\bigoplus_{\{n_0,n\}=0}\mathfrak{g}_n.$$

These give rise to subgroups $G_\pm^{n_0},G_0^{n_0}$ of G. We drop the n_0 when it is clear from context. Again, this allows us to factor any $g\in G$ as $g=g_+\circ g_0\circ g_-$ with $g_\pm\in G_\pm$, $g_0\in G_0$. We can further decompose $\mathfrak{g}_0=\mathfrak{g}_0^{\|}\oplus\mathfrak{g}_0^\perp$, where $\mathfrak{g}_0^{\|}:=\mathfrak{g}_{n_0}^{\|}$, while $\mathfrak{g}_0^\perp$ involves those summands of $\mathfrak{g}_0$ coming from n not proportional to n_0. Note that $[\mathfrak{g}_0,\mathfrak{g}_0^\perp]\subseteq\mathfrak{g}_0^\perp$. Indeed, if $n_1+n_2=kn_0$ with $\{n_i,n_0\}=0$ for $i=1,2$, we then have $\{n_1,n_2\}=0$ so that $[\mathfrak{g}_{n_1},\mathfrak{g}_{n_2}]=0$ by (1.1). Thus we have a projection homomorphism $G_0\to G_0^{\|}$ with kernel $G_0^\perp$. In particular, the factorization $g=g_+\circ g_0\circ g_-$ yields an element $g_0^{\|}\in G_0^{\|}$ via this projection. We then have a map (of sets)

$$\Psi:G\to\prod_{n_0\,\in\,N^+\text{ primitive}}G_{n_0}^{\|}.$$

Proposition 1.20. *Ψ is a set bijection.*

Proof. Ψ is induced by an analogous map to order k,

$$\Psi_k:G^{\leq k}\to\prod\exp(\mathfrak{g}_{n_0}^{\|}/\mathfrak{g}_{n_0}^{\|}\cap\mathfrak{g}^{>k}).$$

One checks easily that this is a bijection order by order. $\square$

Theorem 1.21. *Let $\mathfrak{D}$ be a consistent scattering diagram corresponding to $\mathfrak{p}_{+,-}\in G$. The following hold:*

(1) *For each $n_0\in N^+$, to any fixed finite order, there is an open neighborhood $U\subset n_0^\perp$ of $p^*(n_0)$ such that $g_x(\mathfrak{D})=\Psi(\mathfrak{p}_{+,-})_{n_0}\in G_0^x=G_{n_0}^{\|}$ for all general $x\in U$. Here $\Psi(g)_{n_0}$ denotes the component of $\Psi(g)$ indexed by n_0.*

(2) *$\mathfrak{D}$ is equivalent to a diagram with only one wall in $n_0^\perp$ containing $p^*(n_0)$ for each $n_0\in N^+$, and the group element attached to this wall is $\Psi(\mathfrak{p}_{+,-})_{n_0}$.*

(3) *Set*

$$\mathfrak{D}_{\mathrm{in}} := \{(n_0^\perp, \Psi(\mathfrak{p}_{+,-})_{n_0}) \,|\, n_0 \in N^+ \text{ primitive}\}.$$

Then $\mathfrak{D}$ is equivalent to a consistent scattering diagram $\mathfrak{D}'$ such that $\mathfrak{D}' \supseteq \mathfrak{D}_{\mathrm{in}}$ and $\mathfrak{D}' \setminus \mathfrak{D}_{\mathrm{in}}$ consists only of outgoing walls. Furthermore, up to equivalence, $\mathfrak{D}'$ is the unique consistent scattering diagram with this property.

(4) *The equivalence class of a consistent scattering diagram is determined by its set of incoming walls.*

We note first that (3) of Theorem 1.21 implies Theorem 1.12. Indeed, let $g_i \in G$ for $i \in I_{\mathrm{uf}}$ be the group element corresponding to $1+z^{v_i}$, so that the initial scattering can be written as $\mathfrak{D}_{\mathrm{in},\mathbf{s}} = \{(e_i^\perp, g_i) \,|\, i \in I_{\mathrm{uf}}\}$. By Proposition 1.20 there is a unique element $g \in G$ with

$$\Psi(g)_n = \begin{cases} g_i & n = e_i, i \in I_{\mathrm{uf}}, \\ 1 & \text{otherwise.} \end{cases}$$

Now apply Theorem 1.21 with $\mathfrak{p}_{+,-} = g$.

Proof of Theorem 1.21. First note that statement (1) implies (2). Further, (1), along with Theorem 1.17 and Proposition 1.20, implies (4), which in turn gives the uniqueness in (3). Note (1) implies that, to the given finite order, $\mathfrak{D}$ is equivalent to a diagram having only one incoming wall contained in $n_0^\perp$, and the attached group element is $\Psi(\mathfrak{p}_{+,-})_{n_0}$. Now we can replace this single wall by an equivalent collection of walls consisting of $(n_0^\perp, \Psi(\mathfrak{p}_{+,-})_{n_0})$ and a number of outgoing walls contained in $n_0^\perp$ with attached group element $\Psi(\mathfrak{p}_{+,-})_{n_0}^{-1}$. This gives the existence in (3).

Thus it suffices to prove (1). We work modulo $\mathfrak{g}^{>k}$, so we may assume $\mathfrak{D}$ is finite, and compare the splittings (1.18) coming from a choice of $x \in n_0^\perp$ near $p^*(n_0)$ and (1.19), after replacing $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{g}^{>k}$. For each $n \in N^+$, there exists an open neighborhood $U_n \subset n_0^\perp$ of $p^*(n_0)$ such that $\langle p^*(n_0), n\rangle > 0$ (resp. < 0) implies $\langle x, n\rangle > 0$ (resp. < 0) for all $x \in U_n$. Since $\mathfrak{g}_\pm^{n_0}$ is now a finite sum of $\mathfrak{g}_n$'s, we can find a single U so that $\mathfrak{g}_\pm^{n_0} \subseteq \mathfrak{g}_\pm^x$ for all $x \in U$. If x is general inside $n_0^\perp$, we also have $\mathfrak{g}_0^x = \mathfrak{g}_{n_0}^{\parallel}$.

Now write

$$\mathfrak{p}_{+,-} = g_+^{n_0} \cdot g_0^{n_0} \cdot g_-^{n_0}$$

as in (1.19). Then we can further factor

$$g_0^{n_0} = h_+^x \cdot h_0^x \cdot h_-^x$$

as in (1.18). Note $h_\pm^x \in G_0^\perp$, $h_0^x \in G_{n_0}^{\parallel} = G_0^x$. Since the projection $G_0 \to G_0^{\parallel}$ is a group homomorphism with kernel $G_0^\perp$, the image of $g_0^{n_0}$ in $G_{n_0}^{\parallel}$ is h_0^x, which thus coincides with $\Psi(\mathfrak{p}_{+,-})_{n_0}$ by definition of the latter. We have

$$\mathfrak{p}_{+,-} = (g_+^{n_0} \cdot h_+^x) \cdot h_0^x \cdot (h_-^x \cdot g_-^{n_0})$$

which is then the (unique) factorisation from (1.18). Thus

$$g_x(\mathfrak{p}_{+,-}) = h_0^x = \Psi(\mathfrak{p}_{+,-})_{n_0}$$

for any general $x \in U$. □

1.3. **Mutation invariance of the scattering diagram.** We now study how the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ constructed from seed data defined in the previous subsection changes under mutation. This is crucial for uncovering the chamber structure of these diagrams and giving the connection with the exchange graph and cluster complex.

Thus let $k \in I_{\mathrm{uf}}$ and $\mathbf{s}' = \mu_k(\mathbf{s})$ be the mutated seed (see, e.g., [GHK13, (2.3)]). To distinguish the two Lie algebras involved, we write $\mathfrak{g}_{\mathbf{s}}$ and $\mathfrak{g}_{\mathbf{s}'}$ for the Lie algebras arising from these two different seeds. We recall that the injectivity assumption is independent of the choice of seed.

Definition 1.22. We set

$$\mathcal{H}_{k,+} := \{m \in M_{\mathbb{R}} \mid \langle e_k, m\rangle \geq 0\}, \quad \mathcal{H}_{k,-} := \{m \in M_{\mathbb{R}} \mid \langle e_k, m\rangle \leq 0\}.$$

For $k \in I_{\mathrm{uf}}$, define the piecewise linear transformation $T_k : M^\circ \to M^\circ$ by, for $m \in M^\circ$,

$$T_k(m) := \begin{cases} m + v_k\langle d_k e_k, m\rangle & m \in \mathcal{H}_{k,+}, \\ m & m \in \mathcal{H}_{k,-}. \end{cases} \tag{1.23}$$

As we will explain in §2, T_k is the tropicalization of μ_k viewed as a birational map between tori. We will write $T_{k,-}$ and $T_{k,+}$ to be the linear transformations used to define T_k in the regions $\mathcal{H}_{k,-}$ and $\mathcal{H}_{k,+}$, respectively.

Define the scattering diagram $T_k(\mathfrak{D}_{\mathbf{s}})$ to be the scattering diagram obtained by the following:

(1) For each wall $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_{\mathbf{s}} \setminus \{\mathfrak{d}_k\}$, where $\mathfrak{d}_k := (e_k^\perp, 1 + z^{v_k})$, we have one or two walls in $T_k(\mathfrak{D}_{\mathbf{s}})$ given as

$$\big(T_k(\mathfrak{d} \cap \mathcal{H}_{k,-}), T_{k,-}(f_{\mathfrak{d}})\big), \quad \big(T_k(\mathfrak{d} \cap \mathcal{H}_{k,+}), T_{k,+}(f_{\mathfrak{d}})\big),$$

throwing out the first or second of these if $\dim \mathfrak{d} \cap \mathcal{H}_{k,-} < \operatorname{rank} M - 1$ or $\dim \mathfrak{d} \cap \mathcal{H}_{k,+} < \operatorname{rank} M - 1$, respectively. Here for $T : M^\circ \to M^\circ$ linear, we write $T(f_{\mathfrak{d}})$ for the formal power series obtained by applying T to each exponent in $f_{\mathfrak{d}}$.

(2) $T_k(\mathfrak{D}_{\mathbf{s}})$ also contains the wall $\mathfrak{d}_k' := (e_k^\perp, 1 + z^{-v_k})$.

The main result of this subsection is:

Theorem 1.24. *Suppose the injectivity assumption is satisfied. Then $T_k(\mathfrak{D}_{\mathbf{s}})$ is a consistent scattering diagram for $\mathfrak{g}_{\mu_k(\mathbf{s})}$ and $N^+_{\mu_k(\mathbf{s})}$. Furthermore, $\mathfrak{D}_{\mu_k(\mathbf{s})}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ are equivalent.*

The main point in the proof, which is not at all obvious from the definition, is that $T_k(\mathfrak{D}_{\mathbf{s}})$ is a scattering diagram for $\mathfrak{g}_{\mathbf{s}'}$, $N^+_{\mathbf{s}'}$, where $\mathbf{s}' = \mu_k(\mathbf{s})$. Formally, consistency will be easy to check using consistency of $\mathfrak{D}_{\mathbf{s}}$. It will follow easily that by construction $\mathfrak{D}_{\mathbf{s}'}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ have the same incoming walls, so the theorem will then follow from the uniqueness in Theorem 1.12.

The main problem to overcome is that the functions attached to walls of $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$ live in two different completed monoid rings, $\widehat{\Bbbk[P]}$ and $\widehat{\Bbbk[P']}$, for P a monoid chosen to contain $v_i, i \in I_{\mathrm{uf}}$, and P' a monoid chosen to contain $v_i', i \in I_{\mathrm{uf}}$. We need first a common monoid $\bar{P}$ containing both P and P'.

Definition 1.25. Let $\sigma \subseteq M^\circ_{\mathbb{R}}$ be a top-dimensional cone containing v_i, $i \in I_{\mathrm{uf}}$, and $-v_k$, and such that $\sigma \cap (-\sigma) = \mathbb{R}v_k$. Set $\bar{P} = \sigma \cap M^\circ$, and $J = \bar{P} \setminus (\bar{P} \cap \mathbb{R}v_k) = \bar{P} \setminus \bar{P}^\times$.

Given such a choice of $\bar{P}$, we can find P, P' contained in $\bar{P}$. However, we have an additional problem that $\mathfrak{D}_{\mathbf{s}}$ is not trivial modulo J. Indeed, $v_k \notin J$, while one of the initial walls of $\mathfrak{D}_{\mathrm{in},\mathbf{s}}$ is $(e_k^\perp, 1+z^{v_k})$. In particular, the wall-crossing automorphism associated to

$$\mathfrak{d}_k := (e_k^\perp, 1+z^{v_k})$$

is not an automorphism of the ring $\widehat{\Bbbk[\bar{P}]}$, but rather of the localized ring $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$. (Here the hats denote completion with respect to J.) This kind of situation is dealt with in [GS11], see especially §4.3. However the current situation is quite a bit simpler, so we will give the complete necessary arguments here and in Appendix C.

We will use the notation $\mathfrak{p}_{\mathfrak{d}_k}$ for the automorphism of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$ associated to crossing the wall $\mathfrak{d}_k$ from $\mathcal{H}_{k,-}$ to $\mathcal{H}_{k,+}$. Explicitly,

$$\mathfrak{p}_{\mathfrak{d}_k}(z^m) = z^m(1+z^{v_k})^{-\langle d_k e_k, m\rangle}. \tag{1.26}$$

In this situation, define

$$N_{\mathbf{s}}^{+,k} := \left\{ \sum_{i\in I_{\mathrm{uf}}} a_i e_i \,\middle|\, a_i \in \mathbb{Z}_{\geq 0} \text{ for } i \neq k,\ a_k \in \mathbb{Z}, \text{ and } \textstyle\sum_{i\in I_{\mathrm{uf}}\setminus\{k\}} a_i > 0 \right\}.$$

We note that by the definition of the mutated seed $\mathbf{s}'$, $N_{\mathbf{s}}^{+,k} = N_{\mathbf{s}'}^{+,k}$, so we indicate it by $N^{+,k}$.

We now extend the definition of scattering diagram.

Definition 1.27. A *wall* for $\bar{P}$ and ideal J is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ with $\mathfrak{d}$ as in Definition 1.4, but with $n_0 \in N^{+,k}$, and $f_{\mathfrak{d}} = 1 + \sum_{k=1}^\infty c_k z^{kp^*(n_0)} \in \widehat{\Bbbk[\bar{P}]}$ congruent to 1 mod J. The *slab* for the seed $\mathbf{s}$ means the pair $\mathfrak{d}_k = (e_k^\perp, 1+z^{v_k})$. Note since $v_k \in \bar{P}^\times$ this does not qualify as a wall. Now a scattering diagram $\mathfrak{D}$ is a collection of walls and possibly this single slab, with the condition that for each $k>0$, $f_{\mathfrak{d}} \equiv 1 \bmod J^k$ for all but finitely many walls in $\mathfrak{D}$.

Note that crossing a wall or slab $(\mathfrak{d}, f_{\mathfrak{d}})$ now induces an automorphism of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$ of the form $\mathfrak{p}_{f_{\mathfrak{d}}}^{\pm 1}$ (with the localization only needed when a slab is crossed).

The following is proved in Appendix C:

Theorem 1.28. *There exists a scattering diagram $\overline{\mathfrak{D}}_{\mathbf{s}}$ in the sense of Definition* 1.27 *such that*

(1) $\overline{\mathfrak{D}}_{\mathbf{s}} \supseteq \mathfrak{D}_{\mathrm{in},\mathbf{s}}$,
(2) $\overline{\mathfrak{D}}_{\mathbf{s}} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ *consisting only of outgoing walls, and*
(3) $\mathfrak{p}_{\gamma,\mathfrak{D}}$ *as an automorphism of* $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$ *only depends on the endpoints of* γ.

Furthermore, $\overline{\mathfrak{D}}_{\mathbf{s}}$ *with these properties is unique up to equivalence.*

Finally, $\overline{\mathfrak{D}}_{\mathbf{s}}$ *is also a scattering diagram for the data* $\mathfrak{g}_{\mathbf{s}}, N_{\mathbf{s}}^+$ *and, as such, is equivalent to* $\mathfrak{D}_{\mathbf{s}}$.

Remark 1.29. Note in particular that the theorem implies $\mathfrak{D}_{\mathbf{s}} \setminus \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ does not contain any walls contained in $e_k^\perp$ besides $\mathfrak{d}_k$. Indeed, no wall of $\overline{\mathfrak{D}}_{\mathbf{s}}$ is contained in $e_k^\perp$: only the slab $\mathfrak{d}_k$ is contained in $e_k^\perp$.

Proof of Theorem 1.24. We write $\mathbf{s}' = \mu_k(\mathbf{s})$, $\mathbf{s}' = (e_i' \mid i \in I)$.

We first note that we can choose representatives for $\mathfrak{D}_{\mathbf{s}}$, $\mathfrak{D}_{\mathbf{s}'}$, which are scattering diagrams in the sense of Definition 1.27, by Theorem 1.28. Furthermore, $T_k(\mathfrak{D}_{\mathbf{s}})$ is also a scattering diagram in the sense of Definition 1.27 for the seed $\mathbf{s}'$: this follows since if $z^m \in J^i$ for some i, we also have $z^{T_{k,\pm}(m)} \in J^i$. Thus by the uniqueness statement of Theorem 1.28, $T_k(\mathfrak{D}_{\mathbf{s}})$ and $\mathfrak{D}_{\mathbf{s}'}$ are equivalent if (1) these diagrams are equivalent to diagrams which have the same set of slabs and incoming walls; (2) $T_k(\mathfrak{D}_{\mathbf{s}})$ is consistent. We carry out these two steps.

Step I. Up to equivalence, $T_k(\mathfrak{D}_{\mathbf{s}})$ and $\mathfrak{D}_{\mathbf{s}'}$ have the same set of slabs and incoming walls.

If $\mathfrak{d} \in \mathfrak{D}_{\mathbf{s}}$ is outgoing, the wall $\mathfrak{d}$ contributes to $T_k(\mathfrak{D}_{\mathbf{s}})$ and is also outgoing, so let us consider the incoming walls of $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$. Setting $v_i' = p^*(e_i')$, already $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$ contains the slab for $\mathbf{s}'$

$$((e_k')^\perp, 1 + z^{v_k'}) = (e_k^\perp, 1 + z^{-v_k}) = \mathfrak{d}_k',$$

which lies in $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$ by construction. Next consider the wall $(e_i^\perp, 1 + z^{v_i})$, for $i \neq k$. We have three cases to consider, based on whether $\langle v_i, e_k\rangle$ is zero, positive, or negative.

First if $\langle v_i, e_k\rangle = 0$, then T_k takes the plane $e_i^\perp$ to itself (in a piecewise linear way), and $T_{k,+}(v_i) = T_{k,-}(v_i) = v_i$. Thus the wall $(e_i^\perp, 1 + z^{v_i})$ contributes two walls $(e_i^\perp \cap \mathcal{H}_{k,\pm}, 1 + z^{v_i})$ whose union is the wall $((e_i')^\perp, 1 + z^{v_i})$, as $e_i' = e_i$ and $v_i' = v_i$ in this case. Up to equivalence, we can replace these two walls with the single wall $((e_i')^\perp, 1 + z^{v_i})$.

If $\langle v_i, e_k\rangle > 0$, then consider the wall

$$\mathfrak{d}_{i,+} := \big(T_k(\mathcal{H}_{k,+} \cap e_i^\perp), 1 + z^{T_{k,+}(v_i)}\big) \in T_k(\mathfrak{D}_{\mathbf{s}}).$$

This wall contains the ray $\mathbb{R}_{\geq 0}T_{k,+}(v_i)$, so this is an incoming wall. Note that if $m \in \mathcal{H}_{k,+} \cap e_i^\perp$, we have, with ϵ as given in (A.1),

$$\begin{aligned}\langle e_i', T_k(m)\rangle &= \langle e_i + [\epsilon_{ik}]_+ e_k, m + v_k\langle d_k e_k, m\rangle\rangle \\ &= \{e_k, e_i\}\langle d_k e_k, m\rangle + d_k\{e_i, e_k\}\langle e_k, m\rangle \\ &= 0.\end{aligned}$$

Thus $T_k(\mathcal{H}_{k,+} \cap e_i^\perp)$ is a half-space contained in $(e_i')^\perp$, and furthermore $1 + z^{T_{k,+}(v_i)} = 1 + z^{v_i'}$ since

$$T_k(v_i) = v_i + v_k\epsilon_{ik} = v_i'.$$

Thus we see that the wall $\mathfrak{d}_{i,+}$ of $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$ is half of the wall $((e_i')^\perp, 1 + z^{v_i'})$ of $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$.

If $\langle v_i, e_k\rangle < 0$, then the wall $\mathfrak{d}_{i,-} := \big(T_k(\mathcal{H}_{k,-} \cap e_i^\perp), 1 + z^{T_{k,-}(v_i)}\big) \in T_k(\mathfrak{D}_{\mathbf{s}})$ coincides with $(\mathcal{H}_{k,-} \cap e_i^\perp, 1 + z^{v_i})$, and $\mathcal{H}_{k,-}$ also contains $\mathbb{R}_{\geq 0}v_i$, so $\mathfrak{d}_{i,-}$ is an incoming wall. But also $v_i' = v_i$, $e_i' = e_i$ in this case. Thus $\mathfrak{d}_{i,-}$ is again half of the wall $((e_i')^\perp, 1 + z^{v_i'})$.

In summary, we find that after splitting some of the walls of $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$ in two, $T_k(\mathfrak{D}_{\mathrm{in},\mathbf{s}})$ and $\mathfrak{D}_{\mathrm{in},\mathbf{s}'}$ have the same set of incoming walls, and thus, making a similar change to $\mathfrak{D}_{\mathbf{s}'}$, we see that $T_k(\mathfrak{D}_{\mathbf{s}})$ and $\mathfrak{D}_{\mathbf{s}'}$ have the same set of incoming walls.

Step II. $\mathfrak{p}_{\gamma,T_k(\mathfrak{D}_{\mathbf{s}})} = \mathrm{id}$ for any loop γ for which this automorphism is defined.

Indeed, the only place a problem can occur is for γ a loop around a joint of $\mathfrak{D}_{\mathbf{s}}$ contained in the slab $\mathfrak{d}_k$, as this is where T_k fails to be linear. To test this, consider a loop γ around a joint contained in $\mathfrak{d}_k$. Assume that it has basepoint in the half-space $\mathcal{H}_{k,-}$ and is split up as $\gamma = \gamma_1\gamma_2\gamma_3\gamma_4$, where γ_1 immediately crosses $\mathfrak{d}_k$, γ_2 is contained entirely in $\mathcal{H}_{k,+}$, crossing all walls of $\mathfrak{D}_{\mathbf{s}}$ which contain $\mathfrak{j}$ and intersect the interior of $\mathcal{H}_{k,+}$, γ_3 crosses $\mathfrak{d}_k$ again, and γ_4 then crosses all relevant walls in the half-space $\mathcal{H}_{k,-}$.

Let $\mathfrak{p}_{\mathfrak{d}_k}$, $\mathfrak{p}_{\mathfrak{d}'_k}$ be the wall-crossing automorphisms for crossing $\mathfrak{d}_k$ or $\mathfrak{d}'_k$ passing from $\mathcal{H}_{k,-}$ to $\mathcal{H}_{k,+}$, as in (1.26). Then by Remark 1.29, $\mathfrak{p}_{\gamma_1,\mathfrak{D}_{\mathbf{s}}} = \mathfrak{p}_{\mathfrak{d}_k}$ and $\mathfrak{p}_{\gamma_3,\mathfrak{D}_{\mathbf{s}}} = \mathfrak{p}_{\mathfrak{d}_k}^{-1}$.

Let $\alpha : \Bbbk[M^\circ] \to \Bbbk[M^\circ]$ be the automorphism induced by $T_{k,+}$, i.e.,

$$\alpha(z^m) = z^{m+v_k\langle d_k e_k, m\rangle}.$$

Then note that

$$\begin{aligned}
\mathfrak{p}_{\gamma_1,T_k(\mathfrak{D}_{\mathbf{s}})} &= \mathfrak{p}_{\mathfrak{d}'_k},\\
\mathfrak{p}_{\gamma_2,T_k(\mathfrak{D}_{\mathbf{s}})} &= \alpha\circ\mathfrak{p}_{\gamma_2,\mathfrak{D}_{\mathbf{s}}}\circ\alpha^{-1},\\
\mathfrak{p}_{\gamma_3,T_k(\mathfrak{D}_{\mathbf{s}})} &= \mathfrak{p}_{\mathfrak{d}'_k}^{-1},\\
\mathfrak{p}_{\gamma_4,T_k(\mathfrak{D}_{\mathbf{s}})} &= \mathfrak{p}_{\gamma_4,\mathfrak{D}_{\mathbf{s}}}.
\end{aligned}$$

Thus to show $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}} = \mathfrak{p}_{\gamma,T_k(\mathfrak{D}_{\mathbf{s}})}$, it is enough to show that

$$\alpha^{-1}\circ\mathfrak{p}_{\mathfrak{d}'_k} = \mathfrak{p}_{\mathfrak{d}_k}.$$

But

$$\begin{aligned}
\alpha^{-1}(\mathfrak{p}_{\mathfrak{d}'_k}(z^m)) &= \alpha^{-1}((1+z^{-v_k})^{-\langle d_k e_k, m\rangle} z^m)\\
&= (1+z^{-v_k})^{-\langle d_k e_k, m\rangle} z^{m-v_k\langle d_k e_k, m\rangle}\\
&= z^m(z^{v_k}+1)^{-\langle d_k e_k, m\rangle}\\
&= \mathfrak{p}_{\mathfrak{d}_k}(z^m),
\end{aligned}$$

as desired. □

Construction 1.30 (The chamber structure)**.** Suppose given fixed data Γ satisfying the injectivity assumption and seed data $\mathbf{s}$. We then obtain for every seed $\mathbf{s}'$ obtained from $\mathbf{s}$ via mutation a scattering diagram $\mathfrak{D}_{\mathbf{s}'}$. In each case we will choose a representative for the scattering diagram with minimal support.

Note by construction and Remark 1.29, irrespective of the representative of $\mathfrak{D}_{\mathbf{s}}$ used, $\mathfrak{D}_{\mathbf{s}}$ contains walls whose union of supports is $\bigcup_{k\in I_{\mathrm{uf}}} e_k^\perp$. Furthermore, we have $\mathcal{C}^\pm \subseteq M_{\mathbb{R}}$ given by (1.16), which can be written more explicitly as

$$\begin{aligned}
\mathcal{C}^+_{\mathbf{s}} := \mathcal{C}^+ &= \{m\in M_{\mathbb{R}} \mid \langle e_i, m\rangle \ge 0 \quad \forall i\in I_{\mathrm{uf}}\},\\
\mathcal{C}^-_{\mathbf{s}} := \mathcal{C}^- &= \{m\in M_{\mathbb{R}} \mid \langle e_i, m\rangle \le 0 \quad \forall i\in I_{\mathrm{uf}}\}.
\end{aligned}$$

Then $\mathcal{C}^\pm_{\mathbf{s}}$ are the closures of connected components of $M_{\mathbb{R}}\setminus\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$. Similarly, we see that taking $\mathcal{C}^\pm_{\mu_k(\mathbf{s})}$ to be the chambers where all e'_i are positive (or negative), we have that $\mathcal{C}^\pm_{\mu_k(\mathbf{s})}$ is the closure of a connected component of $M_{\mathbb{R}}\setminus\mathrm{Supp}(\mathfrak{D}_{\mu_k(\mathbf{s})})$, so that $T_k^{-1}(\mathcal{C}^\pm_{\mu_k(\mathbf{s})})$ is the closure of a connected component of $M_{\mathbb{R}}\setminus\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$. Note that the closures of $T_k^{-1}(\mathcal{C}^+_{\mu_k(\mathbf{s})})$ and $\mathcal{C}^+_{\mathbf{s}}$ have a common codimension 1 face given by the intersection with $e_k^\perp$. This gives rise to the following chamber structure for a subset of $M_{\mathbb{R}}\setminus\mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$.

We refer the reader to Appendix A for the definition of the infinite oriented tree $\mathfrak{T}$ (or $\mathfrak{T}_{\mathbf{s}}$) used for parameterizing seeds obtained via mutation of $\mathbf{s}$. In particular, for any vertex v of $\mathfrak{T}$, there is a simple path from the root vertex to v, indicating a sequence of mutations $\mu_{k_1},\ldots,\mu_{k_p}$ and hence a piecewise linear transformation

$$T_v = T_{k_p}\circ\cdots\circ T_{k_1} : M_{\mathbb{R}} \to M_{\mathbb{R}}.$$

Note that T_{k_i} is defined using the basis vector e_{k_i} of the seed $\mu_{k_{i-1}} \circ \cdots \circ \mu_{k_1}(\mathbf{s})$, not the basis vector e_{k_i} of the original seed $\mathbf{s}$. By applying Theorem 1.24 repeatedly, we see that

$$T_v(\mathfrak{D}_{\mathbf{s}}) = \mathfrak{D}_{\mathbf{s}_v} \tag{1.31}$$

(where T_v applied to the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ is interpreted as the composition of the actions of each T_{k_i}) and

$$\mathcal{C}_v^{\pm} := T_v^{-1}(\mathcal{C}_{\mathbf{s}_v}^{\pm})$$

is the closure of a connected component of $M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$.

Note that the map from vertices of $\mathfrak{T}$ to chambers of $M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$ is never one-to-one. Indeed, if v is the vertex obtained by following the edge labeled k twice starting at the root vertex, one checks that $\mathcal{C}_v^{\pm} = \mathcal{C}_{\mathbf{s}}^{\pm}$, even though $\mu_k(\mu_k(\mathbf{s})) \neq \mathbf{s}$ (see [GHK13, Remark 2.5]).

Thus we have a chamber structure on a subset of $M_{\mathbb{R}}$; in general, the union of the cones $\mathcal{C}_v^{\pm}$ do not form a dense subset of $M_{\mathbb{R}}$.

Since we will often want to compare various aspects of this geometry for different seeds, we will write the short-hand $v \in \mathbf{s}$ for an object parameterized by a vertex v where the root of the tree is labeled with the seed $\mathbf{s}$. In particular:

Definition 1.32. We write $\mathcal{C}_{v\in\mathbf{s}}^{\pm}$ for the chamber of $M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$ corresponding to the vertex v. We write $\Delta_{\mathbf{s}}^{\pm}$ for the set of chambers $\mathcal{C}_{v\in\mathbf{s}}^{\pm}$ for v running over all vertices of $\mathfrak{T}_{\mathbf{s}}$. We call elements of $\Delta_{\mathbf{s}}^{+}$ *cluster chambers.*

2. Basics on tropicalization and the Fock–Goncharov cluster complex

We now explain that the chamber structure of Construction 1.30 coincides with the Fock–Goncharov cluster complex. To do so, we first recall the basics of tropicalization.

For a lattice N with $M = \mathrm{Hom}(N, \mathbb{Z})$, let $Q_{\mathrm{sf}}(N)$ be the subset of elements of the field of fractions of $\Bbbk[M] = H^0(T_N, \mathcal{O}_{T_N})$ which can be expressed as a ratio of Laurent polynomials with nonnegative integer coefficients. Then Q_{sf} is a semifield under ordinary multiplication and addition. For any semifield P, restriction to the monomials $M \subset Q_{\mathrm{sf}}(N)$ gives a canonical bijection

$$\mathrm{Hom}_{\mathrm{sf}}(Q_{\mathrm{sf}}(N), P) \to \mathrm{Hom}_{\mathrm{groups}}(M, P^{\times}) = N \otimes_{\mathbb{Z}} P^{\times},$$

where the first Hom is maps of semifields, $P^{\times}$ means the multiplicative group of P, and in the last tensor product we mean $P^{\times}$ viewed as $\mathbb{Z}$-module. Following [FG09] we define the P-valued points of T_N to be $T_N(P) := \mathrm{Hom}_{\mathrm{sf}}(Q_{\mathrm{sf}}(N), P)$. A *positive* birational map $\mu : T_N \dashrightarrow T_N$ means a birational map for which the pullback μ^* induces an isomorphism on $Q_{\mathrm{sf}}(N)$. Obviously, it gives an isomorphism on P-valued points. Thus it makes sense to talk about $X(P)$ for any variety X with a positive atlas of tori, for example many of the various flavors of cluster variety.

There are two equally good semifield structures on $\mathbb{Z}$, the max-plus and the min-plus structures. Here addition is either maximum or minimum, and multiplication is addition. We notate these as $\mathbb{Z}^T$ and $\mathbb{Z}^t$, respectively, thinking of capital T for the max-plus tropicalization and little t for the min-plus tropicalization. We similarly define $\mathbb{R}^T$ and $\mathbb{R}^t$. Thus taking $P = \mathbb{Z}^T$ or $\mathbb{Z}^t$, we obtain the sets of tropical points $X(\mathbb{Z}^T)$ or $X(\mathbb{Z}^t)$. The former is the convention used by Fock and Goncharov in [FG09], so we refer to this as the Fock–Goncharov tropicalization. The latter choice in fact coincides with $X^{\mathrm{trop}}(\mathbb{Z})$ as defined in [GHK13, Def. 1.7], defined as a subset of the set of discrete valuations. We refer to this as the geometric tropicalization. It will turn out both are useful. There is the obvious isomorphism of semifields $x \mapsto -x$ from $\mathbb{Z}^T \to \mathbb{Z}^t$. This induces a canonical sign-change identification $i : X(\mathbb{Z}^T) \to X(\mathbb{Z}^t)$.

Given a positive birational map $\mu : T_N \dashrightarrow T_N$, we use $\mu^T : N \to N$ and $\mu^t : N \to N$ to indicate the induced maps $T_N(\mathbb{Z}^T) \to T_N(\mathbb{Z}^T)$ and $T_N(\mathbb{Z}^t) \to T_N(\mathbb{Z}^t)$, respectively. For the geometric tropicalization, this coincides with the map on discrete valuations induced by pullback of functions; see [GHK13, §1]. For cluster varieties the two types of tropicalization are obviously equivalent. The geometric tropicalization has the advantage that it makes sense for any log Calabi–Yau variety, while the Fock–Goncharov tropicalization is restricted to (Fock–Goncharov) *positive spaces*, i.e., spaces obtained by gluing together algebraic tori via positive birational maps. We will use both notions: $X(\mathbb{R}^t)$ because in many cases it is more natural to think in terms of valuations/boundary divisors, and $X(\mathbb{R}^T)$ because, as we indicate below, the scattering diagram for building $\mathcal{A}_{\mathrm{prin}}$ lives naturally in $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ (because of already established cluster sign conventions).

One computes easily that for the basic mutation

$$\mu_{(n,m)} : T_N \dashrightarrow T_N, \quad \mu^*_{(n,m)}(z^{m'}) = z^{m'}(1+z^m)^{\langle m',n\rangle} \tag{2.1}$$

the Fock–Goncharov tropicalization is

$$\mu^T_{(n,m)} : N = T_N(\mathbb{Z}^T) \to T_N(\mathbb{Z}^T) = N, \quad x \mapsto x + [\langle m,x\rangle]_+ n \tag{2.2}$$

while the geometric tropicalization (see [GHK13, (1.4)]) is

$$\mu^t_{(n,m)} : N = T_N(\mathbb{Z}^t) \to T_N(\mathbb{Z}^t) = N, \quad x \mapsto x + [\langle m,x\rangle]_- n. \tag{2.3}$$

Thus:

Proposition 2.4. *$T_k : M^\circ \to M^\circ$ defined in* (1.23) *is the Fock–Goncharov tropicalization of*

$$\mu_{(v_k, d_k e_k)} : T_{M^\circ} \dashrightarrow T_{M^\circ}.$$

A rational function f on a cluster variety V is called *positive* if its restriction to each seed torus is positive, i.e., can be expressed as a ratio of sums of characters with positive integer coefficients. We can then define its Fock–Goncharov tropicalization $f^T : V(\mathbb{R}^T) \to \mathbb{R}$ by $f^T(p) = -p(f)$. Similarly, for f positive, we have its geometric tropicalization $f^t : V(\mathbb{R}^t) \to \mathbb{R}$ which for each $v \in V(\mathbb{R}^t)$ has value $f^t(v) = v(f)$. Using the identification of $V(\mathbb{Z}^t)$ with $V^{\mathrm{trop}}(\mathbb{Z})$, v is interpreted as a valuation and $f^t(v)$ coincides with $v(f)$, the value of v on f. In particular, this value is defined regardless of whether f is positive. We have a commutative diagram

$$\begin{tikzcd} V(\mathbb{R}^T) \arrow[r, "i"] \arrow[d, "f^T"'] & V(\mathbb{R}^t) \arrow[d, "f^t"'] \\ \mathbb{R} \arrow[r, equal] & \mathbb{R} \end{tikzcd} \tag{2.5}$$

where i is the canonical isomorphism determined by the sign-change isomorphism. The definition of f^t in terms of valuations extends the definition of f^t, and hence

f^T via this diagram, to any nonzero rational function. We note that

$$(2.6)\qquad \begin{aligned}(z^m)^T(a) &= \langle m, -r(a)\rangle, \quad m \in M, a \in T_N(\mathbb{Z}^T),\\ (z^m)^t(a) &= \langle m, r(a)\rangle, \quad m \in M, a \in T_N(\mathbb{Z}^t),\\ (z^m)^T(a) &= (z^m)^t(i(a)),\end{aligned}$$

where

$$(2.7)\qquad r : T_N(P) = \mathrm{Hom}_{\mathrm{sf}}(Q_{\mathrm{sf}}(N), P) \to \mathrm{Hom}_{\mathrm{groups}}(M, P^\times) = N \otimes P^\times$$

is the canonical restriction isomorphism. We will almost always leave r, i out of the notation.

Lemma 2.8.

(1) *For a positive Laurent polynomial* $g := \sum_{m\in M} c_m z^m \in Q_{\mathrm{sf}}(N)$ (*i.e.*, $c_m \in \mathbb{Z}_{\geq 0}$), *and* $x \in T_N(\mathbb{R}^T)$

$$g^T(x) = \min_{m, c_m \neq 0} \langle m, -r(x)\rangle,$$

where r *is the canonical isomorphism* (2.7).

(2) *If* $v \in T_N^{\mathrm{trop}}(\mathbb{Z})$ *is a divisorial discrete valuation and* $g = \sum c_m z^m$ *is any Laurent polynomial* (*so now* $c_m \in \Bbbk$), *then*

$$v(g) =: g^t(v) = \min_{m, c_m \neq 0} v(z^m) = \min_{m, c_m \neq 0} \langle m, r(v)\rangle.$$

Proof. By definition

$$g^T(x) = -\max_{m, c_m \neq 0} \langle m, r(x)\rangle = \min_{m, c_m \neq 0} \langle m, -r(x)\rangle.$$

This gives the first statement. For the second, we can assume $v \in T_N^{\mathrm{trop}}(\mathbb{Z}) = N$ is primitive, so part of a basis. Then the statement reduces to an obvious statement about the X_1 degree of a linear combination of monomials in $\Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. □

Note the mutations $\mu_{(v_k, d_k e_k)}$ are precisely the mutations between the tori in the atlas for $\mathcal{A}^\vee$ (see Appendix A for the definition of the Fock–Goncharov dual $\mathcal{A}^\vee$, and [GHK13, (2.5)] for the mutations between $\mathcal{X}$ tori in our notation). Thus by Theorem 1.24 and Proposition 2.4, the support of $\mathfrak{D}_{\mathbf{s}}$ viewed as a subset of $\mathcal{A}^\vee(\mathbb{R}^T)$ under the identification $M^\circ_{\mathbb{R},\mathbf{s}} = \mathcal{A}^\vee(\mathbb{R}^T)$ (induced canonically from the open set $T_{M^\circ,\mathbf{s}} \subset \mathcal{A}^\vee$) is independent of seed. In particular it makes sense to talk about $\mathcal{A}^\vee(\mathbb{R}^T) \setminus \mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$ as being completely canonically defined without choosing any seed. For any seed, the chambers $\mathcal{C}^\pm_{\mathbf{s}} \subset M^\circ_{\mathbb{R},\mathbf{s}} = \mathcal{A}^\vee(\mathbb{R}^T)$ are connected components of $\mathcal{A}^\vee(\mathbb{R}^T) \setminus \mathrm{Supp}(\mathfrak{D}_{\mathbf{s}})$.

We recall from [FG11]:

Definition 2.9. Suppose we are given fixed data Γ and an initial seed. For a seed $\mathbf{s} = (e_1, \ldots, e_n)$ obtained by mutation from the initial seed, the *Fock–Goncharov cluster chamber associated to* $\mathbf{s}$ is the subset

$$\{x \in \mathcal{A}^\vee(\mathbb{R}^T) \mid (z^{e_i})^T(x) \leq 0 \text{ for all } i \in I_{\mathrm{uf}}\},$$

identified with

$$\{x \in \mathcal{A}^\vee(\mathbb{R}^t) \mid (z^{e_i})^t(x) \leq 0 \text{ for all } i \in I_{\mathrm{uf}}\}$$

via i. The (*Fock–Goncharov*) *cluster complex* Δ^+ is the set of all such chambers.

Lemma 2.10. *Suppose we are given fixed data Γ satisfying the injectivity assumption, and suppose we are given an initial seed. For a seed $\mathbf{s} = (e_1, \ldots, e_n)$ obtained by mutation from the initial seed, the chamber $\mathcal{C}^+_{\mathbf{s}} \subset M^\circ_{\mathbb{R},\mathbf{s}} = \mathcal{A}^\vee(\mathbb{R}^T)$ (also identified with $\mathcal{A}^\vee(\mathbb{R}^t)$ via i) is the Fock–Goncharov cluster chamber associated to $\mathbf{s}$. Hence the Fock–Goncharov cluster chambers are the maximal cones of a simplicial fan (of not necessarily strictly convex cones). In particular Δ^+ is identified with $\Delta^+_{\mathbf{s}}$ for any choice of seed $\mathbf{s}$ giving an identification of $\mathcal{A}^\vee(\mathbb{R}^T)$ with $M^\circ_{\mathbb{R},\mathbf{s}}$.*

Proof. The identification of the chamber is immediate from the definition. The result then follows from the chamber structure of Construction 1.30 and the fact that the T_k are the Fock–Goncharov tropicalizations of the mutations μ_k for $\mathcal{A}^\vee$. It is obvious each maximal cone is simplicial, and each adjacent pair of maximal cones meets along a codimension 1 face of each. Hence we obtain a simplicial fan. □

Construction 2.11. See Appendix B for a review of the cluster variety with principal coefficients, $\mathcal{A}_{\mathrm{prin}}$. Any seed $\mathbf{s}$ gives rise to a scattering diagram $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ living in

$$\widetilde{M}^\circ_{\mathbb{R},\mathbf{s}} = (M^\circ \oplus N)_{\mathbb{R},\mathbf{s}} = (\widetilde{N}^\vee)^*_{\mathbb{R},\mathbf{s}},$$

the second equality by Proposition B.2(3). Indeed in this situation, the injectivity assumption is satisfied since the form $\{\cdot,\cdot\}$ on $\widetilde{N} = N \oplus M^\circ$ is nondegenerate (which is the reason we use $\mathcal{A}_{\mathrm{prin}}$ instead of $\mathcal{A}$ or $\mathcal{X}$). Indeed, the vectors $\tilde{v}_i := \{(e_i, 0), \cdot\} = (v_i, e_i)$ are linearly independent. Note by Theorem 1.21, $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ contains the scattering diagram

$$\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathrm{in},\mathbf{s}} := \{\big((e_i,0)^\perp, 1 + z^{(v_i,e_i)}\big) \mid i \in I_{\mathrm{uf}}\}. \tag{2.12}$$

Recall from Proposition B.2 that we have a canonical map $\rho : \mathcal{A}^\vee_{\mathrm{prin}} \to \mathcal{A}^\vee$ which is defined on cocharacter lattices by the canonical projection $M^\circ \oplus N \to M^\circ$; see (B.4). Thus the tropicalization

$$\rho^T : \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) \to \mathcal{A}^\vee(\mathbb{R}^T)$$

coincides with this projection, which can be viewed as the quotient of an action of translation by N. By Definition 1.4, walls of $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ are of the form $(n,0)^\perp$ for $n \in N^+$. Thus all walls are invariant under translation by N, and thus are inverse images of walls under ρ^T. So even though $\mathcal{A}$ may not satisfy the injectivity assumption necessary to build a scattering diagram, we see that $\mathrm{Supp}(\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}})$ is the inverse image of a subset of $\mathcal{A}^\vee(\mathbb{R}^T)$ canonically defined independently of the seed. In particular, note that the Fock–Goncharov cluster chamber in $\mathcal{A}^\vee(\mathbb{R}^T)$ associated to the seed $\mathbf{s}$ (where $(z^{e_i})^T \le 0$ for all $i \in I_{\mathrm{uf}}$) pulls back to the corresponding Fock–Goncharov cluster chamber in $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$.

The following was conjectured by Fock and Goncharov, [FG11, §1.5]:

Theorem 2.13. *For any initial data the Fock–Goncharov cluster chambers in $\mathcal{A}^\vee(\mathbb{R}^T)$ are the maximal cones of a simplicial fan.*

Proof. When the injectivity assumption holds, this follows from Lemma 2.10. In particular it holds for $\mathcal{A}_{\mathrm{prin}}$. Now the general case follows by the above invariance of $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ under the translation by N. □

Example 2.14. Consider the rank 3 skew-symmetric cluster algebra given by the matrix

$$\epsilon = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}.$$

Then projecting the walls of $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\text{prin}}}$ to $M^\circ_{\mathbb{R}}$ via ρ^T, one obtains a collection of walls in a three-dimensional vector space. One can visualize this by intersecting the walls with the affine hyperplane $\langle e_1 + e_2 + e_3, \cdot\rangle = 1$. The collection of resulting rays and lines appears on the first page of [FG11]. While Fock and Goncharov were not aware of scattering diagrams in this context, in fact there the picture represents the same slice of the cluster complex, and hence coincides with the scattering diagram.

The cluster complex in fact fills up the half-space $\langle e_1 + e_2 + e_3, \cdot\rangle \geq 0$. There is no path through chambers connecting $\mathcal{C}^-_{\mathbf{s}}$ and $\mathcal{C}^+_{\mathbf{s}}$.

This example is particularly well known in cluster theory, and gives the cluster algebra associated with triangulations of the once-punctured torus.

3. Broken lines

We will explain how a scattering diagram determines a class of piecewise straight paths which will allow for the construction of theta functions. The notion of broken line was introduced in [G09] and was developed from the point of view of defining canonical functions in [CPS] and [GHK11].

We choose fixed data Γ and a seed $\mathbf{s}$ as described in Appendix A, and assume it satisfies the injectivity assumption. This gives rise to the group G described in §1.1 which acts by automorphisms of $\widehat{\Bbbk[P]}$ for a choice of monoid P containing $p_1^*(N^+)$ and with $P^\times = \{0\}$. The group G also acts on the rank 1 free $\widehat{\Bbbk[P]}$-module $z^{m_0}\widehat{\Bbbk[P]}$ for any $m_0 \in M^\circ$, with a log derivation $f\partial_n$ acting on z^{m_0} as usual to give $f\langle n, m_0\rangle z^{m_0}$.

We then have:

Definition 3.1. Let $\mathfrak{D}$ be a scattering diagram in the sense of Definition 1.6, and let $m_0 \in M^\circ \setminus \{0\}$ and $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$. A *broken line* for m_0 with endpoint Q is a piecewise linear continuous proper path $\gamma : (-\infty, 0] \to M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$ with a finite number of domains of linearity. This path comes along with a monomial $c_L z^{m_L} \in \Bbbk[M^\circ]$ for each domain of linearity $L \subseteq (-\infty, 0]$ of γ. This data satisfies the following properties:

(1) $\gamma(0) = Q$.
(2) If L is the first (and therefore unbounded) domain of linearity of γ, then $c_L z^{m_L} = z^{m_0}$.
(3) For t in a domain of linearity L, $\gamma'(t) = -m_L$.
(4) Let $t \in (-\infty, 0)$ be a point at which γ is not linear, passing from domain of linearity L to L'. Let
$$\mathfrak{D}_t = \{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \mid \gamma(t) \in \mathfrak{d}\}.$$
Then $c_{L'} z^{m_{L'}}$ is a term in the formal power series $\mathfrak{p}_{\gamma|_{(t-\epsilon,t+\epsilon)},\mathfrak{D}_t}(c_L z^{m_L})$.

Remark 3.2. Note that since a broken line does not pass through a singular point of $\mathfrak{D}$, we can write

$$\mathfrak{p}_{\gamma|_{(t-\epsilon,t+\epsilon)},\mathfrak{D}_t}(c_L z^{m_L}) = c_L z^{m_L} \prod_{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_t} f_{\mathfrak{d}}^{\langle n_0, m_L\rangle},$$

where $n_0 \in N^\circ$ is primitive, vanishes on each $\mathfrak{d} \in \mathfrak{D}_t$, and $\langle n_0, m_L \rangle$ is positive by item (3) of the definition of broken line. It is an important feature of broken lines that we never need to invert.

Definition 3.3. Let $\mathfrak{D}$ be a scattering diagram, and let $m_0 \in M^\circ \setminus \{0\}$ and $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$. For a broken line γ for m_0 with endpoint Q, define

$$I(\gamma) = m_0$$

(where I is for initial),

$$b(\gamma) = Q,$$

and

$$\mathrm{Mono}(\gamma) = c(\gamma) z^{F(\gamma)}$$

to be the monomial attached to the final (where F is for final) domain of linearity of γ. Define

$$\vartheta_{Q,m_0} = \sum_\gamma \mathrm{Mono}(\gamma),$$

where the sum is over all broken lines for m_0 with endpoint Q.

For $m_0 = 0$, we define for any endpoint Q

$$\vartheta_{Q,0} = 1.$$

In general, ϑ_{Q,m_0} is an infinite sum, but makes sense formally:

Proposition 3.4. $\vartheta_{Q,m_0} \in z^{m_0} \widehat{\Bbbk[P]}$.

Proof. It is clear by construction that for any broken line γ with $I(\gamma) = m_0$, we have $\mathrm{Mono}(\gamma) \in z^{m_0} \Bbbk[P]$. So it is enough to show that for any $k > 0$, there are only a finite number of broken lines γ such that $I(\gamma) = m_0$, $b(\gamma) = Q$, and $\mathrm{Mono}(\gamma) \notin z^{m_0} J^k$.

First note by the assumption that $J = P \setminus \{0\}$, there are only a finite number of choices for $F(\gamma)$ such that $\mathrm{Mono}(\gamma) \notin z^{m_0} J^k$. Fix a choice m for $F(\gamma)$. Second, to test that there are finitely many broken lines with $I(\gamma) = m_0$, $b(\gamma) = Q$, and $F(\gamma) = m$, we can throw out any wall $\mathfrak{d} \in \mathfrak{D}$ with $f_{\mathfrak{d}} \equiv 1 \mod J^k$, so we can assume $\mathfrak{D}$ is finite. Third, no broken line γ with $\mathrm{Mono}(\gamma) \notin z^{m_0} J^k$ can bend more than k times. Thus there are only a finite number of possible ordered sequences of walls $\mathfrak{d}_1, \ldots, \mathfrak{d}_s$ at which γ can bend. Fix one such sequence. One then sees there are at most a finite number of broken lines with $b(\gamma) = Q$, $F(\gamma) = m$ bending at $\mathfrak{d}_1, \ldots, \mathfrak{d}_s$. Indeed, one can start at Q and trace a broken line backward, using that the final direction is $-m$. Crossing a wall $\mathfrak{d}_i$ and passing from domain of linearity L (for smaller t) to domain of linearity L' (for larger t), one sees that knowing the monomial attached to L' restricts the choices of monomial on L to a finite number of possibilities. This shows the desired finiteness. □

The most important general feature of broken lines is the following:

Theorem 3.5. *Let $\mathfrak{D}$ be a consistent scattering diagram, and let $m_0 \in M^\circ \setminus \{0\}$ and $Q, Q' \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$ be two points. Suppose further the coordinates of Q are linearly independent over $\mathbb{Q}$, and the same is true for Q'. Then for any path γ with endpoints Q and Q' for which $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is defined, we have*

$$\vartheta_{Q',m_0} = \mathfrak{p}_{\gamma,\mathfrak{D}}(\vartheta_{Q,m_0}).$$

Proof. This is a special case of results of [CPS, §4]. The generality condition on Q and Q' guarantees that we do not have to worry about broken lines which pass through joints (which we are not allowing). Indeed, the dimension of the $\mathbb{Q}$-span of the coordinates can drop by at most one along a line with rational slope, and a point in a joint has two independent $\mathbb{Q}$-linear relations in its coordinates. $\square$

Let us next consider how broken lines change under mutation. Let $\mathbf{s}$ be a seed, and let $\bar{P}$ be as in Definition 1.25.

Proposition 3.6. *T_k defines a one-to-one correspondence $\gamma \mapsto T_k(\gamma)$ between broken lines for m_0 with endpoint Q for $\mathfrak{D}_{\mathbf{s}}$ and broken lines for $T_k(m_0)$ with endpoint $T_k(Q)$ for $\mathfrak{D}_{\mu_k(\mathbf{s})}$. This correspondence satisfies, depending on whether $Q \in \mathcal{H}_{k,+}$ or $\mathcal{H}_{k,-}$,*

$$\mathrm{Mono}(T_k(\gamma)) = T_{k,\pm}(\mathrm{Mono}(\gamma)),$$

where $T_{k,\pm}$ acts linearly on the exponents. In particular, we have

$$\vartheta^{\mu_k(\mathbf{s})}_{T_k(Q),T_k(m_0)} = T_{k,\pm}(\vartheta^{\mathbf{s}}_{Q,m_0}),$$

where the superscript indicates which scattering diagram is used to define the theta function.

Remark 3.7. By Propositions 3.6 and 2.4, when the injectivity assumption holds, broken lines make sense in $\mathcal{A}^\vee(\mathbb{R}^T)$ independent of a choice of seed.

Proof. Given a broken line γ for $\mathfrak{D}_{\mathbf{s}}$, we define $T_k(\gamma)$ to have underlying map $T_k \circ \gamma : (-\infty, 0] \to M_{\mathbb{R}}$. Subdivide domains of linearity of γ so that we can assume any domain of linearity L satisfies $\gamma(L) \subseteq \mathcal{H}_{k,+}$ or $\mathcal{H}_{k,-}$. In the two cases, the attached monomial $c_L z^{m_L}$ becomes $c_L z^{T_{k,+}(m_L)}$ or $c_L z^{T_{k,-}(m_L)}$, respectively. We show that $T_k(\gamma)$ is a broken line for $T_k(m_0)$ with endpoint $T_k(Q)$, with respect to the scattering diagram $\mathfrak{D}_{\mu_k(\mathbf{s})}$, which is equal to $T_k(\mathfrak{D}_{\mathbf{s}})$, by Theorem 1.24. Indeed, the only thing to do is to analyze what happens when γ crosses $e_k^\perp$. So suppose in passing from a domain of linearity L_1 to a domain of linearity L_2, γ crosses $e_k^\perp$, so that $c_{L_2} z^{m_{L_2}}$ is a term in

$$c_{L_1} z^{m_{L_1}} (1 + z^{v_k})^{|\langle d_k e_k, m_{L_1}\rangle|}.$$

Assume first that γ passes from $\mathcal{H}_{k,-}$ to $\mathcal{H}_{k,+}$. Then $c_{L_2} z^{T_{k,+}(m_{L_2})}$ is a term in

$$\begin{aligned} c_{L_1} z^{T_{k,+}(m_{L_1})}(1+z^{v_k})^{-\langle d_k e_k, m_{L_1}\rangle} &= c_{L_1} z^{m_{L_1}+\langle d_k e_k, m_{L_1}\rangle v_k}(1+z^{v_k})^{-\langle d_k e_k, m_{L_1}\rangle} \\ &= c_{L_1} z^{m_{L_1}}(1+z^{-v_k})^{-\langle d_k e_k, m_{L_1}\rangle}, \end{aligned}$$

showing that $T_k(\gamma)$ satisfies the correct rules for bending as it crosses the slab $\mathfrak{d}'_k = (e_k^\perp, 1 + z^{-v_k})$ of $T_k(\mathfrak{D}_{\mathbf{s}})$.

If instead γ crosses from $\mathcal{H}_{k,+}$ to $\mathcal{H}_{k,-}$, then $c_{L_2} z^{T_{k,-}(m_{L_2})} = c_{L_2} z^{m_{L_2}}$ is a term in

$$\begin{aligned} c_{L_1} z^{T_{k,-}(m_{L_1})}(1+z^{v_k})^{\langle d_k e_k, m_{L_1}\rangle} &= c_{L_1} z^{m_{L_1}+\langle d_k e_k, m_{L_1}\rangle v_k}(1+z^{-v_k})^{\langle d_k e_k, m_{L_1}\rangle} \\ &= c_{L_1} z^{T_{k,+}(m_{L_1})}(1+z^{-v_k})^{\langle d_k e_k, m_{L_1}\rangle}, \end{aligned}$$

so again $T_k(\gamma)$ satisfies the bending rule at the slab $\mathfrak{d}'_k$.

The map T_k on broken lines is then shown to be a bijection by observing T_k^{-1}, similarly defined, is the inverse to T_k on the set of broken lines. $\square$

The following, which shows that cluster variables are theta functions, is the key observation for proving positivity of the Laurent phenomenon.

Proposition 3.8. *Let $Q \in \mathrm{Int}(\mathcal{C}^+_{\mathbf{s}})$ be a basepoint, and let $m \in \mathcal{C}^+_{\mathbf{s}} \cap M^\circ$. Then $\vartheta_{Q,m} = z^m$.*

Proof. This says the only broken line with asymptotic direction m and basepoint Q has image $Q + \mathbb{R}_{\geq 0} m$, with attached monomial z^m. To see this, suppose we are given a broken line $\gamma : (-\infty, 0] \to M_{\mathbb{R}}$ with asymptotic direction m which bends successively at walls $\mathfrak{d}_1, \ldots, \mathfrak{d}_q$. For each i, there is an $n_i \in N^+$ such that $\mathfrak{d}_i \subseteq n_i^\perp$. Multiplying n_i by a positive integer if necessary, we can assume that the monomial attached to γ upon crossing the wall $\mathfrak{d}_i$ changes by a factor $c_i z^{p^*(n_i)}$. Now if $L_i \subseteq M_{\mathbb{R}}$ is the image of the ith linear segment of γ, we show inductively that

$$L_{i+1} \subseteq H_i = \left\{ m \,\middle|\, \left\langle \sum_{j=1}^{i} n_j, m \right\rangle \leq 0 \right\}.$$

Indeed, $L_1 = q + \mathbb{R}_{\geq 0} m$ for some q, so initially L_1 is contained on the positive side of $n_1^\perp$, i.e., n_1 is positive on L_1, and hence after bending at $n_1^\perp$, we see $L_2 \subseteq H_1$. Next, assume true for $i = k-1$. Then $L_k \subseteq H_{k-1}$, and if t_k is the time when γ bends at the wall $\mathfrak{d}_k$, we have $\langle n_k, \gamma(t_k)\rangle = 0$ and $\langle \sum_{j=1}^{k-1} n_j, \gamma(t_k)\rangle \leq 0$ by the induction hypothesis. Thus $\langle \sum_{j=1}^{k} n_j, \gamma(t_k)\rangle \leq 0$. In addition, the derivative γ' of γ along L_{k+1} is $-m - \sum_{j=1}^{k} p^*(n_j)$, and

$$\begin{aligned}
\left\langle \sum_{j=1}^{k} n_j, -m - \sum_{j=1}^{k} p^*(n_j) \right\rangle &= -\left\langle \sum_{j=1}^{k} n_j, m \right\rangle - \left\{ \sum_{j=1}^{k} n_j, \sum_{j=1}^{k} n_j \right\} \\
&= -\left\langle \sum_{j=1}^{k} n_j, m \right\rangle \leq 0
\end{aligned}$$

by skew symmetry of $\{\cdot, \cdot\}$ and $m \in \mathcal{C}^+_{\mathbf{s}}$. Thus

$$L_{k+1} \subseteq \gamma(t_k) - \mathbb{R}_{\geq 0} \left(m + \sum_{j=1}^{k} p^*(n_j) \right) \subseteq H_k.$$

Since $\mathrm{Int}(\mathcal{C}^+_{\mathbf{s}}) \cap H_i = \emptyset$ for all i, any broken line with asymptotic direction m which bends cannot terminate at the basepoint $Q \in \mathrm{Int}(\mathcal{C}^+_{\mathbf{s}})$. This shows that there is only one broken line for m terminating at $Q \in \mathrm{Int}(\mathcal{C}^+_{\mathbf{s}})$. □

Corollary 3.9. *Let $\sigma \in \Delta^+_{\mathbf{s}}$ be a cluster chamber, and let $Q \in \mathrm{Int}(\sigma)$, $m \in \sigma \cap M^\circ$. Then $\vartheta_{Q,m} = z^m$.*

Proof. Note $\sigma = \mathcal{C}^+_{v \in \mathbf{s}}$ for some vertex v of $\mathfrak{T}_{\mathbf{s}}$, with associated seed $\mathbf{s}_v$. There is then a piecewise linear map $T_v : M_{\mathbb{R}} \to M_{\mathbb{R}}$ with $T_v(\mathfrak{D}_{\mathbf{s}}) = \mathfrak{D}_{\mathbf{s}_v}$; see (1.31). Then the result follows by applying Proposition 3.8 to $T_v(m)$, $T_v(Q)$ and Proposition 3.6. □

In the next section, we will identify theta functions which are polynomials with universal Laurent polynomials, i.e., elements of the cluster algebra associated to the fixed and seed data. It will follow from the above corollary that cluster monomials are in fact theta functions.

Example 3.10. Figures 3.1 and 3.2 show some examples of broken lines in the case of Example 1.15 with $b = c = 2$. In the first figure, we take $m = (1, -1)$; in the second, $m = (2, -2)$. Neither of these lie in the cluster complex: the union of

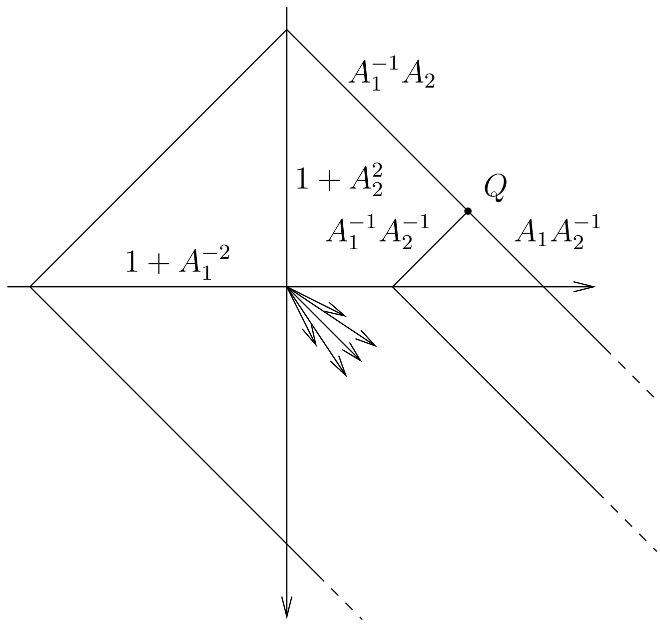

FIGURE 3.1. Broken lines defining $\vartheta_{Q,(1,-1)}$

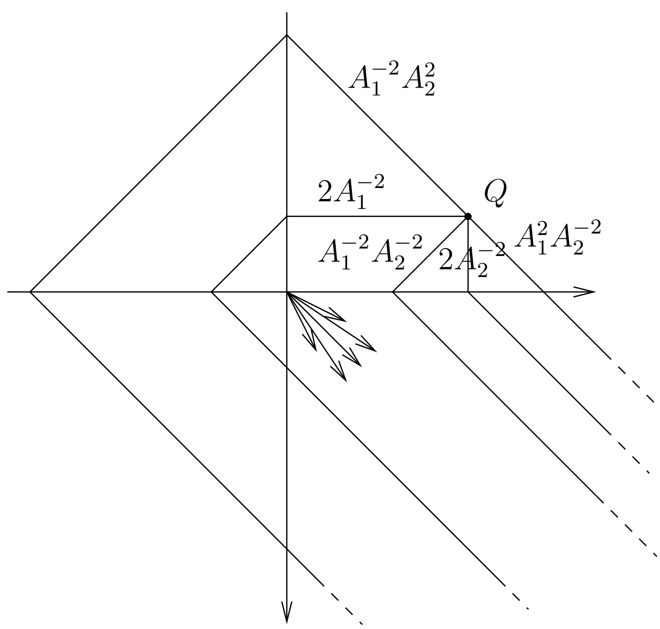

FIGURE 3.2. Broken lines defining $\vartheta_{Q,(2,-2)}$

all cones in the cluster complex is $M_{\mathbb{R}} \setminus \mathbb{R}_{\geq 0}(1,-1)$. In this case the only bends occur on the original lines of $\mathfrak{D}_{\text{in}}$, as any bending along the additional rays of the scattering diagram will result in the broken line shooting back out, unable to reach the first quadrant containing the basepoint Q. In the figures, the final line segment is labeled with its attached monomial, so that the theta function is a sum of these labels. One finds

$$\vartheta_{Q,(1,-1)} = A_1A_2^{-1} + A_1^{-1}A_2^{-1} + A_1^{-1}A_2,$$
$$\vartheta_{Q,(2,-2)} = A_1^2A_2^{-2} + 2A_2^{-2} + A_1^{-2}A_2^{-2} + 2A_1^{-2} + A_1^{-2}A_2^2.$$

In [CGMMRSW] it was shown that for any b, c, with Q lying in the first quadrant, the $\vartheta_{Q,m}$ with m ranging over all elements of m coincides with the greedy basis [LLZ13].

4. Building $\mathcal{A}$ from the scattering diagram and positivity of the Laurent phenomenon

Throughout this section we work with initial data Γ satisfying the injectivity assumption, so we obtain the cluster chamber structure $\Delta^+_{\mathbf{s}}$ from $\mathfrak{D}_{\mathbf{s}}$ described in Construction 1.30. In particular, this condition holds for initial data Γ_{prin}; see Appendix B.

In what follows, we will often want to deal with multiple copies of N, M, etc., indexed either by vertices v of $\mathfrak{T}_{\mathbf{s}}$ or chambers $\sigma \in \Delta^+_{\mathbf{s}}$. To distinguish these (identical) copies, we will use subscripts v or σ; e.g., the scattering diagram $\mathfrak{D}_{\mathbf{s}_v}$ lives in $M^\circ_{\mathbb{R},v}$, and chambers in $\mathfrak{D}_{\mathbf{s}_v}$ give, under the identification $M^\circ_{\mathbb{R},\mathbf{s}_v} = \mathcal{A}^\vee(\mathbb{R}^T)$, the Fock–Goncharov cluster complex Δ^+ by Lemma 2.10. In particular the cluster chambers of $\mathfrak{D}_{\mathbf{s}_v}$ and $\mathfrak{D}_{\mathbf{s}_{v'}}$ are in canonical bijection.

Construction 4.1. Fix a seed $\mathbf{s}$. We use the cluster chambers to build a positive space. We attach a copy of the torus $T_{N^\circ,\sigma} := T_{N^\circ}$ to each cluster chamber $\sigma \in \Delta^+_{\mathbf{s}}$.

Given any two cluster chambers σ', σ of $\Delta^+_{\mathbf{s}}$, we can choose a path γ from σ' to σ. We then get an automorphism $\mathfrak{p}_{\gamma,\mathfrak{D}} : \widehat{\Bbbk[P]} \to \widehat{\Bbbk[P]}$ which is independent of choice of path. If we choose the path to lie in the support of the cluster complex, then by Remark 1.29 (which shows in particular that the scattering functions on walls of the cluster complex are polynomials, as opposed to formal power series), the wall crossings give birational maps of the torus, and hence we can view $\mathfrak{p}_{\gamma,\mathfrak{D}}$ as giving a well-defined map of fields of fractions

$$\mathfrak{p}_{\gamma,\mathfrak{D}} : \Bbbk(M^\circ) \to \Bbbk(M^\circ).$$

This induces a birational map

$$\mathfrak{p}_{\sigma,\sigma'} : T_{N^\circ,\sigma} \dashrightarrow T_{N^\circ,\sigma'},$$

which is in fact positive.

We can then construct a space $\mathcal{A}_{\text{scat},\mathbf{s}}$ by gluing together all the tori $T_{N^\circ,\sigma}$, $\sigma \in \Delta^+_{\mathbf{s}}$ via these birational maps; see [GHK13, Proposition 2.4]. We call this space (with its atlas of tori) $\mathcal{A}_{\text{scat},\mathbf{s}}$.

We write $T_{N^\circ,\sigma\in\mathbf{s}} := T_{N^\circ,\sigma}$ if we need to make clear which seed $\mathbf{s}$ is being used.

We check first that mutation equivalent seeds give canonically isomorphic spaces.

We recall first something of the construction of $\mathcal{A}$. Fix a seed $\mathbf{s}$. Then we have positive spaces

$$\mathcal{A}_{\mathbf{s}} = \bigcup_v T_{N^\circ,v}, \quad \mathcal{A}^\vee_{\mathbf{s}} = \bigcup_v T_{M^\circ,v},$$

where each atlas is parameterized by vertices v of the infinite tree $\mathfrak{T}_{\mathbf{s}}$. We write, e.g., $T_{M^\circ,v\in\mathbf{s}} \subset \mathcal{A}^\vee_{\mathbf{s}}$ for the open subset parameterized by v. If we obtain a seed $\mathbf{s}' = \mathbf{s}_v$ by mutation from $\mathbf{s}$, then we can think of the tree $\mathfrak{T}_{\mathbf{s}'}$ as a subtree of $\mathfrak{T}_{\mathbf{s}}$ rooted at v, and thus we obtain natural open immersions

$$\mathcal{A}_{\mathbf{s}'} \hookrightarrow \mathcal{A}_{\mathbf{s}}, \quad \mathcal{A}^\vee_{\mathbf{s}'} \hookrightarrow \mathcal{A}^\vee_{\mathbf{s}}. \tag{4.2}$$

These are easily seen to be isomorphisms. Under this immersion, the open cover of $\mathcal{A}_{\mathbf{s}'}$ is identified canonically with the subcover of $\mathcal{A}_{\mathbf{s}}$ indexed by vertices of $\mathfrak{T}_{\mathbf{s}'}$ (but in either atlas there are many tori identified with the same open set of the union). Because of this we view $\mathcal{A}$ as independent of the choice of seed in a given mutation equivalence class.

Given vertices v, v' of $\mathfrak{T}_{\mathbf{s}}$, we have birational maps

$$\mu_{v,v'} : T_{N^\circ,v} \dashrightarrow T_{N^\circ,v'}, \quad \mu_{v,v'} : T_{M^\circ,v} \dashrightarrow T_{M^\circ,v'}$$

induced by the inclusions $T_{N^\circ,v}, T_{N^\circ,v'} \subseteq \mathcal{A}_{\mathbf{s}}$ and $T_{M^\circ,v}, T_{M^\circ,v'} \subseteq \mathcal{A}^\vee_{\mathbf{s}}$, respectively.

In what follows, we use the same notation for the restriction of a piecewise linear map to a maximal cone on which it is linear and the unique linear extension of this restriction to the ambient vector space.

Proposition 4.3. *Let $\mathbf{s}$ be a seed. Let v be the root of $\mathfrak{T}_{\mathbf{s}}$, v' any other vertex. Consider the Fock–Goncharov tropicalization $\mu^T_{v',v} : M^\circ_{v'} \to M^\circ_v$ of $\mu_{v',v} : T_{M^\circ,v'} \dashrightarrow T_{M^\circ,v}$. Its restriction $\mu^T_{v',v}|_{\sigma'}$ to each cluster chamber $\sigma' \in \Delta^+_{\mathbf{s}_{v'}}$ is a linear isomorphism onto the corresponding chamber $\sigma := \mu^T_{v',v}(\sigma') \in \Delta^+_{\mathbf{s}}$. The linear map*

$$\mu^T_{v',v}|_{\sigma'} : M^\circ_{\sigma'\in\mathbf{s}_{v'}} \to M^\circ_{\sigma\in\mathbf{s}}$$

induces an isomorphism

$$T_{v',\sigma} : T_{N^\circ,\sigma\in\mathbf{s}} \to T_{N^\circ,\sigma'\in\mathbf{s}_{v'}}.$$

These glue to give an isomorphism of positive spaces $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \to \mathcal{A}_{\mathrm{scat},\mathbf{s}_{v'}}$.

In view of Proposition 4.3, we can view $\mathcal{A}_{\mathrm{scat}} = \mathcal{A}_{\mathrm{scat},\mathbf{s}}$ as independent of the seed in a given mutation class.

Proof. It is enough to treat the case where v' is adjacent to v via an edge labeled with $k \in I_{\mathrm{uf}}$, so that $\mathbf{s}' := \mathbf{s}_{v'} = \mu_k(\mathbf{s})$, as in general $\mu_{v',v}$ is the inverse of a composition of mutations $\mu_{k_p} \circ \cdots \circ \mu_{k_1}$. Note in this special case $\mu^T_{v',v} = T_k^{-1}$ by Proposition 2.4, the definition of $\mathcal{A}^\vee$ in Appendix A, and the formula for the $\mathcal{X}$-cluster mutation μ_k (see, e.g., [GHK13, (2.5)]). So

$$T_{v',\sigma} : T_{N^\circ,\sigma\in\mathbf{s}} \to T_{N^\circ,T_k(\sigma)\in\mathbf{s}'}$$

is the isomorphism determined by the linear map $T_k^{-1}|_{T_k(\sigma)}$. Proposition 4.3 amounts to showing commutativity of the diagram, for $\sigma, \tilde{\sigma} \in \Delta^+_{\mathbf{s}}$, $\sigma' = T_k(\sigma)$, $\tilde{\sigma}' = T_k(\tilde{\sigma})$,

$$\begin{tikzcd} T_{N^\circ,\sigma\in\mathbf{s}} \arrow[r, "T_{v',\sigma}"] \arrow[d, dashed, "\mathfrak{p}_{\sigma,\tilde{\sigma}}"'] & T_{N^\circ,\sigma'\in\mathbf{s}'} \arrow[d, dashed, "\mathfrak{p}_{\sigma',\tilde{\sigma}'}"] \\ T_{N^\circ,\tilde{\sigma}\in\mathbf{s}} \arrow[r, "T_{v',\tilde{\sigma}}"'] & T_{N^\circ,\tilde{\sigma}'\in\mathbf{s}'} \end{tikzcd}$$

where in the left column $\mathfrak{p}$ indicates wall crossings in $\mathfrak{D}_{\mathbf{s}}$, while in the right column the wall crossings are in $\mathfrak{D}_{\mathbf{s}'}$.

If σ and $\tilde{\sigma}$ are on the same side of the wall $e_k^\perp$, then commutativity follows immediately from Theorem 1.24. So we can assume that σ and $\tilde{\sigma}$ are adjacent cluster chambers separated by the wall $e_k^\perp$, and further without loss of generality that e_k is nonnegative on σ. Now by Remark 1.29 there is only one wall of $\mathfrak{D}_{\mathbf{s}}$ ($\mathfrak{D}_{\mathbf{s}'}$) contained in $e_k^\perp$, with support $e_k^\perp$ itself and attached function $1 + z^{v_k}$ (resp. $1 + z^{-v_k}$). Now it is a simple calculation:

$$\begin{aligned} T^*_{v',\sigma}(\mathfrak{p}^*_{\sigma',\tilde{\sigma}'}(z^m)) &= T^*_{v',\sigma}(z^m(1+z^{-v_k})^{-\langle d_k e_k, m\rangle}) \\ &= z^{m-v_k\langle d_k e_k, m\rangle}(1+z^{-v_k})^{-\langle d_k e_k, m\rangle} \\ &= z^m(1+z^{v_k})^{-\langle d_k e_k, m\rangle} \\ &= \mathfrak{p}^*_{\sigma,\tilde{\sigma}}(T^*_{v',\tilde{\sigma}}(z^m)). \end{aligned}$$

This gives the desired commutativity. □

Next we explain how to identify $\mathcal{A}_{\mathrm{scat}}$ with $\mathcal{A}$.

Recall for each vertex v of $\mathfrak{T}_{\mathbf{s}}$ there is an associated cluster chamber $\mathcal{C}^+_v \in \Delta^+_{\mathbf{s}}$ in the cluster complex. While the atlas for $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$ is parameterized by chambers of $\Delta^+_{\mathbf{s}}$, we can use a more redundant atlas indexed by vertices of $\mathfrak{T}_{\mathbf{s}}$, equating $T_{N^\circ,v}$ with $T_{N^\circ,\mathcal{C}^+_v}$. The open sets and the gluing maps in this redundant atlas are the same as in the original, but in the redundant atlas a given open set might be repeated many times.

Theorem 4.4. *Fix a seed* $\mathbf{s}$. *Let* v *be the root of* $\mathfrak{T}_{\mathbf{s}}$, *and let* v' *be any other vertex. Let* $\psi^*_{v,v'} : M^\circ_{v'} \to M^\circ_{v'}$ *be the linear map* $\mu^T_{v,v'}|_{\mathcal{C}^+_{v'\in\mathbf{s}}}$. *Let* $\psi_{v,v'} : T_{N^\circ,v'} \to T_{N^\circ,v'}$ *be the associated map of tori. These glue to give an isomorphism of positive spaces*

$$\mathcal{A}_{\mathbf{s}} := \bigcup_{v'} T_{N^\circ,v'} \to \mathcal{A}_{\mathrm{scat},\mathbf{s}} := \bigcup_{v'} T_{N^\circ,v'}.$$

Furthermore, the diagram

$$\begin{tikzcd} \mathcal{A}_{\mathbf{s}} \arrow[r] \arrow[d] & \mathcal{A}_{\mathrm{scat},\mathbf{s}} \arrow[d] \\ \mathcal{A}_{\mathbf{s}_{v'}} \arrow[r] & \mathcal{A}_{\mathrm{scat},\mathbf{s}_{v'}} \end{tikzcd} \tag{4.5}$$

is commutative, where the right-hand vertical map is the isomorphism of Proposition 4.3, the left-hand vertical map the isomorphism given in (4.2), *and the horizontal maps are the isomorphisms just described.*

Proof. Let $v', v'' \in \mathfrak{T}_{\mathbf{s}}$. The desired isomorphism is equivalent to commutativity of the diagram

$$\begin{tikzcd} T_{N^\circ,v'} \subset \mathcal{A}_{\mathbf{s}} \arrow[r, "\psi_{v,v'}"] \arrow[d, dashed, "\mu_{v',v''}"'] & T_{N^\circ,v'} \subset \mathcal{A}_{\mathrm{scat},\mathbf{s}} \arrow[d, dashed, "\mathfrak{p}_{\mathcal{C}^+_{v'},\mathcal{C}^+_{v''}}"] \\ T_{N^\circ,v''} \subset \mathcal{A}_{\mathbf{s}} \arrow[r, "\psi_{v,v''}"'] & T_{N^\circ,v''} \subset \mathcal{A}_{\mathrm{scat},\mathbf{s}} \end{tikzcd} \tag{4.6}$$

where the right-hand vertical arrow is given by wall crossings in $\mathfrak{D}_{\mathbf{s}}$ between the cluster chambers for v', v''. For this we may assume there is an oriented path from v' to v'' in $\mathfrak{T}_{\mathbf{s}}$ and, thus, that $v'' \in \mathfrak{T}_{\mathbf{s}_{v'}} \subset \mathfrak{T}_{\mathbf{s}}$.

The commutativity of (4.5) is equivalent to the commutativity of

$$\begin{tikzcd} T_{N^\circ,v''} \subset \mathcal{A}_{\mathbf{s}} \arrow[r, "\psi_{v,v''}"] \arrow[d, equal] & T_{N^\circ,v''} \subset \mathcal{A}_{\mathrm{scat},\mathbf{s}} \arrow[d] \\ T_{N^\circ,v''} \subset \mathcal{A}_{\mathbf{s}_{v'}} \arrow[r, "\psi_{v',v''}"] & T_{N^\circ,v''} \subset \mathcal{A}_{\mathrm{scat},\mathbf{s}_{v'}} \end{tikzcd} \tag{4.7}$$

where the right-hand vertical map is the restriction of the isomorphism $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \to \mathcal{A}_{\mathrm{scat},\mathbf{s}_{v'}}$ of Proposition 4.3. We argue the commutativity of (4.7) first and then show that this implies the commutativity of (4.6).

Each map in (4.7) is an isomorphism, induced by the restrictions of tropicalizations of various $\mu_{w,w'}$ to various cluster chambers. Explicitly, on character lattices, we have the corresponding diagram

$$\begin{tikzcd} M^\circ \arrow[d, equal] & M^\circ \arrow[l, "\mu^T_{v,v''}|_{\mathcal{C}^+_{v''}\in \mathbf{s}}"'] \\ M^\circ & M^\circ \arrow[l, "\mu^T_{v',v''}|_{\mathcal{C}^+_{v''}\in \mathbf{s}_{v'}}"] \arrow[u, "\mu^T_{v',v}|_{\mathcal{C}^+_{v''\in \mathbf{s}_{v'}}}"'] \end{tikzcd}$$

which is obviously commutative as tropicalization is functorial and $\mu_{v',v''} = \mu_{v,v''} \circ \mu_{v',v}$.

Now for the commutativity of (4.6). It is enough to check the case when there is an oriented edge from v' to v'' in $\mathfrak{T}_{\mathbf{s}}$ labeled by $k \in I_{\mathrm{uf}}$. We claim we may also assume $\mathbf{s} = \mathbf{s}_{v'}$. Indeed, assume we have proven commutativity in this case. We draw a cube, whose back vertical face is the diagram (4.6), and whose front vertical face is the analogous diagram for $\mathbf{s}_{v'}$, which is commutative by assumption. The top and bottom horizontal faces are instances of (4.7), and the right-hand vertical face is the commutative diagram of atlas tori giving the isomorphism $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \to \mathcal{A}_{\mathrm{scat},\mathbf{s}_{v'}}$ of Proposition 4.3. Finally, the left-hand vertical face consists of equality of charts or birational maps coming from inclusions of these tori in $\mathcal{A}_{\mathbf{s}}$ or $\mathcal{A}_{\mathbf{s}_v}$, and thus is commutative. Now the commutativity of the back vertical face (4.6) follows.

Finally, to show (4.6) when $\mathbf{s} = \mathbf{s}_{v'}$, i.e., $v = v'$, we note $\psi_{v,v'}$ is automatically the identity, and $\psi_{v,v''}$ is also the identity, by Definition 1.22, and the identification of T_k as Fock–Goncharov tropicalization of the birational map of tori $\mu_{v',v''} = \mu_k : T_{M^\circ,v'} \dashrightarrow T_{M^\circ,v''}$. Thus the commutativity amounts to showing that the wall-crossing automorphism $\Bbbk(M^\circ) \to \Bbbk(M^\circ)$ of fraction fields, given by crossing the wall $e_k^\perp$ from the negative to the positive side, is the pullback on rational functions of the birational mutation $\mu_k : T_{N^\circ} \dashrightarrow T_{N^\circ}$. Note the only scattering function on the wall is $1+z^{v_k}$, so this follows from the coordinate free formula for the birational mutation; see, e.g., [GHK13, (2.6)]. $\square$

We can now make precise the relationship between theta functions and cluster monomials mentioned at the end of §3.

Definition 4.8. Given fixed and initial data $\Gamma, \mathbf{s}$, if a seed $\mathbf{s}_w = (e'_1, \ldots, e'_n)$ is given, with $(e'_i)^*$ the dual basis and $f'_i = d_i^{-1}(e'_i)^*$, a *cluster monomial* in this seed is a monomial on $T_{N^\circ,w} \subset \mathcal{A}$ of the form z^m with $m = \sum_{i=1}^n a_i f'_i$ and the a_i nonnegative for $i \in I_{\mathrm{uf}}$. By the Laurent phenomenon [FZ02b], such a monomial always extends to a regular function on $\mathcal{A}$. A *cluster monomial on* $\mathcal{A}$ is a regular function which is a cluster monomial in some seed.

Theorem 4.9. *Let Γ be fixed data satisfying the injectivity assumption, and let* $\mathbf{s}$ *be an initial seed. Let $Q \in \mathcal{C}^+_{\mathbf{s}}$ and $m \in \sigma \cap M^\circ$ for some $\sigma \in \Delta^+_{\mathbf{s}}$. Then $\vartheta_{Q,m}$ is a positive Laurent polynomial which expresses a cluster monomial of $\mathcal{A}$ in the initial seed* $\mathbf{s}$. *Further, all cluster monomials can be expressed in this way.*

Proof. By Theorem 4.4 and Proposition 4.3 we have a canonical isomorphism of positive spaces $\varphi : \mathcal{A} \to \mathcal{A}_{\mathrm{scat}} = \mathcal{A}_{\mathrm{scat},\mathbf{s}}$. Let v be the root of $\mathfrak{T}_{\mathbf{s}}$, and let v' be any vertex of $\mathfrak{T}_{\mathbf{s}}$. Then we have $T_{N^\circ,v'} \subset \mathcal{A}$, and the cluster monomials for the seed $\mathbf{s}_{v'}$ are just the monomials z^m on $T_{N^\circ,v'}$ with $m \in \mathcal{C}^+_{\mathbf{s}_{v'}} \cap M^\circ_{v'}$. By Theorem 4.4, this is identified with the monomial $(\psi^{-1}_{v,v'})^*(z^m) = z^{\mu^T_{v',v}(m)}$ on $T_{N^\circ,v'} \subset \mathcal{A}_{\mathrm{scat},\mathbf{s}}$, as $\mu^T_{v',v}$ takes $\mathcal{C}^+_{\mathbf{s}_{v'}} \in \Delta^+_{\mathbf{s}_{v'}}$ to $\mathcal{C}^+_{v'\in\mathbf{s}} \in \Delta^+_{\mathbf{s}}$ by Proposition 4.3. So the cluster monomials for the chart indexed by v' in $\mathcal{A}_{\mathrm{scat}}$ are of the form z^m with $m \in \mathcal{C}^+_{v'\in\mathbf{s}}$. Furthermore, if for each vertex w of $\mathfrak{T}_{\mathbf{s}}$, $Q_w \in \mathcal{C}^+_{w\in\mathbf{s}}$ is a general basepoint, we have $\vartheta_{Q_{v'},m} = z^m$ for $m \in \mathcal{C}^+_{v'\in\mathbf{s}}$ by Corollary 3.9. By the definition of $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$ in Construction 4.1, the corresponding rational function on the open set $T_{N^\circ,v} \subset \mathcal{A}_{\mathrm{scat},\mathbf{s}}$ is $\mathfrak{p}_{\gamma,\mathfrak{D}}(\vartheta_{Q_{v'},m})$, where γ is a path from $Q_{v'}$ to $Q \in \mathcal{C}^+_{\mathbf{s}}$ lying in the support of $\Delta^+_{\mathbf{s}}$. But $\vartheta_{Q,m} = \mathfrak{p}_{\gamma,\mathfrak{D}}(\vartheta_{Q_{v'},m})$ by Theorem 3.5. Finally, $\vartheta_{Q,m}$ is a positive Laurent series by Theorem 1.13 and the definition of broken lines. By the Laurent phenomenon, it is also a polynomial. $\square$

We can now remove the injectivity assumption to prove:

Theorem 4.10 (Positivity of the Laurent phenomenon)**.** *Each cluster variable of an $\mathcal{A}$-cluster algebra is a Laurent polynomial with nonnegative integer coefficients in the cluster variables of any given seed.*

Proof. Since, as explained in Proposition B.11, each cluster variable lifts canonically from $\mathcal{A}$ to $\mathcal{A}_{\mathrm{prin}}$, we can replace the initial data Γ with Γ_{prin}, for which the injectivity assumption holds. The result then immediately follows from Theorem 4.9. $\square$

Remark 4.11. When fixed and initial data Γ, $\mathbf{s}$ have frozen variables, there is a partial compactification of cluster varieties $\mathcal{A} \subset \overline{\mathcal{A}}$; see Construction B.9. We have an analogous partial compactification $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \subset \overline{\mathcal{A}}_{\mathrm{scat},\mathbf{s}}$, given by an atlas of toric varieties $T_{N^\circ,v\in\mathbf{s}} \subset \mathrm{TV}(\Sigma_{v\in\mathbf{s}})$. The choice of fans is forced by the identifications of Proposition 4.3: for v the root of $\mathfrak{T}_{\mathbf{s}}$, $\Sigma_{v\in\mathbf{s}} := \Sigma^{\mathbf{s}}$ ($\Sigma^{\mathbf{s}}$ as in Construction B.9) and then $\Sigma_{v'\in\mathbf{s}} := \mu^t_{v,v'}(\Sigma_{v\in\mathbf{s}})$. Now Proposition 4.3 and Theorem 4.4 (and their proofs) extend to the partial compactifications without change. One checks easily that all mutations in the positive spaces $\mathcal{A}, \mathcal{A}_{\mathrm{scat}}$, and all the linear isomorphisms between corresponding tori in the atlases for $\mathcal{A}, \mathcal{A}_{\mathrm{scat},\mathbf{s}}, \mathcal{A}_{\mathrm{scat},\mathbf{s}_v}$ preserve the monomials $A_i = z^{f_i}$, $i \notin I_{\mathrm{uf}}$ (these are the frozen cluster variables), so that all the spaces come with canonical projection to $\mathbb{A}^{\#(I \setminus I_{\mathrm{uf}})}$, preserved by the isomorphisms between these positive spaces. We shall see in the next section that in the special case of the partial compactification of $\mathcal{A}_{\mathrm{prin}}$, the relevant fans are particularly well-behaved.

5. Sign coherence of c- and g-vectors

We begin with some philosophy concerning log Calabi–Yau varieties following on from the discussion of [GHK13, §1]. Suppose $V \subset U$ are both log Calabi–Yau and V is a Zariski open subset of U, both having maximal boundary ([GHK13, Definition 1.5]). The tropical sets (which are expected to parameterize the theta function basis of functions on the mirror) of U and V are canonically equal, and we expect the mirror $U^\vee$ to degenerate to the mirror $V^\vee$. In particular when $V = T$ is an algebraic torus, we expect a canonical degeneration of $U^\vee$ to the dual torus $T^\vee$, under which the theta functions degenerate to monomials (i.e., characters). When $U = \mathcal{A}$ is an $\mathcal{A}$-cluster variety and $T = T_{N^\circ,\mathbf{s}} \subset \mathcal{A}$ is a cluster torus, it turns out this degeneration has a purely cluster construction: the choice of seed $\mathbf{s}$ determines a canonical partial compactification $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$ of $\pi : \mathcal{A}_{\mathrm{prin}} \to T_M$; see Proposition B.2 and Remark B.10. The main point of this section is to show that $\pi^{-1}(0) = T_{N^\circ}$; see Corollary 5.3. This degeneration is central to what follows in this paper. For example, we prove linear independence of theta functions by showing they restrict to different characters on T_{N°, and the Fock–Goncharov conjecture, false in general, is true in a formal neighborhood of this fiber. There are analogous degenerations (identified with this one when the Fock–Goncharov conjecture holds) for, e.g., $\mathrm{can}(\mathcal{A})$, and here they are even more central, being the main tool we have for proving properties of this algebra (e.g., that its spectrum gives a Gorenstein log Calabi–Yau of the right dimension); see Theorem 8.32. The equality $\pi^{-1}(0) = T_{N^\circ}$, while not at all obvious from the cluster atlas, is immediate using the alternative description $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$ of the previous section, as we now explain. Further, there are some immediate benefits, such as sign coherence of c-vectors.

For the remainder of the paper, the only scattering diagram we will ever consider is $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$; see Construction 2.11. So we will often omit the superscript from the notation.

Construction 5.1. Fix a seed $\mathbf{s}$ for fixed data Γ. By Construction 4.1, the scattering diagram $\mathfrak{D}_{\mathbf{s}} = \mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ gives an atlas for the space $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$. (Technically, we should write $\mathcal{A}_{\mathrm{prin},\mathrm{scat},\mathbf{s}}$ to indicate we are constructing something isomorphic to $\mathcal{A}_{\mathrm{prin}}$; however, this will make the notation even less readable.) This was constructed by attaching a copy $T_{\widetilde{N}^\circ,\sigma}$ of the torus $T_{\widetilde{N}^\circ}$ to each cluster chamber $\sigma \in \Delta^+_{\mathbf{s}}$, and (compositions of) wall-crossing automorphisms give the birational maps between them. By Theorem 4.4 this space is canonically identified with $\mathcal{A}_{\mathrm{prin}}$: $\mathcal{A}_{\mathrm{prin}}$ has an atlas of tori $T_{\widetilde{N}^\circ,w}$ parameterized by vertices w of $\mathfrak{T}_{\mathbf{s}}$, and we have canonical isomorphisms $\psi_{v,w} : T_{\widetilde{N}^\circ,w} \to T_{\widetilde{N}^\circ,\mathcal{C}^+_w}$ for each vertex w which induce the isomorphism $\mathcal{A}_{\mathrm{prin}} \to \mathcal{A}_{\mathrm{scat},\mathbf{s}}$.

In what follows, if w is a vertex of $\mathfrak{T}_{\mathbf{s}}$, we write $\tilde{\mathbf{s}}_w$ for the seed obtained by mutating $\tilde{\mathbf{s}}$ (see (B.1)) via the sequence of mutations dictated by the path from the root v of $\mathfrak{T}_{\mathbf{s}}$ to w. As described in Remark B.10, the initial seed $\mathbf{s}$ determines the partial compactification $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$, given by the atlas of toric varieties

$$T_{\widetilde{N}^\circ,w} \subset \mathrm{TV}(\Sigma^{\mathbf{s}}_w),$$

where $\Sigma^{\mathbf{s}}_w$ is the cone generated by the subset of basis vectors of $\tilde{\mathbf{s}}_w$ corresponding to the second copy of I.

By Remark 4.11, the seed $\mathbf{s}$ also determines a partial compactification $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$ (the superscript, thus the seed *close* to the overline in the notation, is responsible for the partial compactification), given by an atlas of toric varieties. Explicitly, if w is a vertex of $\mathfrak{T}_{\mathbf{s}}$, the fan $\Sigma^{\mathbf{s}}_w$ yields the partial compactification of $T_{\widetilde{N}^\circ,w}$ in $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$, and this is identified with $T_{\widetilde{N}^\circ,\mathcal{C}^+_w\in\Delta^+_{\mathbf{s}}}$ via $\psi_{v,w}$ under the isomorphism $\mathcal{A}_{\mathrm{prin}} \cong \mathcal{A}_{\mathrm{scat},\mathbf{s}}$ of Theorem 4.4. Thus the fan giving the partial compactifaction of $T_{\widetilde{N}^\circ,\mathcal{C}^+_w\in\Delta^+_{\mathbf{s}}}$ is

$$\Sigma^{\mathbf{s}}_{\mathrm{scat},w} := \psi^t_{v,w}(\Sigma^{\mathbf{s}}_w).$$

In fact, this fan is easily calculated:

Lemma 5.2. *The cones $\Sigma^{\mathbf{s}}_{\mathrm{scat},w}$, and thus the toric varieties in the atlas for the partial compactification $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$, are the same for all w. Each is equal to the cone spanned by the vectors $(0,e_1^*),\dots,(0,e_n^*) \in \widetilde{N}^\circ$, where $\mathbf{s} = (e_1,\dots,e_n)$ and $e_1^*,\dots,e_n^*$ denotes the dual basis.*

Proof. $\Sigma^{\mathbf{s}}_{\mathrm{scat},v}$ is the given cone, by definition of the seed $\tilde{\mathbf{s}}$. By Construction B.9 the other fans are given by applying the geometric tropicalization of the birational gluing of the tori in the atlas for $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$. These birational maps are given by wall crossings in $\mathfrak{D}_{\mathbf{s}}$. But for each wall between cluster chambers, the wall crossing is a standard mutation $\mu_{(\tilde{n},\tilde{m})}$ (notation as in §2), for some $\tilde{n} \in \widetilde{N}^\circ, \tilde{m} \in \widetilde{M}^\circ$. The attached scattering function is $1+z^{p^*(n,0)}$ for some n in the convex hull of $\{e_i \mid i \in I\}$, and $\tilde{m} = p^*(n,0)$. But then $\langle \tilde{m},(0,e_i^*)\rangle = \{(n,0),(0,e_i^*)\} \geq 0$. Thus the geometric tropicalization $\mu^t_{(\tilde{n},\tilde{m})}$ fixes all the e_i^* by (2.3), and so the fan is constant. $\square$

Corollary 5.3. *Fix a seed $\mathbf{s}$, and let v be the root of $\mathfrak{T}_{\mathbf{s}}$. The following hold:*

1. *The fiber of $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\dots,X_n}$ over 0 is T_{N°; see Proposition* B.2 *for the definition of π.*
2. *The mutation maps*

$$\mathrm{TV}(\Sigma^{\mathbf{s}}_w) \dashrightarrow \mathrm{TV}(\Sigma^{\mathbf{s}}_{w'})$$

for the atlas of toric varieties defining $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ are isomorphisms in a neighborhood of the fiber over $0 \in \mathbb{A}^n_{X_1,\dots,X_n}$.

3. *For the partial compactification $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$ with atlas corresponding to cluster chambers of $\mathfrak{D}_{\mathbf{s}}$, the corresponding mutation map between two charts* (*which by Lemma* 5.2 *has the same domain and range*) *is an isomorphism in a neighborhood of the fiber $0 \in \mathbb{A}^n_{X_1,\dots,X_n}$ and restricts to the identity on this fiber.*

Proof. It is clear that (3) implies (2) implies (1).

For (3), the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ is trivial modulo the X_i (which pulls back to $z^{(0,e_i)}$), because this holds for the initial walls, with attached functions $1+z^{(v_i,e_i)}$. Now for any adjacent vertices $w,w' \in \mathfrak{T}_{\mathbf{s}}$, the birational gluing map $\mathrm{TV}(\Sigma^{\mathbf{s}}_{\mathrm{scat},w}) \dashrightarrow \mathrm{TV}(\Sigma^{\mathbf{s}}_{\mathrm{scat},w'})$ is given on the level of monomials by $z^{\tilde{m}} \mapsto z^{\tilde{m}} f^{\langle \tilde{n},\tilde{m}\rangle}$ for a regular function f on $\mathrm{TV}(\Sigma^{\mathbf{s}}_{\mathrm{scat},w'})$ and some $\tilde{n} \in \widetilde{N}^\circ$ and any $\tilde{m} \in \widetilde{M}^\circ$, and by the above

f is identically 1 when restricted to the torus where the X_i are zero. On the other hand, this birational map gives an isomorphism between the open subsets of $\mathrm{TV}(\Sigma^{\mathbf{s}}_{\mathrm{scat},w})$ and $\mathrm{TV}(\Sigma^{\mathbf{s}}_{\mathrm{scat},w'})$ where f is nonzero. In particular, the gluing maps are isomorphisms in the neighborhood of the fiber where all X_i vanish and are the identity on that fiber. $\square$

The proof of the corollary shows the utility of constructing $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ as the positive space $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$ associated to the cluster chambers in the scattering diagram $\mathfrak{D}_{\mathbf{s}}$. Next we show that sign coherence of c-vectors follows easily from the corollary.

In what follows, given a seed $\tilde{\mathbf{s}}_w = (\tilde{e}_1, \ldots, \tilde{e}_{2n})$ obtained via mutation from $\tilde{\mathbf{s}}$, we write $\tilde{\epsilon}^w$ for the $n \times 2n$ exchange matrix for this seed, with

$$\tilde{\epsilon}^w_{ij} = \begin{cases} \{\tilde{e}_i, \tilde{e}_j\} d_j, & 1 \le j \le n, \\ \{\tilde{e}_i, \tilde{e}_j\} d_{j-n}, & n+1 \le j \le 2n. \end{cases} \tag{5.4}$$

The c-vectors of this seed are the rows of the right-hand $n \times n$ submatrix.

Corollary 5.5 (Sign coherence of c-vectors). *For any vertex w of $\mathfrak{T}_{\mathbf{s}}$ and fixed k satisfying $1 \le k \le n$, either the entries $\tilde{\epsilon}^w_{k,j}$, $n+1 \le j \le 2n$ are all nonpositive or these entries are all nonnegative.*

Proof. The result follows directly from Corollary 5.3 by writing down the mutation in cluster coordinates. Following the notation given in Appendix B, we have the fixed seed $\mathbf{s} = (e_1, \ldots, e_n)$ which determines $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ and the family $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$. The corresponding initial seed for $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ is

$$\tilde{\mathbf{s}} = ((e_1, 0), \ldots, (e_n, 0), (0, f_1), \ldots, (0, f_n)),$$

and the coordinate X_i on $\mathbb{A}^n$ pulls back to $z^{(0,e_i)}$ on $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$. These are the frozen cluster variables for $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$. Note that $X_i = z^{g_{n+i}}$, where g_i is the dual basis to the basis $(d_1 e_1, 0), \ldots, (d_n e_n, 0), (0, e_1^*), \ldots, (0, e_n^*)$ of $\widetilde{N}^\circ$.

A vertex w' corresponds to a seed $\mathbf{s}_{w'} = (e'_1, \ldots, e'_n)$ for N with corresponding seed $\tilde{\mathbf{s}}_{w'} = ((e'_1, 0), \ldots, (e'_n, 0), h'_1, \ldots, h'_n)$ for $\widetilde{N}$, with $\tilde{\mathbf{s}}_{w'}$ obtained from $\tilde{\mathbf{s}}$ by a sequence of mutations. The h_i are no longer necessarily given by the f'_i. Write $\tilde{f}'_i$, $1 \le i \le 2n$, for the corresponding basis of $\widetilde{M}^\circ$. The cluster variables on the corresponding torus $T_{\widetilde{N}^\circ, w'}$ are $A'_i := z^{\tilde{f}'_i}$. Say w'' is a vertex of $\mathfrak{T}_{\mathbf{s}}$ adjacent to w' along an edge labeled by k. Then

$$\tilde{\mathbf{s}}_{w''} = ((e''_1, 0), \ldots, (e''_n, 0), h''_1, \ldots, h''_n),$$

and the cluster coordinates are $A''_i = z^{\tilde{f}''_i}$. Since the last n cluster variables are frozen, $A'_{n+i} = A''_{n+i} = X_i$, $1 \le i \le n$.

The fan $\Sigma^{\mathbf{s}}_{w'}$ determining a toric variety in the atlas for $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ consists of a single cone spanned by $h'_1, \ldots, h'_n$, and

$$\mathrm{TV}(\Sigma^{\mathbf{s}}_{w'}) = (\mathbb{G}^n_m)_{A'_1,\ldots,A'_n} \times \mathbb{A}^n_{X_1,\ldots,X_n}.$$

Similarly

$$\mathrm{TV}(\Sigma^{\mathbf{s}}_{w''}) = (\mathbb{G}^n_m)_{A''_1,\ldots,A''_n} \times \mathbb{A}^n_{X_1,\ldots,X_n}.$$

The mutation $\mu_k : \mathrm{TV}(\Sigma^{\mathbf{s}}_{w'}) \dashrightarrow \mathrm{TV}(\Sigma^{\mathbf{s}}_{w''})$ is given by the exchange relation [FZ07, (2.15)] (see [GHK13, (2.8)] in our notation) which is, with $\tilde\epsilon = \tilde\epsilon^{w'}$,

$$\begin{aligned}
\mu_k^*(A_i'') &= A_i' \text{ for } i \neq k,\\
\mu_k^*(A_k'') &= (A_k')^{-1}\Big(\prod_{i=1}^{2n}(A_i')^{[\tilde\epsilon_{ki}]_+} + \prod_{i=1}^{2n}(A_i')^{-[\tilde\epsilon_{ki}]_-}\Big)\\
&= (A_k')^{-1}\Big(p_k^+\prod_{i=1}^{n}(A_i')^{[\tilde\epsilon_{ki}]_+} + p_k^-\prod_{i=1}^{n}(A_i')^{-[\tilde\epsilon_{ki}]_-}\Big),\\
\mu_k^*(X_i) &= X_i,
\end{aligned}$$

where

$$p_k^+ := \prod_{\substack{1\le i\le n\\ \tilde\epsilon_{k,n+i}\ge 0}} X_i^{\tilde\epsilon_{k,n+i}}, \quad p_k^- := \prod_{\substack{1\le i\le n\\ -\tilde\epsilon_{k,n+i}\ge 0}} X_i^{-\tilde\epsilon_{k,n+i}}.$$

Now μ_k fails to be an isomorphism exactly along the vanishing locus of

$$p_k^+\prod_{i=1}^{n}(A_i')^{[\tilde\epsilon_{ki}]_+} + p_k^-\prod_{i=1}^{n}(A_i')^{-[\tilde\epsilon_{ki}]_-}.$$

This locus is disjoint from the central fiber $0 \in \mathbb{A}^n_{X_1,\dots,X_n}$ by Corollary 5.3. On the other hand it is disjoint from the central fiber if and only if exactly one of p_k^+, p_k^- is the empty product, i.e., the constant monomial 1. Sign coherence is the statement that at least one of p_k^+, p_k^- is the empty product. $\square$

Recall from Definition 4.8 the notion of cluster monomial, and also note from Proposition B.2(2) the T_{N°-action on $\mathcal{A}_{\mathrm{prin}}$.

Definition 5.6. By Proposition B.11, the choice of seed $\mathbf{s}$ provides a canonical extension of each cluster monomial on $\mathcal{A}$ to a cluster monomial on $\mathcal{A}_{\mathrm{prin}}$. Each cluster monomial on $\mathcal{A}_{\mathrm{prin}}$ is a T_{N°-eigenfunction under the above T_{N° action. The *g-vector* with respect to a seed $\mathbf{s}$ (see [FZ07, (6.4)]) associated to a cluster monomial of $\mathcal{A}$ is the T_{N°-weight of its lift determined by $\mathbf{s}$.

We now give an alternative description of g-vectors, which will lead to a more intrinsic definition of g-vector (Definition 5.8). This in turn generalizes to all the different flavors of cluster varieties (Definition 5.10).

Proposition 5.7. *Fix a seed* $\mathbf{s}$*, giving the partial compactification* $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ *and* T_{N°*-equivariant* $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\dots,X_n}$. *The central fiber* $\pi^{-1}(0)$ *is a* T_{N°*-torsor. Let* $\overline{A}$ *be a cluster monomial on* $\mathcal{A} = \pi^{-1}(1,1,\dots,1)$*, and let* A *be the corresponding lifted cluster monomial on* $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$*. This restricts to a regular nonvanishing* T_{N°*-eigenfunction along* $\pi^{-1}(0)$ *and so canonically determines an element of* M° (*its weight*)*. This is the* g*-vector associated to* $\overline{A}$.

Proof. Let $w \in \mathfrak{T}_{\mathbf{s}}$ determine the seed in which $\overline{A}$ is defined as a monomial. By Corollary 5.3, all mutations are isomorphisms near the central fiber of π, so it is enough to check that $\overline{A}$ is regular on the toric variety $\mathrm{TV}(\Sigma^{\mathbf{s}}_w)$, and it restricts to a character on its central fiber. But this is true by construction: if the seed $\tilde s_w$ is

$(\tilde{e}_1, \ldots, \tilde{e}_{2n})$, then the cluster variables for the seed $\tilde{\mathbf{s}}_w$ on the torus $T_{\widetilde{N}^\circ,w}$ are $z^{\tilde{f}_k}$, and $\Sigma_w^{\mathbf{s}}$ is the fan with rays spanned by the $\tilde{e}_{n+1}, \ldots, \tilde{e}_{2n}$. Thus the lift A of $\overline{A}$ is regular on $\mathrm{TV}(\Sigma_w^{\mathbf{s}})$, and hence is regular in a neighborhood of $\pi^{-1}(0) \subset \mathcal{A}_{\mathrm{prin},\mathbf{s}}$. Furthermore, it is nonzero on $\pi^{-1}(0)$ since the canonical lift only involves monomials $z^{\tilde{f}_1}, \ldots, z^{\tilde{f}_n}$, which are nonvanishing on the strata of $\mathrm{TV}(\Sigma_w^{\mathbf{s}})$. The final statement follows since the restriction of the variable to the central fiber will have the same T_{N°-weight, as the map π is T_{N°-equivariant, and T_{N° fixes $0 \in \mathbb{A}^n$. $\square$

Definition 5.8. Writing $\mathcal{A} = \bigcup_{\mathbf{s}} T_{N^\circ,\mathbf{s}}$, let $\overline{A}$ be a cluster monomial of the form z^m on a chart $T_{N^\circ,\mathbf{s}'}$, $\mathbf{s}' = (e'_1, \ldots, e'_n)$. Note that $(z^{e'_i})^T(m) \leq 0$ for all i, so after identifying $\mathcal{A}^\vee(\mathbb{R}^T)$ with $M^\circ_{\mathbb{R},\mathbf{s}'}$, m yields a point in the Fock–Goncharov cluster chamber $\mathcal{C}^+_{\mathbf{s}'} \subseteq \mathcal{A}^\vee(\mathbb{R}^T)$, as defined in Lemma 2.10. We define $\mathbf{g}(\overline{A})$ to be this point of $\mathcal{C}^+_{\mathbf{s}'} \subseteq \mathcal{A}^\vee(\mathbb{R}^T)$.

Corollary 5.9. *Let $\overline{A}$ be a cluster monomial on $\mathcal{A}$, and fix a seed $\mathbf{s}$ giving an identification $\mathcal{A}^\vee(\mathbb{R}^T) = M^\circ_{\mathbb{R},\mathbf{s}}$. Then under this identification, $\mathbf{g}(\overline{A})$ is the g-vector of the cluster monomial $\overline{A}$ with respect to $\mathbf{s}$.*

Proof. We first note that if $\overline{A}$ is a monomial z^m on the chart $T_{N^\circ,\mathbf{s}'}$ with $\mathbf{s}' = \mathbf{s}_w$, $\mathbf{s} = \mathbf{s}_v$, then the image of $\mathbf{g}(\overline{A})$ under the identification $\mathcal{A}^\vee(\mathbb{R}^T) = M^\circ_{\mathbb{R},v}$ is $\mu^T_{w,v}(m)$, where as usual $\mu_{w,v} : T_{M^\circ,w} \dashrightarrow T_{M^\circ,v}$ is the rational map induced by the inclusions $T_{M^\circ,w}, T_{M^\circ,v} \subset \mathcal{A}^\vee$.

The choice of the seed $\mathbf{s}$ gives the lift of $\overline{A}$ to a cluster monomial A on $\mathcal{A}_{\mathrm{prin}}$. Using the identification of $\mathcal{A}_{\mathrm{prin}}$ with $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$, A is identified with a monomial of the form $z^{(m',n')}$ on the chart $T_{\widetilde{N}^\circ,w}$ (or $T_{\widetilde{N}^\circ,\mathcal{C}^+_{w\in\mathbf{s}}}$, depending on how one chooses to parameterize charts of $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$). Let v be the root of $\mathfrak{T}_{\mathbf{s}}$. By Lemma 5.2, the corresponding chart of $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$ is the toric variety defined by the fan $\Sigma^{\mathbf{s}}_{\mathrm{scat},w}$. By Proposition 5.7, A is a regular function on $\mathrm{TV}(\Sigma^{\mathbf{s}}_{\mathrm{scat},w})$ which is nonvanishing along $\pi^{-1}(0)$. The T_{N°-weight is the g-vector. Since $\Sigma^{\mathbf{s}}_{\mathrm{scat},w}$ is the cone spanned by $(0,e_1^*), \ldots, (0,e_n^*)$ in $\widetilde{N}^\circ_{\mathbb{R}}$, where $\mathbf{s} = (e_1, \ldots, e_n)$, one sees that $(m',n') = (g,0)$.

Thus to show the corollary, it is enough to show that $m = \mu^T_{v,w}(g) \in M^\circ_w$. Note however a similar statement is already true at the level of $\mathcal{A}_{\mathrm{prin}}$. Indeed, in the chart $T_{\widetilde{N}^\circ,w}$ of $\mathcal{A}_{\mathrm{prin}}$, the monomial A takes the form $z^{(m,n'')}$ for some $n'' \in N$, and (m,n'') lies in the positive chamber of $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}_w}$. But $\mathcal{C}^+_{w\in\mathbf{s}}$ is the image of this positive chamber under the map $\mu^T_{w,v}$, where now $\mu_{w,v} : T_{\widetilde{M}^\circ,w} \dashrightarrow T_{\widetilde{M}^\circ,v}$ is the map induced by the inclusions $T_{\widetilde{M}^\circ,v}, T_{\widetilde{M}^\circ,w} \subset \mathcal{A}^\vee_{\mathrm{prin}}$. Now $(g,0) = (\psi^*_{v,w})^{-1}(m,n'')$ by Theorem 4.4, and $(\psi^*_{v,w})^{-1} = (\mu^T_{v,w}|_{\mathcal{C}^+_{w\in\mathbf{s}}})^{-1}$, so we see that $(m,n'') = \mu^T_{v,w}(g,0)$.

Now because there is a well-defined map $\rho : \mathcal{A}^\vee_{\mathrm{prin}} \to \mathcal{A}^\vee$ by Proposition B.2(4), with ρ^T given by projection onto M°, this projection ρ^T is compatible with the tropicalizations $\mu^T_{v,w} : \widetilde{M}^\circ \to \widetilde{M}^\circ$ and $\mu^T_{v,w} : M^\circ \to M^\circ$, i.e., $\mu^T_{v,w} \circ \rho^T = \rho^T \circ \mu^T_{v,w}$. Thus $\mu^T_{v,w}(m) = g$, as desired. $\square$

This corollary shows us how to generalize the notion of g-vector to any cluster variety:

Definition 5.10. Let $V = \bigcup_{\mathbf{s}} T_{L,\mathbf{s}}$ be a cluster variety, suppose that f is a global monomial (see Definition 0.1) on V, and let $\mathbf{s}$ be a seed such that $f|_{T_{L,\mathbf{s}} \subset V}$ is the character z^m, $m \in \mathrm{Hom}(L, \mathbb{Z}) = L^*$. Define the *g-vector* of f to be the image of m under the identifications of §2:

$$V^\vee(\mathbb{Z}^T) = T_{L^*,\mathbf{s}}(\mathbb{Z}^T) = L^*.$$

We write the g-vector of f as $\mathbf{g}(f)$.

Note that the definition as given is not clearly independent of the choice of seed $\mathbf{s}$, but for a cluster variety of $\mathcal{A}$ type, the previous corollary shows this. This independence will be shown in general in Lemma 7.10.

By [NZ], the sign coherence for c-vectors (proved in Corollary 5.5 here), implies a sign coherence for g-vectors. Here we give a much shorter proof using the above description of g-vectors.

Theorem 5.11 (Sign coherence of g-vectors). *Fix initial seed $\mathbf{s} = (e_1, \ldots, e_n)$, with $f_i = d_i^{-1} e_i^*$ as usual. Given any mutation equivalent seed $\mathbf{s}'$, the ith-coordinates of the g-vectors for the cluster variables of this seed, expressed in the basis $(f_1, \ldots, f_n)$, are either all nonnegative or all nonpositive.*

Proof. By Corollary 5.9, the g-vectors in question are the generators of a chamber in the cluster complex of $\mathbf{s}$, defined as the images of the cluster chambers of $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\mathrm{prin}}}$ under the projection ρ^T, by Theorem 2.13. The hyperplanes $e_i^\perp$ are thus walls in the cluster complex. In particular, e_i is either nonnegative everywhere on a chamber or nonpositive everywhere on a chamber. The theorem follows. □

For future reference, we record the relationship between c-vectors and the cluster chambers in the case of no frozen variables. Fix a seed $\mathbf{s}$. By Lemma 2.10, each mutation equivalent seed $\mathbf{s}' = \mathbf{s}_w$ has an associated cluster chamber $\mathcal{C}^+_{w\in\mathbf{s}} \subset M^\circ_{\mathbb{R},\mathbf{s}}$. This is a full-dimensional strictly simplicial cone, generated by a basis of M° consisting of g-vectors of the cluster variables $A'_1, \ldots, A'_n$ of $\mathbf{s}_w$. The facets of $\mathcal{C}^+_{w\in\mathbf{s}}$ are thus in natural bijection with the elements of $\mathbf{s}$ (or the indices in $I = I_{\mathrm{uf}}$).

Lemma 5.12. *The facet of $\mathcal{C}^+_{w\in\mathbf{s}}$ corresponding to $i \in I$ is the intersection of $\mathcal{C}^+_{w\in\mathbf{s}}$ with the orthogonal complement of the c-vector for the corresponding element of $\mathbf{s}_w^\vee$ (the corresponding mutation of the Langlands dual seed $\mathbf{s}^\vee$; see Appendix* A*). Furthermore, each c-vector for $\mathbf{s}_w^\vee$ is nonnegative on $\mathcal{C}^+_{w\in\mathbf{s}}$.*

Proof. This is the content of [NZ, Theorem 1.2] with the condition [NZ, (1.8)] holding by our Corollary 5.5. The g-vectors used in [NZ] are precisely the g-vectors of the cluster variables A'_i. □

6. The formal Fock–Goncharov conjecture

In this section we associate in a canonical way to every universal Laurent polynomial g on $\mathcal{A}_{\mathrm{prin}}$ a formal sum $\sum_{q\in\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)} \alpha(g)(q)\vartheta_q$, $\alpha(g)(q) \in \Bbbk$, which, roughly speaking, converges to g at infinity in each partial compactification $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$. To give such an expression for a single $\mathbf{s}$ is quite easy; see Propositions 6.4 and 6.5. The crucial point, remarkably, is that these coefficients are independent of $\mathbf{s}$; see Theorem 6.8, our alternative to the Fock–Goncharov conjecture (which fails in general). This establishes the connection between $\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ and $\mathrm{can}(\mathcal{A}_{\mathrm{prin}})$, and is key to one of our main technical results; see the proof of Proposition 8.22.

Choose a seed $\mathbf{s} = (e_1, \ldots, e_n)$. We let $X_i := z^{e_i}$, let $I_{\mathbf{s}} = (X_1, \ldots, X_n) \subset \Bbbk[X_1, \ldots, X_n]$, set

$$\mathbb{A}^n_{(X_1,\ldots,X_n),k} = \operatorname{Spec} \Bbbk[X_1, \ldots, X_n]/I_{\mathbf{s}}^{k+1}, \quad \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k} = \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \times_{\mathbb{A}^n_{X_1,\ldots,X_n}} \mathbb{A}^n_{(X_1,\ldots,X_n),k},$$

and write the map induced by $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$ also as

$$\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k} \to \mathbb{A}^n_{(X_1,\ldots,X_n),k}.$$

We use the notation $\mathrm{up}(Y) := H^0(Y, \mathcal{O}_Y)$ for a variety Y, so that, e.g., $\mathrm{up}(\mathcal{A})$ is the upper cluster algebra. We define

$$\widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} = \varprojlim \mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k}).$$

Note that for any $g \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, $z^n g \in \mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})$ for some monomial z^n in the X_i. This induces a canonical inclusion

$$\mathrm{up}(\mathcal{A}_{\mathrm{prin}}) \subset \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N], \tag{6.1}$$

where $N^+_{\mathbf{s}} \subset N$ is the monoid generated by $e_1, \ldots, e_n$. Let $\pi_N : \widetilde{M}^\circ \to N$ be the projection, and set

$$\widetilde{M}^{\circ,+}_{\mathbf{s}} = \pi_N^{-1}(N^+_{\mathbf{s}}).$$

Recall that a choice of seed $\mathbf{s} = (e_1, \ldots, e_n)$ determines a scattering diagram $\mathfrak{D}_{\mathbf{s}} = \mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}} \subset \widetilde{M}^\circ_{\mathbb{R},\mathbf{s}}$ with initial walls $(e_i^\perp, 1 + z^{(v_i,e_i)})$ for $i \in I_{\mathrm{uf}}$. We let $P_{\mathbf{s}} \subset \widetilde{M}^\circ_{\mathbf{s}}$ be the monoid generated by $(v_1, e_1), \ldots, (v_n, e_n)$. We have the cluster complex $\Delta^+_{\mathbf{s}}$ of cones in $\widetilde{M}^\circ_{\mathbb{R},\mathbf{s}}$, with cones $\mathcal{C}^+_v \in \Delta^+_{\mathbf{s}}$ for each vertex v of $\mathfrak{T}_{\mathbf{s}}$.

Similarly to the above discussion, $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$ induces maps

$$\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s},k} := \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}} \times_{\mathbb{A}^n} \mathbb{A}^n_{(X_1,\ldots,X_n),k} \to \mathbb{A}^n_{(X_1,\ldots,X_n),k}.$$

The isomorphism between $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ and $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$ discussed in Construction 5.1 restricts to give an isomorphism between $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k}$ and $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s},k}$. Furthermore, as $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$ is described by gluing charts isomorphic to $\mathrm{TV}(\Sigma)$ with Σ the cone generated by $(0, e_1^*), \ldots, (0, e_n^*)$ for every chart by Lemma 5.2, in fact $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s},k}$ is described by gluing charts parameterized by $\sigma \in \Delta^+_{\mathbf{s}}$ isomorphic to

$$V_{\mathbf{s},\sigma,k} := \mathrm{TV}(\Sigma) \times_{\mathbb{A}^n} \mathbb{A}^n_{(X_1,\ldots,X_n),k}.$$

Note for $\sigma, \sigma' \in \Delta^+_{\mathbf{s}}$, the birational map $\mathfrak{p}_{\sigma,\sigma'} : \mathrm{TV}(\Sigma) \dashrightarrow \mathrm{TV}(\Sigma)$ between the charts of $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$ indexed by σ and σ' restrict to isomorphisms $V_{\mathbf{s},\sigma,k} \to V_{\mathbf{s},\sigma',k}$; this is implied by Corollary 5.3(3).

We choose a generic basepoint $Q_\sigma \in \sigma$ for each $\sigma \in \Delta^+_{\mathbf{s}}$. Then for any $q \in \widetilde{M}^\circ_{\mathbf{s}}$, by Proposition 3.4, we obtain as a sum over broken lines a well-defined series

$$\vartheta_{Q_\sigma,q} \in z^q \widehat{\Bbbk[P_{\mathbf{s}}]}$$

satisfying by Theorem 3.5

$$\vartheta_{Q_\sigma,q} = \mathfrak{p}^*_{\sigma,\sigma'}(\vartheta_{Q_{\sigma'},q}).$$

The following definition will yield the structure constants for the theta functions:

Definition-Lemma 6.2. *Let $p_1, p_2, q \in \widetilde{M}^{\circ}_{\mathbf{s}}$. Let $z \in \widetilde{M}^{\circ}_{\mathbb{R},\mathbf{s}}$ be chosen generally. There are at most finitely many pairs of broken lines γ_1, γ_2 with $I(\gamma_i) = p_i$, $b(\gamma_i) = z$, and $F(\gamma_1) + F(\gamma_2) = q$ (see Definition 3.3 for this notation). We can then define*

$$\alpha_z(p_1, p_2, q) = \sum_{\substack{(\gamma_1,\gamma_2) \\ I(\gamma_i)=p_i, b(\gamma_i)=z \\ F(\gamma_1)+F(\gamma_2)=q}} c(\gamma_1)c(\gamma_2).$$

The integers $\alpha_z(p_1, p_2, q)$ are nonnegative.

Proof. By definition of scattering diagram for $\mathfrak{D}_{\mathbf{s}}$, all walls $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_{\mathbf{s}}$ have $f_{\mathfrak{d}} \in \widehat{\Bbbk[P_{\mathbf{s}}]}$. Note also that because $P_{\mathbf{s}}$ comes from a strictly convex cone, any element of $P_{\mathbf{s}}$ can only be written as a finite sum of elements in $P_{\mathbf{s}}$ in a finite number of ways. In particular, as $F(\gamma_i) \in I(\gamma_i) + P_{\mathbf{s}}$, we can write $F(\gamma_i) = I(\gamma_i) + m_i$ for $m_i \in P_{\mathbf{s}}$. Thus we have

$$I(\gamma_1) + I(\gamma_2) + m_1 + m_2 = q,$$

and there are only a finite number of possible m_1, m_2. So with p_1, p_2, q fixed, there are only finitely many possible monomial decorations that can occur on either γ_i. From this, finiteness is clear; cf. the proof of Proposition 3.4. The nonnegativity statement follows from Theorem 1.13, which implies $c(\gamma) \in \mathbb{Z}_{\geq 0}$ for any broken line γ. □

Definition 6.3. For a monoid $C \subset L$ a lattice, we write $C_k \subset C$ for the subset of elements which can be written as a sum of k noninvertible elements of C.

Proposition 6.4. *Notation as above. The following hold:*

(1) *For $q \in \widetilde{M}^{\circ,+}_{\mathbf{s}}$, $\vartheta_{Q_\sigma,q}$ is a regular function on $V_{\mathbf{s},\sigma,k}$, and the $\vartheta_{Q_\sigma,q}$ as σ varies glue to give a canonically defined function $\vartheta_{q,k} \in \operatorname{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k})$.*

(2) *For each $q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ and $k' \geq k$, we have $\vartheta_{q,k'}|_{\mathcal{A}^{\mathbf{s}}_{\mathrm{prin},k}} = \vartheta_{q,k}$, and thus the $\vartheta_{q,k}$ for $k \geq 0$ canonically define*

$$\vartheta_q \in \widehat{\operatorname{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N].$$

The ϑ_q are linearly independent; i.e., we have a canonical inclusion of $\Bbbk$-vector spaces

$$\operatorname{can}(\mathcal{A}_{\mathrm{prin}}) := \bigoplus_{q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)} \Bbbk \cdot \vartheta_q \subset \widehat{\operatorname{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N].$$

(3)

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{q \in \widetilde{M}^{\circ}_{\mathbf{s}}} \alpha_{z(q)}(p_1, p_2, q)\vartheta_q \in \widehat{\operatorname{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N]$$

for $z(q)$ chosen sufficiently close to q. In particular, $\alpha_z(p_1, p_2, q)$ is independent of the choice of z sufficiently near q, and we define

$$\alpha(p_1, p_2, q) := \alpha_z(p_1, p_2, q)$$

for z chosen sufficiently close to q.

(4)

$$\{\vartheta_q \mid q \in \widetilde{M}^{\circ,+}_{\mathbf{s}} \setminus \widetilde{M}^{\circ,+}_{\mathbf{s},k+1}\} \text{ and } \{\vartheta_q \mid q \in \pi_N^{-1}(0)\}$$

restrict to bases of $\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k})$ *as a* $\Bbbk$*-vector space and* $\Bbbk[N^+_{\mathbf{s}}]/I^{k+1}_{\mathbf{s}}$*-module, respectively.*

Proof. Using the isomorphism $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ with $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s}}$, the basic compatibility Theorem 3.5 gives the gluing statement (1). To prove (4), using the N-linearity, it is enough to prove the given ϑ_q restrict to basis as $\Bbbk[N^+_{\mathbf{s}}]/(X_1,\ldots,X_n)^{k+1}$-module. By Corollary 5.3, the central fiber of $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$ is the torus T_{N°. If $q \in \pi_N^{-1}(0)$, the only broken lines with $q = I(\gamma)$ and $\mathrm{Mono}(\gamma) \notin (X_1,\ldots,X_n)$ are straight lines. Thus these ϑ_q restrict to the basis of characters on the central fiber. Now the result follows from the nilpotent Nakayama lemma (see [Ma89, pg. 58, Theorem 8.4]).

(2) follows immediately from (1) and (4).

For (3), it is enough to prove the equality in $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k}$ for each k. The argument is the same as the proof of the multiplication rule in [GHK11, Theorem 2.38], which, as it is very short, we recall for the reader's convenience. We work with the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ modulo $I^{k+1}_{\mathbf{s}}$, which has only finitely many walls with nontrivial attached function. Expressing $\vartheta_{p_1}\cdot\vartheta_{p_2}$ in the basis $\{\vartheta_q\}$ of (4), we examine the coefficient of ϑ_q. We choose a general point $z \in \widetilde{M}^\circ_{\mathbb{R}}$ very close to q, so that z, q lie in the closure of the same connected component of $\widetilde{M}^\circ_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_{\mathbf{s},k})$ (where $\mathfrak{D}_{\mathbf{s},k}$ denotes the finitely many walls nontrivial modulo $I^{k+1}_{\mathbf{s}}$). By definition of α_z,

$$\vartheta_{z,p_1}\cdot\vartheta_{z,p_2} = \sum_r \alpha_z(p_1,p_2,r)z^r.$$

Now observe first that there is only one broken line γ with endpoint z and $F(\gamma)=q$: this is the broken line whose image is $z+\mathbb{R}_{\geq 0}q$. Indeed, the final segment of such a γ is on this ray, and this ray meets no walls, other than walls containing q, so the broken line cannot bend. Thus the coefficient of $\vartheta_{z,q}$ can be read off by looking at the coefficient of the monomial z^q on the right-hand side of the above equation. This gives the desired formula to order k. The finiteness argument of Definition-Lemma 6.2 then shows that for any given q, z chosen sufficiently close to q will work for all k. $\square$

By the proposition, each $g \in \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})}$ has a unique expression as a convergent formal sum $\sum_{q\in\widetilde{M}^{\circ,+}_{\mathbf{s}}} \alpha_{\mathbf{s}}(g)(q)\vartheta_q$, with coefficients $\alpha_{\mathbf{s}}(g)(q)\in\Bbbk$. This immediately implies:

Proposition 6.5. *Notation as in Proposition* 6.4. *There is a unique inclusion*

$$\alpha_{\mathbf{s}} : \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N] \hookrightarrow \mathrm{Hom}_{\mathrm{sets}}(\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) = \widetilde{M}^\circ_{\mathbf{s}}, \Bbbk)$$

given by

$$g \mapsto (q \mapsto \alpha_{\mathbf{s}}(g)(q)).$$

We have $\alpha_{\mathbf{s}}(z^n\cdot g)(q+n) = \alpha_{\mathbf{s}}(g)(q)$ *for all* $n\in N$.

Definition 6.6. For $g \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, write $g = \sum_{q\in\widetilde{M}^\circ_{\mathbf{s}}} \beta_s(g)(q)z^q$ on the torus chart $T_{\widetilde{N}^\circ,\mathbf{s}}$ of $\mathcal{A}_{\mathrm{prin}}$ corresponding to a seed $\mathbf{s}$. We also have a formal expansion

$g = \sum_{q\in\widetilde{M}^\circ_{\mathbf{s}}} \alpha_{\mathbf{s}}(g)(q)\vartheta_q$ as $z^m g \in \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})}$ for some $m \in \widetilde{M}^\circ_{\mathbf{s}}$. Set

$$\overline{S}_{g,\mathbf{s}} = \{q \in \widetilde{M}^\circ_{\mathbf{s}} \,|\, \beta_{\mathbf{s}}(g)(q) \neq 0\},$$
$$S_{g,\mathbf{s}} = \{q \in \widetilde{M}^\circ_{\mathbf{s}} \,|\, \alpha_{\mathbf{s}}(g)(q) \neq 0\}.$$

If $P_{\mathbf{s}}$ is the monoid generated by $\{(v_i, e_i) \,|\, i \in I_{\mathrm{uf}}\}$, it is easy to check from the construction of theta functions that

$$S_{g,\mathbf{s}} \subseteq \overline{S}_{g,\mathbf{s}} + P_{\mathbf{s}}.$$

Remark 6.7. Note that $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s},k}$ is constructed by gluing together various $V_{\mathbf{s},\sigma,k} \cong T_{M^\circ} \times \operatorname{Spec} \Bbbk[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^{k+1}$ via isomorphisms, and hence $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{scat},\mathbf{s},k} \cong T_{M^\circ} \times \operatorname{Spec} \Bbbk[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^{k+1}$. Thus by Proposition 6.4(4), a collection of the ϑ_q yields a basis for regular functions on $T_{M^\circ} \times \operatorname{Spec} \Bbbk[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^{k+1}$ with the property that this basis restricts to a monomial basis on the underlying reduced space. It remains a mystery about theta functions in general whether they satisfy some other interesting characterizing properties, such as the heat equation satisfied by theta functions on abelian varieties.

The main point of the following theorem is that on $\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, $\alpha_{\mathbf{s}}$ is independent of the seed $\mathbf{s}$.

Theorem 6.8. *There is a unique function*

$$\alpha : \mathrm{up}(\mathcal{A}_{\mathrm{prin}}) \to \mathrm{Hom}_{\mathrm{sets}}(\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T), \Bbbk),$$

with all the following properties:

(1) *α is compatible with the $\Bbbk[N]$-module structure on $\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ and the N-translation action on $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ in the sense that*
$$\alpha(z^n \cdot g)(x + n) = \alpha(g)(x)$$
for all $g \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, $n \in N$, $x \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$.

(2) *For each choice of seed $\mathbf{s}$, the formal sum $\sum_{q\in\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)} \alpha(g)(q)\vartheta_q$ converges to g in $\widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N]$.*

(3) *If $z^n \cdot g \in \mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})$, then $\alpha(z^n \cdot g)(q) = 0$ unless $\pi_N(q) \in N^+_{\mathbf{s}}$, and*
$$z^n \cdot g = \sum_{\pi_{N,\mathbf{s}}(q)\in N^+_{\mathbf{s}}\setminus(N^+_{\mathbf{s}})_{k+1}} \alpha(z^n \cdot g)(q)\vartheta_q \mod I^{k+1}_{\mathbf{s}}$$
and the coefficients $\alpha(z^n \cdot g)(q)$ are the coefficients for the expansion of $z^n \cdot g$ viewed as an element of $\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin},k})$ in the basis of theta functions from Proposition 6.4.

(4) *For any seed $\mathbf{s}'$ obtained via mutations from $\mathbf{s}$, α is the composition of the inclusions*
$$\mathrm{up}(\mathcal{A}_{\mathrm{prin}}) \subset \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}'}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}'}]} \Bbbk[N] \subset \mathrm{Hom}_{\mathrm{sets}}(\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) = \widetilde{M}^\circ_{\mathbf{s}'}, \Bbbk)$$
given by (6.1) *and Proposition* 6.5*. This sends a cluster monomial $A \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ to the delta function $\delta_{\mathbf{g}(A)}$ for its g-vector $\mathbf{g}(A) \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$.*

In the notation of Definition 6.6*, $\alpha(g)(m) = \alpha_{\mathbf{s}'}(g)(m)$ for any seed $\mathbf{s}'$. In particular the sets $S_{g,\mathbf{s}'} \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ of Definition* 6.6 *are independent of the seed, depending only on g.*

Proof. It is easy to see from Proposition 6.4 that $\alpha_{\mathbf{s}}$ is the unique function which satisfies conditions (1)–(3) of Theorem 6.4 for the given seed $\mathbf{s}$. Moreover, it satisfies condition (4) for $\mathbf{s} = \mathbf{s}'$. Thus it is enough to show that $\alpha_{\mathbf{s}}$ is independent of the choice of seed.

The basic idea is that $\alpha_{\mathbf{s}}$ expresses g as a sum of theta functions. As the theta functions are linearly independent, the expression is unique. But as the sums can be infinite, we make the argument in the appropriate formal neighborhood.

For a seed $\mathbf{s} = (e_i \,|\, i \in I)$, we write $\Sigma^{\mathbf{s}}$ for the fan in $\widetilde{N}^\circ = N^\circ \oplus M$ with rays spanned by the $(0, d_i f_i)$. We write $\bar{\Sigma}^{\mathbf{s}}$ for the fan in M with rays spanned by the $d_i f_i$.

Clearly, for the invariance it is enough to consider two adjacent seeds, say $\mathbf{s} = (e_i \,|\, i \in I)$ and $\mathbf{s}' = (e_i' \,|\, i \in I)$ obtained, without loss of generality, by mutation of the first basis vector e_1.

We consider the union of the two tori $T_{\widetilde{N}^\circ, \mathbf{s}}, T_{\widetilde{N}^\circ, \mathbf{s}'}$ in the atlas for $\mathcal{A}_{\text{prin}}$, glued by the mutation μ_1, which we recall is given by

$$\mu_1^* : z^{(m,n)} \mapsto z^{(m,n)} \cdot (1 + z^{(v_1, e_1)})^{-\langle (d_1 e_1, 0), (m,n) \rangle},$$

where $(m,n) \in \widetilde{M}^\circ = M^\circ \oplus N$; see [GHK13, (2.6)]. We will partially compactify this union by gluing the toric varieties

$$\mu_1 : \mathrm{TV}(\Sigma^{\mathbf{s}}) \dashrightarrow \mathrm{TV}(\Sigma^{\mathbf{s}'}),$$

writing

$$U := \mathrm{TV}(\Sigma^{\mathbf{s}}) \cup \mathrm{TV}(\Sigma^{\mathbf{s}'})$$

under this gluing. Note this union of toric varieties is *not* part of the atlas for either $\overline{\mathcal{A}}_{\text{prin}}^{\mathbf{s}}$ or $\overline{\mathcal{A}}_{\text{prin}}^{\mathbf{s}'}$ (for either of these, the fans determining the atlases for the toric compactifications are related by geometric tropicalization of the birational mutation, but here $\mu_1^t(f_1) \neq f_1'$, and thus $\Sigma^{\mathbf{s}'} \neq \mu_1^t(\Sigma^{\mathbf{s}})$).

Note $f_i' = f_i$ for $i \neq 1$, while $f_1' = -f_1 + \sum_j [\{e_j, e_1\}]_+ d_j f_j$, (see, e.g., [GHK13, (2.3)]). Thus the two cones $\bar{\Sigma}^{\mathbf{s}}, \bar{\Sigma}^{\mathbf{s}'}$ share a codimension 1 face and form a fan, $\bar{\Sigma}$. Let $V = \mathrm{TV}(\bar{\Sigma})$. By construction the rational map $\mathrm{TV}(\Sigma^{\mathbf{s}}) \to \mathrm{TV}(\bar{\Sigma}^{\mathbf{s}})$ is regular, and the same holds for the seed $\mathbf{s}'$. Observe that μ_1 commutes with the second projection $\pi : T_{\widetilde{N}^\circ} \to T_M$. From this it follows that $\pi : U \to V$ is regular. Note the toric boundary ∂V has a unique complete one-dimensional stratum $\mathbb{P}^1$ and two zero strata $0_{\mathbf{s}}$, $0_{\mathbf{s}'}$, whose complements in the $\mathbb{P}^1$ we write as $\mathbb{A}^1_{\mathbf{s}'}, \mathbb{A}^1_{\mathbf{s}}$, respectively. We write, e.g., $\mathbb{A}^1_{\mathbf{s},k} \subset V$ for the kth-order neighborhood of this curve, and, e.g., $U_{\mathbb{A}^1_{\mathbf{s}},k}$ for the scheme-theoretic inverse image $\pi^{-1}(\mathbb{A}^1_{\mathbf{s},k}) \subset U$. Finally, let

$$U_{\mathbb{G}_m,k} = U_{\mathbb{A}^1_{\mathbf{s}},k} \cap U_{\mathbb{A}^1_{\mathbf{s}'},k} \subset U.$$

We will show that theta functions give a basis of functions on these formal neighborhoods. To make the computation transparent, we introduce coordinates.

We let $X_i := z^{(0,e_i)}$, $X_i' := z^{(0,e_i')}$, observing that $\mu_1^*(z^{(0,n)}) = z^{(0,n)}$ for all $n \in N$. In particular $\mu_1^*(X_i) = X_i$, $\mu_1^*(X_i') = X_i'$. Since there is a map of fans from $\bar{\Sigma}$ to the fan defining $\mathbb{P}^1$ by dividing out by the subspace spanned by $\{d_i f_i \,|\, i \in I \setminus \{1\}\}$, there is a map $V \to \mathbb{P}^1$. We can pull back $\mathcal{O}_{\mathbb{P}^1}(1)$ to V, getting a line bundle with monomial sections X, X' pulled back from $\mathbb{P}^1$ with $X'/X = X_1'$ in the above notation. The open subset of U where $X' \neq 0$ is given explicitly up to codimension 2 by the hypersurface

$$A_1 \cdot A_1' = X_1 \prod_{j:\epsilon_{1j}\geq 0} A_i^{\epsilon_{1j}} + \prod_{j:\epsilon_{1j}\leq 0} A_j^{-\epsilon_{1j}} \subset \mathbb{A}^n_{X_1,\ldots,X_n} \times \mathbb{A}^2_{A_1,A_1'} \times (\mathbb{G}_m)^{n-1}_{A_2,\ldots,A_n},$$

where $A_i = z^{(f_i,0)}$ and $A_1' = z^{(f_1',0)}$.

Note the points

$$(f_i,0),(0,e_i) \in (M^\circ \oplus N)_{\mathbf{s}} = \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T), \quad (f_1',0) \in (M^\circ \oplus N)_{\mathbf{s}'} = \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$$

lie in the chambers of $\Delta^+_{\mathbf{s}}$ corresponding to $\mathbf{s}$ and $\mathbf{s}'$, respectively, and thus by Proposition 3.8 these points determine theta functions in $\operatorname{up}(\mathcal{A}_{\text{prin}})$, which are of course the corresponding cluster monomials A_i, X_i, A_1'.

We have the exactly analogous description for the open subset $X \neq 0$.

Next observe that all but one of the functions attached to walls in $\mathfrak{D}_{\mathbf{s}}$ is trivial modulo the ideal $J = (X_i \,|\, i \in I \setminus \{1\})$. Indeed, the unique nontrivial wall is $((e_1,0)^\perp, 1+z^{(v_1,e_1)})$. It follows from Theorem 1.28 that modulo J^k the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ has only finitely many nontrivial walls, and $\vartheta_{Q,m}$ is regular on $U_{\mathbb{A}^1_{\mathbf{s}},k}$, for Q a basepoint in the distinguished chamber $\mathcal{C}^+_{\mathbf{s}}$, so long as $\pi_N(m) \in \operatorname{Span}(e_1,\ldots,e_n)$, $\pi_N : \widetilde{M}^\circ \to N$ the projection.

Let $C := \sum_{k=1}^n \mathbb{N}e_i$, $C' := \sum_{k=1}^n \mathbb{N}e_i'$. Noting $e_1' = -e_1$, we can set

$$\widetilde{C} := \mathbb{Z}e_1 + \sum_{k=2}^n \mathbb{N}e_k = \mathbb{Z}e_1' + \sum_{k=2}^n \mathbb{N}e_k'.$$

Observe $U_{\mathbb{G}_m,k}$ is the subscheme of U defined by the ideal J^k in the open subset $XX' \neq 0 \subset U$. Note that the open subset of U defined by $XX' \neq 0$, $\prod_{i\neq 1} X_i \neq 0$ is the union of the two tori $T_{\widetilde{N}^\circ,\mathbf{s}}, T_{\widetilde{N}^\circ,\mathbf{s}'}$.

Claim 6.9. *The following hold:*

(1) *The collection* $\vartheta_{Q,m}$, $m \in \widetilde{M}^\circ$, $\pi_N(m) \in C \setminus (\widetilde{C}_{k+1} \cap C)$ *forms a* $\Bbbk$*-basis of the vector space* $\operatorname{up}(U_{\mathbb{A}^1_{\mathbf{s},k}})$.

(2) *The collection* $\vartheta_{Q,(m,0)}$, $m \in M^\circ$, *forms a basis of* $\operatorname{up}(U_{\mathbb{A}^1_{\mathbf{s},k}})$ *as an* $H^0(\mathbb{A}^1_{\mathbf{s},k}, \mathcal{O}_{\mathbb{A}^1_{\mathbf{s},k}})$*-module.*

(3) *The collection* $\vartheta_{Q,m}$, $\pi_N(m) \in \widetilde{C} \setminus \widetilde{C}_k$, *forms a* $\Bbbk$*-basis of* $\operatorname{up}(U_{\mathbb{G}_m,k})$.

Proof. (2) implies (1) using the N-linearity of the scattering diagram and multiplication rule with respect to the N-translation. Similarly, (2) implies (3) by inverting X_1.

For the second claim, by [GHK11, Lemma 2.30], we need only prove the statement for $k = 0$. To prove linear independence it is enough to show linear independence modulo $(X_1^r, X_2, \ldots, X_n)$ for all r. For this, again by [GHK11, Lemma 2.30], it is enough to check just over the fiber $X_1 = \cdots = X_n = 0$. This is the torus T_{N°, and the theta functions restrict to the basis of characters.

So it remains only to show the given theta functions generate modulo J. Here we use the explicit description of the open subset of U where $X' \neq 0$ above. This is an affine variety, and the ring of functions is clearly generated by the $A_1, A_1', A_2^{\pm 1}, \ldots, A_n^{\pm 1}$ as a $\Bbbk[X_1,\ldots,X_n]$-algebra. On the other hand, by the explicit description of $\mathfrak{D}_{\mathbf{s}}$ modulo the ideal J, for $m = \sum a_i f_i = \sum a_i' f_i' \in M^\circ$,

$$\vartheta_{Q,(m,0)} = \begin{cases} \prod_i A_i^{a_i}, & a_1 \geq 0, \\ (A_1')^{a_1'} \prod_{i\neq 1} A_i^{a_i'}, & a_1 = -a_1' \leq 0. \end{cases}$$

This shows theta functions generate $\mathrm{up}(U_{\mathbb{A}^1_{\mathbf{s},0}})$ as an $H^0(\mathbb{A}^1_{\mathbf{s},0}, \mathcal{O}_{\mathbb{A}^1_{\mathbf{s},0}})$-module, hence the result. $\square$

Of course there is an analogous claim for $\mathbf{s}'$.

Now we can prove $S_{g,\mathbf{s}} = S_{g,\mathbf{s}'}$ for $g \in \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$.

By the N-translation action on $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ (and the corresponding N-linearity of the scattering diagrams), to prove the equality, we are free to multiply g by a monomial from the base of $\mathcal{A}_{\mathrm{prin}} \to T_M$. Multiplying by a monomial in the X_i, $i \neq 1$, we can then assume g is a regular function on the open subset of U where $XX' \neq 0$. Now in the notation of Definition 6.6, $\pi_N(m) \in \widetilde{C}$ for $m \in P_{\mathbf{s}} + \overline{S}_{g,\mathbf{s}}$ or $m \in P_{\mathbf{s}'} + \overline{S}_{g,\mathbf{s}'}$. It follows now from the fact that $\mathfrak{D}_{\mathbf{s}}$ is finite modulo J^k for any k that each $\vartheta_{Q,m}, \vartheta_{Q',m}$ for $m \in S_{g,\mathbf{s}}, S_{g,\mathbf{s}'}$ is a finite Laurent polynomial modulo J^k. Here Q, Q' are basepoints in the chambers indexed by $\mathbf{s}$ and $\mathbf{s}'$, respectively.

Claim 6.10. *Modulo J^k, the sums $\sum_{m \in S_{g,\mathbf{s}}} \alpha_m \vartheta_{Q,m}$, $\sum_{m \in S_{g,\mathbf{s}'}} \alpha'_m \vartheta_{Q',m}$ are finite and coincide with g in the charts indexed by $\mathbf{s}$ and $\mathbf{s}'$, respectively.*

Proof. By symmetry it is enough to treat $\mathbf{s}$. We can multiply both sides by a power of X_1, and so we may assume g is regular on $X' \neq 0$ and $\pi_N(m) \in C$ for each $\alpha_m \neq 0$. Note $\pi_N(P_{\mathbf{s}} \setminus \{0\}) = C \setminus \{0\}$, thus by construction modulo $(X_1, \ldots, X_n)^r$ for any r the sum $\sum_{m \in S_{g,\mathbf{s}}} \alpha_m \vartheta_{Q,m}$ is finite and equal to g. By Claim 6.9(1), we have a (finite) expression modulo J^k,

$$g = \sum_{\pi_N(m) \in C \setminus (C \cap \widetilde{C}_{k+1})} \beta(m) \vartheta_{Q,m}.$$

Thus, for fixed k and arbitrary $r \geq 1$, we have modulo $J^k + (X_1^r)$,

$$\begin{aligned} g &= \sum_{\pi_N(m) \in C \setminus (C \cap \widetilde{C}_{k+1})} \beta(m) \vartheta_{Q,m} \\ &= \sum_{m \in S_{g,\mathbf{s}}} \alpha_m \vartheta_{Q,m}. \end{aligned}$$

By linear independence these expressions are the same, for all r, thus the expressions are the same modulo J^k. $\square$

Note that by Theorem 3.5, for $m \in \pi_N^{-1}(\widetilde{C})$, $\vartheta_{Q,m}$ and $\vartheta_{Q',m}$ induce the same regular function ϑ_m on $U_{\mathbb{G}_m,k}$. Thus we have by Claim 6.10 that

$$g = \sum_{m \in S_{g,\mathbf{s}}} \alpha_m \vartheta_m = \sum_{m \in S_{g,\mathbf{s}'}} \alpha'_m \vartheta_m \mod J^k.$$

Now by Claim 6.9(3) (varying k) the coefficients in the sums are the same. $\square$

The theorem implies that the theta functions are a topological basis for a natural topological $\mathbb{k}$-algebra completion of $\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$:

Corollary 6.11. *For $n \in N$, let*

$$n^* : \mathrm{Hom}_{\mathrm{sets}}(\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T), \mathbb{k}) \to \mathrm{Hom}_{\mathrm{sets}}(\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T), \mathbb{k})$$

denote precomposition by the action of translation by n on $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$. Let

$$\overline{\mathrm{up}(\mathcal{A}_{\mathrm{prin}})} \subset \mathrm{Hom}_{\mathrm{sets}}(\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T), \mathbb{k})$$

be the vector subspace of functions f such that for each seed $\mathbf{s}$, there exists $n \in N$ for which the restriction of $n^*(f)$ to $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) \setminus \pi^{-1}_{N,\mathbf{s}}((N^+_{\mathbf{s}})_k)$ has finite support for all $k > 0$. Then we have

$$\mathrm{up}(\mathcal{A}_{\mathrm{prin}}) \subset \overline{\mathrm{up}(\mathcal{A}_{\mathrm{prin}})} = \bigcap_{\mathbf{s}} \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N] \subset \mathrm{Hom}_{\mathrm{sets}}(\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T), \Bbbk).$$

$\overline{\mathrm{up}(\mathcal{A}_{\mathrm{prin}})}$ *is a complete topological vector space under the weakest topology so that each inclusion* $\overline{\mathrm{up}(\mathcal{A}_{\mathrm{prin}})} \subset \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N]$ *is continuous. Let* $\vartheta_q = \delta_q \in \overline{\mathrm{up}(\mathcal{A}_{\mathrm{prin}})}$ *be the delta function associated to* $q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$. *The* ϑ_q *are a topological basis for* $\overline{\mathrm{up}(\mathcal{A}_{\mathrm{prin}})}$. *There is a unique topological* $\Bbbk$*-algebra structure on* $\overline{\mathrm{up}(\mathcal{A}_{\mathrm{prin}})}$ *such that* $\vartheta_p \cdot \vartheta_q = \sum_r \alpha(p,q,r)\vartheta_r$ *with structure constants given by Definition-Lemma* 6.2.

7. The middle cluster algebra

In this section, we prove one of the main theorems of the paper, Theorem 0.3. This is done in two steps. First, it follows from the results of the previous section and properties of theta functions in the $\mathcal{A}_{\mathrm{prin}}$ case. This is easiest since the scattering diagram technology works best for $\mathcal{A}_{\mathrm{prin}}$. Second, we descend to the $\mathcal{A}$ or $\mathcal{A}_t$ case and the $\mathcal{X}$ case, with the $\mathcal{A}$-type varieties appearing as fibers of $\mathcal{A}_{\mathrm{prin}} \to T_M$ and the $\mathcal{X}$ variety as a quotient of $\mathcal{A}_{\mathrm{prin}}$ by T_{N°, using the $\mathcal{A}_{\mathrm{prin}}$ case to deduce the result for these other cases.

7.1. **The middle algebra for $\mathcal{A}_{\mathrm{prin}}$.** Recall from Definition 1.32 that $\Delta^+_{\mathbf{s}}$ is the collection of chambers forming the cluster complex. Abstractly, by Lemma 2.10, this can be viewed as giving a collection of chambers Δ^+ in $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$.

Proposition 7.1. *Choose $m_0 \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$. If for some generic basepoint $Q \in \sigma \in \Delta^+$ there are only finitely many broken lines γ with $I(\gamma) = m_0$ and $b(\gamma) = Q$, then the same is true for any generic $Q' \in \sigma' \in \Delta^+$. In particular, $\vartheta_{Q,m_0} \in \Bbbk[\widetilde{M}^\circ]$ is a positive universal Laurent polynomial.*

Proof. By positivity of the scattering diagram, Theorem 1.13, for any basepoint Q, ϑ_{Q,m_0} has only nonnegative coefficients (though it may have infinitely many terms). Also, we know that for basepoints in different chambers, the ϑ_{Q,m_0} are related by wall crossings by Theorem 3.5, which in turn are identified with the mutations of tori in the atlas for $\mathcal{A}_{\mathrm{prin}}$. So the ϑ_{Q,m_0} determine a universal positive Laurent polynomial if and only if we have finiteness of broken lines ending at any Q in any chamber of $\Delta^+_{\mathbf{s}}$. If we vary Q in the chamber, ϑ_{Q,m_0} does not change. So it is enough to check that if ϑ_{Q,m_0} is a polynomial, the same is true of ϑ_{Q',m_0} for Q' in an adjacent chamber σ' to σ close to the wall $\sigma \cap \sigma'$. We can work in some seed. Let the wall be contained in $n_0^\perp$, $n_0 \in \widetilde{N}^\circ$, with $\langle n_0, Q\rangle > 0$, and denote the wall-crossing automorphism from Q to Q' as $\mathfrak{p}$. Recall that $\mathfrak{p}(z^m) = z^m f^{\langle n_0, m\rangle}$, where for walls between cluster chambers, f is some positive Laurent polynomial (in fact it has the form $1 + z^q$ for some $q \in n_0^\perp \subset \widetilde{M}^\circ$).

Monomials $m \in \widetilde{M}^\circ$ are then divided into three groups, according to the sign of $\langle n_0, m\rangle$. This sign is preserved by $\mathfrak{p}$, as n_0 takes the same value on each exponent of a monomial term in $\mathfrak{p}(z^m)$ as n_0 takes on m.

Monomials with $\langle n_0, m\rangle = 0$ are then invariant under $\mathfrak{p}$, so these terms in ϑ_{Q',m_0} coincide with those in ϑ_{Q,m_0}. Hence there are only a finite number of such terms in ϑ_{Q',m_0}.

The sum of terms of the form cz^m in ϑ_{Q,m_0} with $\langle n_0, m\rangle > 0$, which we know forms a Laurent polynomial, is, by the explicit formula for $\mathfrak{p}$, sent to a polynomial. So it only remains to show that there are only finitely many terms cz^m in ϑ_{Q',m_0} with $\langle n_0, m\rangle < 0$. Suppose the contrary is true. The direction vector of each broken line contributing to such terms at Q' is toward the wall $\sigma\cap\sigma'$, and so we can extend the final segment of any such broken line to obtain a broken line terminating at some point Q'' (depending on m) in the same chamber as Q. As there are no cancellations because of the positivity of all coefficients and ϑ_{Q,m_0} does not depend on the location of Q inside the chamber by Theorem 3.5, we see that ϑ_{Q,m_0} has an infinite number of terms, a contradiction. $\square$

Definition 7.2. Let $\Theta \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ be the collection of m_0 such that for some (or equivalently, by Proposition 7.1, any) generic $Q \in \sigma \in \Delta^+$ there are only finitely many broken lines γ with $I(\gamma) = m_0$, $b(\gamma) = Q$.

Definition 7.3. We call a subset $S \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ *intrinsically closed under addition* if $p, q \in S$ and $\alpha(p, q, r) \neq 0$ implies $r \in S$.

Lemma 7.4. *Let $S \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ be intrinsically closed under addition. The image of S in $\widetilde{M}^\circ_{\mathbf{s}}$ (under the identification $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) = \widetilde{M}^\circ_{\mathbf{s}}$ induced by the seed $\mathbf{s}$) is closed under addition for any seed $\mathbf{s}$. If for some seed $S \subset \widetilde{M}^\circ_{\mathbf{s}}$ is a toric monoid (i.e., the integral points of a convex rational polyhedral cone), then this holds for any seed.*

Proof. Choose a seed $\mathbf{s}$. Then straight lines in Definition-Lemma 6.2 show $\alpha(p, q, p + q) \neq 0$. This gives closure under addition. Now suppose $S \subset \widetilde{M}^\circ_{\mathbf{s}}$ is a toric monoid, generating the convex rational polyhedral cone $W \subset \widetilde{M}^\circ_{\mathbf{s},\mathbb{R}}$. Then $\mu_{\mathbf{s},\mathbf{s}'}(W) \subset \widetilde{M}^\circ_{\mathbf{s}',\mathbb{R}}$ is a rational polyhedral cone with integral points $S \subset \widetilde{M}^\circ_{\mathbf{s}'}$. As this set of integral points is closed under addition, $\mu_{\mathbf{s},\mathbf{s}'}(W)$ is convex, and so its integral points are a toric monoid. $\square$

Recall from the introduction the definition of global monomial (Definition 0.1).

Theorem 7.5. *Let*

$$\Delta^+(\mathbb{Z}) = \bigcup_{\sigma\in\Delta^+} \sigma \cap \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$$

be the set of integral points in chambers of the cluster complex. Then

(1) $\Delta^+(\mathbb{Z}) \subset \Theta$.

(2) *For $p_1, p_2 \in \Theta$*

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_r \alpha(p_1, p_2, r)\vartheta_r$$

is a finite sum (i.e., $\alpha(p_1, p_2, r) = 0$ for all but finitely many r) with non-negative integer coefficients. If $\alpha(p_1, p_2, r) \neq 0$, then $r \in \Theta$.

(3) *The set Θ is intrinsically closed under addition. For any seed $\mathbf{s}$, the image of $\Theta \subset \widetilde{M}^\circ_{\mathbf{s}}$ is a saturated monoid.*

(4) *The structure constants* $\alpha(p,q,r)$ *of Definition-Lemma* 6.2 *make the* $\Bbbk$-*vector space with basis indexed by* Θ,

$$\mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) := \bigoplus_{q\in\Theta} \Bbbk \cdot \vartheta_q$$

into an associative commutative $\Bbbk[N]$-*algebra. There are canonical inclusions of* $\Bbbk[N]$-*algebras*

$$\begin{aligned}\mathrm{ord}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}}) \\ \subset \mathrm{up}(\widehat{\mathcal{A}_{\mathrm{prin},\mathbf{s}}}) \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N].\end{aligned}$$

Under the first inclusion a cluster monomial Z *is identified with* $\vartheta_{\mathbf{g}(Z)}$ *for* $\mathbf{g}(Z) \in \Delta^+(\mathbb{Z})$ *its g-vector. Under the second inclusion each* ϑ_q *is identified with a universal positive Laurent polynomial.*

Proof. (1) is immediate from Corollary 3.9. For (2), first note that the coefficients $\alpha(p_1,p_2,r)$ are nonnegative by Definition-Lemma 6.2. Suppose $p_1,p_2\in\Theta$. Take a generic basepoint Q in some cluster chamber. Then $\vartheta_{Q,p_1}\cdot\vartheta_{Q,p_2}$ is the product of two Laurent polynomials, thus it is a Laurent polynomial. It is equal to $\sum_r \alpha(p_1,p_2,r)\vartheta_{Q,r}$ by (3) of Proposition 6.4, and hence this sum must be finite, as it involves a positive linear combination of series with positive coefficients. Further, each $\vartheta_{Q,r}$ appearing with $\alpha(p_1,p_2,r)\neq 0$ must be a Laurent polynomial for the same reason. Thus $r\in\Theta$ by Definition 7.2. (2) then immediately implies Θ is intrinsically closed under addition.

For (4), note each $\vartheta_{Q,p}$, $p\in\Theta$ is a universal positive Laurent polynomial by Proposition 7.1. For $p\in\Delta^+(\mathbb{Z})$, $\vartheta_p\in\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ is the corresponding cluster monomial by Theorem 6.8(4). The inclusions of algebras, and the associativity of the multiplication on mid follow from Proposition 6.4.

Finally, we complete the proof of (3) by checking that Θ is saturated. Assume $k\cdot q\in\Theta$ for some integer $k\geq 1$. Take Q to be a generic basepoint in some cluster chamber. We show that the set of final monomials $S(q):=\{F(\gamma)\}$ for broken lines γ with $I(\gamma)=q$, $b(\gamma)=Q$ is finite. By assumption (and the positivity of the scattering diagram), this holds with q replaced by kq. So it is enough to show $m\in S(q)$ implies $km\in S(kq)$. Indeed, it is easy to see that for every broken line γ for q ending at Q, there is a broken line γ' for kq with the same underlying path, such that for every domain of linearity L of γ, the exponents m_L and m'_L of the monomial decorations of L for γ and γ', respectively, satisfy $m'_L = km_L$. This completes the proof of (3), hence the theorem. ∎

The above theorem immediately implies:

Corollary 7.6. *Theorem* 0.3 *is true for* $V=\mathcal{A}_{\mathrm{prin}}$.

The following shows our theta functions are well behaved with respect to the canonical torus action on $\mathcal{A}_{\mathrm{prin}}$.

Proposition 7.7. *Let* $q\in\Theta\subset\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z})$. *Then* $\vartheta_q\in\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ *is an eigenfunction for the natural* $T_{\widetilde{K}^\circ}$ *action on* $\mathcal{A}_{\mathrm{prin}}$ *(see Proposition* B.2(2)*), with weight* $w(q)$ *given by the canonical map* $w:\widetilde{M}^\circ=(\widetilde{N}^\circ)^*\to(\widetilde{K}^\circ)^*$ *(the map being dual to the inclusion* $\widetilde{K}^\circ\subset\widetilde{N}^\circ$*). In particular* ϑ_q *is an eigenfunction for the subtorus* $T_{N^\circ}\subset T_{\widetilde{K}^\circ}$ *with weight* $w(q)$*, where* $w:\widetilde{M}^\circ\to M^\circ$ *is given by* $(m,n)\mapsto m-p^*(n)$.

Proof. Pick a seed $\mathbf{s}$, giving an identification $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z})$ with $\widetilde{M}^\circ$. Pick also a general basepoint $Q \in \mathcal{C}^+_{\mathbf{s}}$. We need to show that for any broken line γ in $\widetilde{M}^\circ_{\mathbb{R}}$ for q with endpoint Q, $\mathrm{Mono}(\gamma)$ is a semi-invariant for the $T_{\widetilde{K}^\circ}$ action with weight $w(q)$. The $T_{\widetilde{K}^\circ}$ action on the seed torus $T_{\widetilde{N}^\circ,\mathbf{s}} \subset \mathcal{A}_{\mathrm{prin}}$ is given on cocharacters by the natural inclusion $\widetilde{K}^\circ \subset \widetilde{N}^\circ$. By definition of $\widetilde{K}^\circ$ we have $w(v_i, e_i) = 0$, $i \in I_{\mathrm{uf}}$, so every monomial appearing in any function $f_{\mathfrak{d}}$ for $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ is in the kernel of w. The result for $T_{\widetilde{K}^\circ}$ follows. The statement for T_{N° now follows from the definitions. $\square$

With more work, we will define the middle cluster algebra for $V = \mathcal{A}_t$ or $\mathcal{X}$.

7.2. **From $\mathcal{A}_{\mathrm{prin}}$ to $\mathcal{A}_t$ and $\mathcal{X}$.** We now show how the various structures we have used to understand $\mathcal{A}_{\mathrm{prin}}$ induce similar structures for $\mathcal{A}_t$ and $\mathcal{X}$.

By [GHK13, §3], each seed $\mathbf{s}$ (in the $\mathcal{X}$, $\mathcal{A}$, and $\mathcal{A}_{\mathrm{prin}}$ cases) gives a *toric model* for V. The seed specifies the data of a fan $\Sigma_{\mathbf{s},V}$, consisting only of rays (so the boundary $\bar{D} \subset \mathrm{TV}(\Sigma_{\mathbf{s},V})$ of the associated toric variety is a disjoint union of tori). The seed also then specifies a blowup $Y \to \mathrm{TV}(\Sigma_{\mathbf{s},V})$ with codimension 2 center, the disjoint union of divisors $Z_i \subset \bar{D}_i$ in each of the disjoint irreducible components $\bar{D}_i \subset \bar{D}$. If D is the proper transform of $\bar{D}$, then there is a birational map $Y \setminus D \dashrightarrow V$. This map is an isomorphism outside of codimension 2 between $Y \setminus D$ and the upper bound (see [GHK13, Remark 3.13] and [BFZ05, Def. 1.1]) $V_{\mathbf{s}} \subset V$, which we recall is the union of $T_{L,\mathbf{s}}$ with $T_{L,\mathbf{s}'}$ for the *adjacent seeds*, $\mathbf{s}' = \mu_k(\mathbf{s}), k \in I_{\mathrm{uf}}$. In the case $V = \mathcal{A}_{\mathrm{prin}}, \mathcal{X}$ or $\mathcal{A}_t$ for very general t, the inclusion $V_{\mathbf{s}} \subset V$ is an isomorphism outside codimension 2. We have

$$\Sigma_{\mathbf{s},\mathcal{A}} = \{\mathbb{R}_{\geq 0} e_i \,|\, i \in I_{\mathrm{uf}}\},$$
$$\Sigma_{\mathbf{s},\mathcal{X}} = \{-\mathbb{R}_{\geq 0} v_i \,|\, i \in I_{\mathrm{uf}}\}.$$

From these toric models it is easy to determine the global monomials:

Lemma 7.8 (Global monomials)**.** *Notation as immediately above. For $m \in \mathrm{Hom}(L_{\mathbf{s}}, \mathbb{Z})$, the character z^m on the torus $T_{L,\mathbf{s}} \subset V$ is a global monomial if and only if z^m is regular on the toric variety $\mathrm{TV}(\Sigma_{\mathbf{s},V})$, which holds if and only if $\langle m, n\rangle \geq 0$ for the primitive generator n of each ray in the fan $\Sigma_{\mathbf{s},V}$. For $\mathcal{A}$-type cluster varieties a global monomial is the same as a cluster monomial, i.e., a monomial in the variables of a single cluster, where the nonfrozen variables have nonnegative exponent.*

Proof. We have a surjection $Y \to \mathrm{TV}(\Sigma_{\mathbf{s},V})$ by construction of Y, and thus a monomial z^m is regular on $\mathrm{TV}(\Sigma_{\mathbf{s},V})$ if and only if its pullback to Y is regular. Certainly such a function is also regular on $Y \setminus D$. Conversely, suppose z^m is not regular on $\mathrm{TV}(\Sigma_{\mathbf{s},V})$. Then it has a pole on some toric boundary divisor $\bar{D}_i$. Now the support of $Z_i \subset \bar{D}_i$ is given by $1 + z^{v_i} = 0$ (resp. $1 + z^{e_i} = 0$) in the $\mathcal{A}$ (resp. $\mathcal{X}$) case, as explained in [GHK13, §3.2]. In particular for $i \in I_{\mathrm{uf}}$, Z_i is nonempty, in the $\mathcal{A}$ case because of the assumption that $v_i \neq 0$ for $i \in I_{\mathrm{uf}}$ stated in Appendix A. As z^m has no zeros on the big torus, the divisor of zeros of z^m will not contain the center $Z_i \subset \bar{D}_i$. It follows that z^m has a pole along the exceptional divisor E_i over Z_i. Since $E_i \cap (Y \setminus D) \neq \emptyset$, z^m is not regular on $Y \setminus D$. Thus we conclude that z^m is regular on $Y \setminus D$ if and only if z^m is regular on $\mathrm{TV}(\Sigma_{\mathbf{s},V})$. Of course, z^m is regular on $\mathrm{TV}(\Sigma_{\mathbf{s},V})$ if and only if $\langle m, n\rangle \geq 0$ for all primitive generators n of rays of $\Sigma_{\mathbf{s},V}$.

Now the rational map $Y \setminus D \dashrightarrow V_{\mathbf{s}}$ to the upper bound is an isomorphism outside codimension 2, so the two varieties have the same global functions. In the $\mathcal{X}$ (or $\mathcal{A}_{\mathrm{prin}}$) case, the inclusion $V_{\mathbf{s}} \subset V$ is an isomorphism outside codimension 2 as well. This gives the theorem for $\mathcal{X}$ or $\mathcal{A}_{\mathrm{prin}}$, and the forward direction for $\mathcal{A}_t$. The reverse direction for $\mathcal{A}_t$ follows from the Laurent phenomenon. Indeed, the final statement of the lemma simply describes the monomials regular on $\mathrm{TV}(\Sigma_{\mathbf{s},\mathcal{A}})$, and a monomial of the given form is the same as a cluster monomial and these are global regular functions by the Laurent phenomenon. □

Note that by Proposition B.2(4) we have canonical maps $\rho : \mathcal{A}^{\vee}_{\mathrm{prin}} \to \mathcal{A}^{\vee}$ and $\xi : \mathcal{X}^{\vee} \to \mathcal{A}^{\vee}_{\mathrm{prin}}$ with tropicalizations

$$\rho^T : (m,n) \mapsto m, \qquad \xi^T : n \mapsto (-p^*(n), -n).$$

Note ρ^T identifies $\mathcal{A}^{\vee}(\mathbb{Z}^T)$ with the quotient of $\mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$ by the natural N-action. Since ξ identifies $\mathcal{X}^{\vee}$ with the fiber over e of $w : \mathcal{A}^{\vee}_{\mathrm{prin}} \to T_{M^\circ}$, ξ^T identifies $\mathcal{X}^{\vee}(\mathbb{Z}^T)$ with $w^{-1}(0)$, where $w : \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T) \to M^\circ$ is the weight map given by $w(m,n) = m - p^*(n)$.

Definition 7.9. Let $V = \bigcup_{\mathbf{s}} T_{L,\mathbf{s}}$ be a cluster variety. Define $\mathcal{C}^+_{\mathbf{s}}(\mathbb{Z}) \subset V^{\vee}(\mathbb{Z}^T)$ to be the set of g-vectors (see Definition 5.10) for global monomials, which are characters on the seed torus $T_{L,\mathbf{s}} \subset V$, and $\Delta^+_V(\mathbb{Z}) \subset V^{\vee}(\mathbb{Z}^T)$ to be the union of all $\mathcal{C}^+_{\mathbf{s}}(\mathbb{Z})$.

Lemma 7.10.

(1) *For V of $\mathcal{A}$-type $\mathcal{C}^+_{\mathbf{s}}(\mathbb{Z})$ is the set of integral points of the cone $\mathcal{C}^+_{\mathbf{s}}$ in the Fock–Goncharov cluster complex corresponding to the seed* $\mathbf{s}$.

(2) *In any case $\mathcal{C}^+_{\mathbf{s}}(\mathbb{Z})$ is the set of integral points of a rational convex cone $\mathcal{C}^+_{\mathbf{s}}$, and the relative interiors of $\mathcal{C}^+_{\mathbf{s}}$ as* $\mathbf{s}$ *varies are disjoint. The g-vector $\mathbf{g}(f) \in V^{\vee}(\mathbb{Z}^T)$ depends only on the function f (i.e., if f restricts to a character on two different seed tori, the g-vectors they determine are the same).*

(3) *For $m \in w^{-1}(0) \cap \Delta^+_{\mathcal{A}_{\mathrm{prin}}}(\mathbb{Z})$, the global monomial ϑ_m on $\mathcal{A}_{\mathrm{prin}}$ is invariant under the T_{N° action and thus gives a global function on $\mathcal{X} = \mathcal{A}_{\mathrm{prin}}/T_{N^\circ}$. This is a global monomial and all global monomials on $\mathcal{X}$ occur this way, and $m = \mathbf{g}(\vartheta_m)$.*

Proof. (1) In the $\mathcal{A}$ case, $\mathcal{C}^+_{\mathbf{s}}$ is the Fock–Goncharov cone by Lemmas 2.10 and 7.8. These cones form a fan by Theorem 2.13, and the fan statement implies that $\mathbf{g}(f)$ depends only on f.

The $\mathcal{A}$ case of (2) immediately follows also from the discussion in §5. The $\mathcal{X}$ case follows from the $\mathcal{A}$-case (applied to $\mathcal{A}_{\mathrm{prin}}$) by recalling that there is a map $\tilde{p} : \mathcal{A}_{\mathrm{prin}} \to \mathcal{X}$ making $\mathcal{A}_{\mathrm{prin}}$ into T_{N°-torsor over $\mathcal{X}$; see Proposition B.2(2). This map is defined on monomials by $\tilde{p}^*(z^n) = z^{(p^*(n),n)}$. Pulling back a monomial for $\mathcal{X}$ under $\tilde{p}$ gives a T_{N°-invariant global monomial for $\mathcal{A}_{\mathrm{prin}}$. Thus there is an inclusion $\Delta^+_{\mathcal{X}}(\mathbb{Z}) \subseteq w^{-1}(0) \cap \Delta^+_{\mathcal{A}_{\mathrm{prin}}}(\mathbb{Z})$ by Proposition 7.7. Conversely, if $m \in w^{-1}(0)$ and $m = \mathbf{g}(f)$ for a global monomial f on $\mathcal{A}_{\mathrm{prin}}$, then there is some seed $\mathbf{s} = (e_1, \ldots, e_n)$ where f is represented by a monomial z^m on $T_{\widetilde{N}^\circ,\mathbf{s}}$. Because $m \in \widetilde{M}^\circ_{\mathbf{s}}$ lies in $w^{-1}(0)$ it is of the form $m = (p^*(n), n)$ for some $n \in N$. By Lemma 7.8, m is nonnegative on the rays $\mathbb{R}_{\geq 0}(e_i, 0)$ of $\Sigma_{\mathbf{s},\mathcal{A}_{\mathrm{prin}}}$, hence n is nonnegative on the rays $-\mathbb{R}_{\geq 0} v_i$ of $\Sigma_{\mathbf{s},\mathcal{X}}$.

Hence z^n defines a global monomial on $\mathcal{X}$. Thus $\Delta^+_{\mathcal{X}}(\mathbb{Z}) = w^{-1}(0) \cap \Delta^+_{\mathcal{A}_{\mathrm{prin}}}(\mathbb{Z})$. Furthermore, one then sees that the Fock–Goncharov cones for $\mathcal{A}_{\mathrm{prin}}$ yield the cones for $\mathcal{X}$ by intersecting with $w^{-1}(0)$. This gives the remaining statements of (2) in the $\mathcal{X}$ case, as well as (3). $\square$

Construction 7.11 (Broken lines for $\mathcal{X}$ and $\mathcal{A}$)**.** *The $\mathcal{X}$ case.* Note that every function $f_{\mathfrak{d}}$ attached to a wall in $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ is a power series in $z^{(p^*(n),n)}$ for some n, thus w is zero on all exponents appearing in these functions. Thus broken lines with both $I(\gamma)$ and initial infinite segment lying in $w^{-1}(0)$ remain in $w^{-1}(0)$. In particular $b(\gamma) \in w^{-1}(0)$, and all their monomial decorations, e.g., $F(\gamma)$, are in $w^{-1}(0)$. We define these to be the broken lines in $\mathcal{X}^\vee(\mathbb{R}^T)$.[2]

The $\mathcal{A}$ case. We define broken lines in $\mathcal{A}^\vee(\mathbb{R}^T)$ to be images of broken lines in $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ under ρ^T (applying the derivative $D\rho^T$ to the decorating monomials).

Definition 7.12. We define

(1) $\Theta(\mathcal{X}) := \Theta(\mathcal{A}_{\mathrm{prin}}) \cap w^{-1}(0) \subset \mathcal{X}^\vee(\mathbb{Z}^T) = \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) \cap w^{-1}(0)$.

(2)
$$\mathrm{mid}(\mathcal{X}) := \mathrm{mid}(\mathcal{A}_{\mathrm{prin}})^{T_{N^\circ}} = \bigoplus_{q \in \Theta(\mathcal{X})} \Bbbk \vartheta_q,$$
where the superscript denotes the invariant part under the group action.

Corollary 7.13. *Theorem* 0.3 *holds for $V = \mathcal{X}$.*

Proof. This follows immediately from the $\mathcal{A}_{\mathrm{prin}}$ case by taking T_{N°-invariants. $\square$

Moving on to the $\mathcal{A}$ case, the following is easily checked:

Definition-Lemma 7.14.

(1) *Define*
$$\Theta(\mathcal{A}_t) := \rho^T(\Theta(\mathcal{A}_{\mathrm{prin}})) \subset \mathcal{A}^\vee(\mathbb{Z}^T).$$
Noting that $\Theta(\mathcal{A}_{\mathrm{prin}}) \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ is invariant under N-translation, we have $\Theta(\mathcal{A}_{\mathrm{prin}}) = (\rho^T)^{-1}(\Theta(\mathcal{A}_t))$. Furthermore, any choice of section $\Sigma : \mathcal{A}^\vee(\mathbb{Z}^T) \to \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ of ρ^T induces a bijection $\Theta(\mathcal{A}_{\mathrm{prin}}) \to \Theta(\mathcal{A}_t) \times N$.

(2) *Define* $\mathrm{mid}(\mathcal{A}_t) = \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) \otimes_{\Bbbk[N]} \Bbbk$*, where $\Bbbk[N] \twoheadrightarrow \Bbbk$ is given by $t \in T_M$. Given a choice of Σ, the collection $\vartheta_m, m \in \Sigma(M^\circ)$ gives a $\Bbbk[N]$-module basis for* $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$ *and thus a $\Bbbk$-vector space basis for* $\mathrm{mid}(\mathcal{A}_t)$*. For* $\mathrm{mid}(\mathcal{A})$ *the basis is independent of the choice of Σ, while for* $\mathrm{mid}(\mathcal{A}_t)$ *it is independent up to scaling each basis vector* (*i.e., the decomposition of the vector space* $\mathrm{mid}(\mathcal{A})$ *into one-dimensional subspaces is canonical*). $\square$

The variety $\mathcal{A}_t$ is a space $\mathcal{A}_t := \bigcup_{\mathbf{s}} T_{N^\circ,\mathbf{s}}$ with the tori glued by birational maps which vary with t. It is then not so clear how to dualize these birational maps to obtain $\mathcal{A}^\vee_t$ as it is not obvious how to deal with these parameters. However, the tropicalizations of these birational maps are all the same (independent of t) and

[2] In fact each $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ induces a collection of walls with attached functions, $\mathfrak{D}^{\mathcal{X}}_{\mathbf{s}}$, living in $N_{\mathbb{R},\mathbf{s}}$, just by intersecting each wall with $w^{-1}(0)$ and taking the same scattering function. This is a consistent scattering diagram, and we are getting exactly the broken lines for this diagram. We will not use this diagram, as we can get whatever we need from $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}$.

thus the tropical sets $\mathcal{A}_t^\vee(\mathbb{Z}^T)$ should all be canonically identified with $\mathcal{A}^\vee(\mathbb{Z}^T)$. So we just take:

Definition 7.15. $\mathcal{A}_t^\vee(\mathbb{Z}^T) := \mathcal{A}^\vee(\mathbb{Z}^T)$.

Theorem 7.16. *For $V = \mathcal{A}_t$ the following modified statements of Theorem* 0.3 *hold.*

(1) *There is a map*

$$\alpha_{\mathcal{A}_t} : V^\vee(\mathbb{Z}^T) \times V^\vee(\mathbb{Z}^T) \times V^\vee(\mathbb{Z}^T) \to \Bbbk \cup \{\infty\},$$

depending on a choice of a section $\Sigma : \mathcal{A}^\vee(\mathbb{Z}^T) \to \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$. This function is given by the formula

$$\alpha_{\mathcal{A}_t}(p,q,r) = \sum_{n\in N} \alpha_{\mathcal{A}_{\text{prin}}}(\Sigma(p),\Sigma(q),\Sigma(r)+n)z^n(t)$$

if this sum is finite; otherwise, we take $\alpha_{\mathcal{A}_t}(p,q,r) = \infty$. This sum is finite whenever $p,q,r \in \Theta(\mathcal{A}_t)$.

(2) *There is a canonically defined subset $\Theta \subset V^\vee(\mathbb{Z}^T)$ given by $\Theta = \Theta(\mathcal{A}_t)$ such that the restriction of the structure constants give the vector subspace* $\text{mid}(V) \subset \text{can}(V)$ *with basis indexed by Θ the structure of an associative commutative $\Bbbk$-algebra.*

(3) $\Delta_V^+(\mathbb{Z}) \subset \Theta$, *i.e., Θ contains the g-vector of each global monomial.*

(4) *For the lattice structure on $V^\vee(\mathbb{Z}^T)$ determined by any choice of seed, $\Theta \subset V^\vee(\mathbb{Z}^T)$ is closed under addition. Furthermore Θ is saturated.*

(5) *There is a $\Bbbk$-algebra map ν :* $\text{mid}(V) \to \text{up}(V)$ *which sends ϑ_p for $p \in \Delta_V^+(\mathbb{Z})$ to a multiple of the corresponding global monomial.*

(6) *There is no analogue of Theorem* 0.3(6) *because the coefficients of the $\vartheta_{Q,p}$ will generally not be integers.*

(7) *ν is injective for very general t and for all t if the vectors v_i, $i \in I_{\text{uf}}$, lie in a strictly convex cone. When ν is injective, we have canonical inclusions*

$$\text{ord}(V) \subset \text{mid}(V) \subset \text{up}(V).$$

Taking $t = e$ gives Theorem 0.3 *for the $V = \mathcal{A}$ case.*

Proof. For (1), note that for $p,q \in \Theta(\mathcal{A}_t)$, we have $\Sigma(p),\Sigma(q) \in \Theta(\mathcal{A}_{\text{prin}})$, and on $\mathcal{A}_{\text{prin}}$

$$\begin{aligned}\vartheta_{\Sigma(p)}\cdot\vartheta_{\Sigma(q)} &= \sum_{r\in\Theta(\mathcal{A}_{\text{prin}})} \alpha_{\mathcal{A}_{\text{prin}}}(\Sigma(p),\Sigma(q),r)\vartheta_r \\ &= \sum_{r\in\Theta(\mathcal{A}_t)}\sum_{n\in N} \alpha_{\mathcal{A}_{\text{prin}}}(\Sigma(p),\Sigma(q),\Sigma(r)+n)\vartheta_{\Sigma(r)+n} \\ &= \sum_{r\in\Theta(\mathcal{A}_t)} \vartheta_{\Sigma(r)}\cdot\left(\sum_{n\in N}\alpha_{\mathcal{A}_{\text{prin}}}(\Sigma(p),\Sigma(q),\Sigma(r)+n)z^n\right)\end{aligned}$$

using $\vartheta_{\Sigma(r)+n} = \vartheta_{\Sigma(r)}z^n$. Note that the sums are finite because $\Sigma(p),\Sigma(q) \in \Theta_{\mathcal{A}_{\text{prin}}}$. Restricting to $\mathcal{A}_t$ gives the formula of (1).

The remaining statements follow easily from the definitions except for the injectivity of (7). To see this, fix a seed $\mathbf{s}$, which gives the second projection $\pi_{N,\mathbf{s}}$: $\mathcal{A}^\vee(\mathbb{Z}^T) = (M^\circ \oplus N)_{\mathbf{s}} \to N$. Choose the section Σ of ρ^T to be $\Sigma(m) = (m,0)$. Note the collection of ϑ_p, $p \in B := \Sigma(M^\circ) \cap \Theta(\mathcal{A}_{\text{prin}})$ are a $\Bbbk[N]$-basis for $\text{mid}(\mathcal{A}_{\text{prin}})$.

By the choice of Σ, the ϑ_p restrict to the basis of monomials on the central fiber T_{N° of $\pi : \mathcal{A}_{\mathrm{prin},\mathbf{s}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$. It follows that for any finite subset $S \subset B$ there is a Zariski open set $0 \in U_S \subset \mathbb{A}^n$ such that ϑ_p, $p \in S$ restrict to linearly independent elements of $\mathrm{up}(\mathcal{A}_t)$, $t \in U_S$. This gives the injectivity of ν for very general t.

Now suppose the $v_i := \{e_i, \cdot\}$, $i \in I_{\mathrm{uf}}$ span a strictly convex cone. We can then pick an $n \in N^\circ \setminus \{0\}$ such that $\{n, e_i\} = -\langle v_i, n\rangle > 0$ for all i. Now pick $m \in N_{\mathrm{uf}}^\perp$ such that $\langle m, e_f\rangle + \langle p^*(n), e_f\rangle > 0$ for $f \in I \setminus I_{\mathrm{uf}}$, and set $\tilde{n} := (n, p^*(n)) + (0, m) \in \widetilde{K}^\circ$; notation is as in Proposition B.2(2). By construction the second projection $\pi_M(\tilde{n}) = m + p^*(n) \in M$ lies in the interior of the orthant generated by the dual basis e_i^*. Take the one-parameter subgroup $T = \tilde{n} \otimes \mathbb{G}_m \subset T_{\widetilde{K}^\circ}$. Now, by Proposition B.2, the map $\mathcal{A}_{\mathrm{prin},\mathbf{s}} \to \mathbb{A}^n$ is $T_{\widetilde{K}^\circ}$-equivariant, where the action on $\mathbb{A}^n$ is given by the map of cocharacters π_M. Thus T has a one-dimensional orbit whose closure contains $0 \in \mathbb{A}^n$. So 0 is in the closure of the orbit $T \cdot x \subset \mathbb{A}^n$ for all $x \in T_M \subset \mathbb{A}^n$. In particular for all x and all S there is some $t_{S,x}$ with $t_{S,x} \cdot x \in U_S$. Now from the $T_{\widetilde{K}^\circ}$-equivariance of the construction, Proposition 7.7, the linear independence holds for all t.

Changing Σ will change the $\Bbbk[N]$-basis for $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$, multiplying each ϑ_p by some character z^n, $n \in N$. The restrictions to $\mathrm{mid}(\mathcal{A}_t)$ are then multiplied by the values $z^n(t)$.

Theorem 0.3 for $V = \mathcal{A}$ now follows from setting $t = e$, where $z^n(t) = 1$ for all n. $\square$

It is natural to wonder:

Question 7.17. Does the equality $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) = \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ always hold?

Our guess is no, but we do not know a counterexample.

Certainly $\Theta \neq \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ in general, for this implies $\Theta(\mathcal{X})$, which is defined to be $\Theta \cap w^{-1}(0)$, coincides with $\mathcal{X}^\vee(\mathbb{Z}^T)$, while we know that in general $\mathcal{X}$ has many fewer global functions, see [GHK13, §7]. So we look for conditions that guarantee $\Theta = \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$, and mid = up. We turn to this in the next section.

Example 7.18. In the cases of Example 1.15, the convex hull of the union of the cones of Δ^+ in $\widetilde{M}^\circ_{\mathbb{R}}$ is all of $\widetilde{M}^\circ_{\mathbb{R}}$. Indeed, the first three quadrants already are part of the cluster complex. It then follows from the fact that Θ is closed under addition and is saturated that $\Theta = \widetilde{M}^\circ$.

In the case of Example 2.14, we know that

$$\Delta^+(\mathbb{Z}) = \{(m,n) \in \widetilde{M}^\circ \mid \langle e_1 + e_2 + e_3, m\rangle \geq 0\}.$$

It then follows again from the fact that Θ is closed under addition that either $\Theta = \Delta^+(\mathbb{Z})$ or $\Theta = \widetilde{M}^\circ$. We believe, partly based on calculations in [M13, §7.1], that in fact the latter holds. $\square$

We show the analogue of Proposition 7.7 for the $\mathcal{A}$ variety:

Proposition 7.19. *If $q \in \Theta(\mathcal{A}) \subset \mathcal{A}^\vee(\mathbb{Z})$, then $\vartheta_q \in \mathrm{up}(\mathcal{A})$ is an eigenfunction for the natural T_{K° action on $\mathcal{A}$.*

Proof. This is essentially the same as the proof of Proposition 7.7, noting that the monomials $z^{v_i} = z^{(v_i,e_i)}|_{\mathcal{A}}$ are invariant under the T_{K° action, as $v_i|_{K^\circ} = 0$ by definition of $K^\circ = \ker p_2^*$. $\square$

We end this section by showing that linear independence of cluster monomials follows easily from our techniques. This was pointed out to us by Gregory Muller. In the skew-symmetric case, this was proved in [CKLP].

Theorem 7.20. *For any $\mathcal{A}$ cluster variety, there are no linear relations between cluster monomials and theta functions in $\nu(\mathrm{mid}(\mathcal{A})) \subset \mathrm{up}(\mathcal{A})$. More precisely, if there is a linear relation*

$$\sum_{q \in \Theta(\mathcal{A})} \alpha_q \vartheta_q = 0$$

in $\mathrm{up}(\mathcal{A})$, then $\alpha_q = 0$ for all $q \in \Delta^+(\mathbb{Z})$. In particular the cluster monomials in $\mathrm{ord}(\mathcal{A}_{\mathrm{prin}})$ are linearly independent.

Proof. Suppose we are given such a relation. We choose a seed $\mathbf{s}$ and a generic basepoint $Q \in \mathcal{C}^+_{\mathbf{s}} \in \Delta^+$. The seed gives an identification $\mathcal{A}^\vee(\mathbb{Z}^T) = M^\circ$. We first show that if $q \in \Delta^+(\mathbb{Z})$ with $q \notin \mathcal{C}^+_{\mathbf{s}}$, then $\vartheta_{Q,q}$ satisfies the *proper Laurent property*, i.e., every monomial $z^m = z^{\sum a_i f_i}$ appearing in $\vartheta_{Q,q}$ has $a_i < 0$ for some i.

Indeed, fix a section $\Sigma : \mathcal{A}^\vee(\mathbb{Z}^T) \to \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ as in Definition-Lemma 7.14. As the restriction to $\mathcal{A} \subset \mathcal{A}_{\mathrm{prin}}$ gives a bijection between the cluster variables for $\mathcal{A}_{\mathrm{prin}}$ and the cluster variables for $\mathcal{A}$, between the theta functions ϑ_q, $q \in \mathrm{Im}(\Sigma)$ and the theta functions for $\mathcal{A}$, and between the corresponding local expressions $\vartheta_{Q,q}$, it is enough to prove the claim in the $\mathcal{A}_{\mathrm{prin}}$ case. This follows immediately from the definition of broken line. Indeed, if γ is a broken line ending at Q and $F(\gamma) = \sum a_i f_i$ with $a_i \geq 0$ for all i, then γ must be wholly contained in $\mathcal{C}^+_{\mathbf{s}}$. But the unbounded direction of γ is parallel to $\mathbb{R}_{\geq 0} m$, so it follows that $q = I(\gamma) \in \mathcal{C}^+_{\mathbf{s}}$.

We then have the relation

$$\sum_{q \in \Theta(\mathcal{A})} \alpha_q \vartheta_{Q,q} = 0 \in \Bbbk[M^\circ],$$

which we rearrange as

$$\sum_{q \in \mathcal{C}^+_{\mathbf{s}}} \alpha_q \vartheta_{Q,q} = -\sum_{q \notin \mathcal{C}^+_{\mathbf{s}}} \alpha_q \vartheta_{Q,q}.$$

The collection of $\vartheta_{Q,q}$ for $q \in \mathcal{C}^+_{\mathbf{s}}$ are exactly the distinct cluster monomials for the seed $\mathbf{s}$. In particular all of their exponents are nonnegative. Thus both sides of the equation are zero. Since the cluster monomials for $\mathbf{s}$ are linearly independent, we conclude $\alpha_q = 0$ for all $q \in \mathcal{C}^+_{\mathbf{s}}$. Varying $\mathbf{s}$, the result follows. □

8. Convexity in the tropical space

As explained in the introduction, the Fock–Goncharov conjecure is in general false, as a cluster variety V has in general too few functions. The conjectured theta functions only exist formally, near infinity, in the sense of §6. The failure of convergence in general manifests itself in the existence of infinitely many broken lines with a given incoming direction and fixed basepoint, and also in nonfiniteness of the multiplication rule (for fixed p, q infinitely many r with $\alpha(p, q, r) \neq 0$). This section and the next is devoted to the question of finding conditions on cluster varieties which guarantee the conjecture holds as stated. One can only expect a theta function basis for $\mathrm{up}(V)$ in cases when V has enough functions. More precisely, a basis should exist when $\mathrm{up}(V)$ is finitely generated, and the natural map $V \to \mathrm{Spec}(\mathrm{up}(V))$ is an open immersion. Note the second condition is automatic by the Laurent phenomenon when V is $\mathcal{A}$-type. Our main (and simple) idea is that we can replace the assumption of enough functions by the existence of a bounded *convex* polytope in $V^\vee(\mathbb{R}^T)$, cut out by the tropicalization of a regular function. However, our notion of convexity is delicate, as $V(\mathbb{R}^T)$ a priori only has a piecewise linear structure. The correct notion is explained in §8.1.

Polytopes will play several roles. The existence of bounded convex polytopes implies various results on convergence of theta functions. We get finiteness of the multiplication rule, and so an algebra structure on $\mathrm{can}(V)$; see Proposition 8.17. For technical reasons (see Remark 8.12) we often have to replace enough global functions by enough global monomials (EGM), and we can make optimal use only of convex polytopes cut out by the tropicalizations of global monomials. EGM implies $\mathrm{up}(V) \subset \mathrm{can}(V)$; see Propositions 8.18 and 8.22. Convex polytopes give partial (full in the bounded polytope case) compactifications of $\mathrm{Spec}(\mathrm{can}(V))$ by copying the familiar toric construction; see §8.5. These compactifications then degenerate to toric compactifications under (the analogue) of the canonical degeneration $\mathcal{A}_{\mathrm{prin},\mathbf{s}} \to \mathbb{A}^n$ of $\mathcal{A}$ to $T_{N^\circ,\mathbf{s}}$. We use these degenerations to prove $\mathrm{Spec}(\mathrm{can}(V))$ is log Calabi-Yau; see Theorem 8.32. Convex cones in $V^\vee(\mathbb{R}^T)$ are intimately related with partial minimal models $V \subset \overline{V}$, and potential functions; see §9.2 and Corollary 9.17. Finally, our methods give several sufficient conditions to guarantee the full Fock–Goncharov conjecture; see Proposition 8.25 and Corollary 9.18. These statements are weaker, and more technical, than one would hope—the reason is our inability to prove in full generality that EGM (or better, the existence of a bounded convex polytope) implies $\Theta(V) = V^\vee(\mathbb{Z}^T)$. We have only optimal control over the subset $\Delta^+ \subset \Theta$ (the cluster complex), and the technical statements are various ways of saying Δ^+ is sufficiently big.

The first issue is to make sense of the notion of convexity in $V(\mathbb{R}^T)$.

8.1. **Convexity conditions.** The following is elementary:

Definition-Lemma 8.1. *By a piecewise linear function on a real vector space W, we mean a continuous function $f : W \to \mathbb{R}$ piecewise linear with respect to a finite fan of (not necessarily strictly) convex cones. For a piecewise linear function $f : W \to \mathbb{R}$, we say f is* min-convex *if it satisfies one of the following three equivalent conditions:*

(1) *There are finitely many linear functions $\ell_1, \ldots, \ell_r \in W^*$ such that $f(x) = \min\{\ell_i(x)\}$ for all $x \in W$.*
(2) *$f(\lambda_1 v_1 + \lambda_2 v_2) \geq \lambda_1 f(v_1) + \lambda_2 f(v_2)$ for all $\lambda_i \in \mathbb{R}_{\geq 0}$ and $v_i \in W$.*
(3) *The differential df is decreasing on straight lines. In other words, for a directed straight line L with tangent vector v, and $x \in L$ general, then*

$$(df)_{x+rv}(v) \leq (df)_x(v),$$

where $r \in \mathbb{R}_{\geq 0}$ is general and the subscript denotes the point at which the differential is calculated.

We now define convexity for functions on $V(\mathbb{R}^T)$ for V a cluster variety by generalizing the third condition above, using broken lines instead of straight lines:

Definition 8.2.

(1) A *piecewise linear function* $f : V(\mathbb{R}^T) \to \mathbb{R}$ is a function which is piecewise linear after fixing a seed $\mathbf{s}$ to get an identification $V(\mathbb{R}^T) = L_{\mathbb{R},\mathbf{s}}$. If the function is piecewise linear for one seed it is clearly piecewise linear for all seeds.

(2) Let $f : V(\mathbb{R}^T) \to \mathbb{R}$ be piecewise linear, and fix a seed $\mathbf{s}$, to view $f : L_{\mathbb{R},\mathbf{s}} \to \mathbb{R}$. We say f is *min-convex for* V (or just *min-convex* if V is clear from context) if for any broken line for V in $L_{\mathbb{R},\mathbf{s}}$, df is increasing on exponents of the decoration monomials (and thus decreasing on their negatives, which are the velocity vectors of the underlying directed path). We note that this notion is independent of mutation, by the invariance of broken lines, Proposition 3.6, and thus an intrinsic property of a piecewise linear function on $V(\mathbb{R}^T)$.

We have a closely related notion, instead defined using the structure constants for multiplication of theta functions.

Definition 8.3. We say that a piecewise linear $f : V(\mathbb{R}^T) \to \mathbb{R}$ is *decreasing* if for $p_1, p_2, r \in V(\mathbb{R}^T)$, with $\alpha(p_1, p_2, r) \neq 0$, $f(r) \geq f(p_1) + f(p_2)$. Here $\alpha(p, q, r)$ are the structure constants of Theorem 0.3.

Lemma 8.4.

(1) *If $f : V(\mathbb{R}^T) \to \mathbb{R}$ is min-convex, then f is decreasing.*

(2) *If $f \colon V(\mathbb{R}^T) \to \mathbb{R}$ is decreasing, then for any seed $\mathbf{s}$, we have $f \colon L_{\mathbb{R},\mathbf{s}} = V(\mathbb{R}^T) \to \mathbb{R}$ min-convex in the sense of Definition-Lemma* 8.1*.*

Proof. (1) Let γ_1, γ_2 be broken lines. Assume f is min-convex and that z very close to r is the endpoint of each broken line, with $F(\gamma_1) + F(\gamma_2) = r$. Then

$$\begin{aligned} f(r) = (df)_z(r) &= (df)_z(F(\gamma_1)) + (df)_z(F(\gamma_2)) \\ &\geq (df)_{\gamma_1(t)}(I(\gamma_1)) + (df)_{\gamma_2(t)}(I(\gamma_2)) \\ &= f(I(\gamma_1)) + f(I(\gamma_2)), \end{aligned}$$

where $t \ll 0$. Thus f is decreasing.

(2) Suppose f is decreasing. For any $a, b \in \mathbb{Z}_{>0}$, and the linear structure on $V(\mathbb{R}^T) = L_{\mathbb{R},\mathbf{s}}$ determined by any choice of seed $\mathbf{s}$, the contribution of straight lines in Definition-Lemma 6.2 (and item (1) of Theorem 7.16 in the $\mathcal{A}$ case) shows $\alpha(a \cdot p, b \cdot q, a \cdot p + b \cdot q) \neq 0$ for all $p, q \in V(\mathbb{Z}^T)$. Thus $f(a \cdot p + b \cdot q) \geq af(p) + bf(q)$ for all positive integers a and b. By rescaling, the same is true for all positive rational numbers a and b and $p, q \in V(\mathbb{Q}^T)$. Min-convexity in the sense of Definition-Lemma 8.1 then follows by continuity of f. $\square$

Remark 8.5. We do not know whether the converse of either statement of Lemma 8.4 holds.

We have a closely related concept, capturing the generalization of the notion of a convex polytope. For $\Xi \subseteq V(\mathbb{R}^T)$ a closed subset, define the *cone* of Ξ

$$\mathbf{C}(\Xi) = \overline{\{(p, r) \mid p \in r\Xi, r \in \mathbb{R}_{\geq 0}\}} \subseteq V(\mathbb{R}^T) \times \mathbb{R}_{\geq 0}.$$

Note the closure is only necessary if Ξ is not compact, in which case $\mathbf{C}(\Xi) \cap (V(\mathbb{R}^T) \times \{0\})$ is an asymptotic form of Ξ. Denote

$$d\Xi(\mathbb{Z}) = \mathbf{C}(\Xi) \cap (V(\mathbb{Z}^T) \times \{d\}),$$

which we view as a subset of $V(\mathbb{Z}^T)$. Note for $d \neq 0$, $d\Xi(\mathbb{Z})$ agrees with the obvious notion $d\Xi \cap V(\mathbb{Z}^T)$.

Definition 8.6. We will call a closed subset $\Xi \subset V(\mathbb{R}^T)$ *positive* if for any non-negative integers d_1, d_2, any $p_1 \in d_1\Xi(\mathbb{Z})$, $p_2 \in d_2\Xi(\mathbb{Z})$, and any $r \in V(\mathbb{Z}^T)$ with $\alpha(p_1, p_2, r) \neq 0$, we have $r \in (d_1 + d_2)\Xi(\mathbb{Z})$.

Note that if Ξ is a cone, i.e., invariant under rescaling, then this definition agrees with Definition 7.3.

For a piecewise linear function $f : V(\mathbb{R}^T) \to \mathbb{R}$, let

$$\Xi_f := \{x \in V(\mathbb{R}^T) \mid f(x) \geq -1\}. \tag{8.7}$$

Definition 8.8. A *convex polytope* $\Xi \subseteq \mathbb{R}^n$ is a subset of $\mathbb{R}^n$ for which there exist a finite collection of affine linear functions $\{\ell_i : \mathbb{R}^n \to \mathbb{R}\}$ with

$$\Xi = \bigcap_i \{v \in \mathbb{R}^n \mid \ell_i(v) \geq 0\}.$$

By Lemma 8.4, if f is min-convex in the sense of Definition 8.2 (or more generally, decreasing in the sense of Definition 8.3), then under any identification $V(\mathbb{R}^T) = L_{\mathbb{R},\mathbf{s}}$ given by any seed, $\Xi_f \subset L_{\mathbb{R},\mathbf{s}}$ is a convex polytope.

Lemma 8.9. *If a piecewise linear function $f : V(\mathbb{R}^T) \to \mathbb{R}$ is decreasing, then Ξ_f is positive. Furthemore, Ξ_f is compact if and only if $f : V(\mathbb{R}^T) \to \mathbb{R}$ is strictly negative away from* 0.

Proof. Note that $d\Xi_f(\mathbb{Z}) = \{p \in V(\mathbb{Z}^T) \mid f(p) \geq -d\}$. Thus if f is decreasing and $p_i \in d_i\Xi_f \cap V(\mathbb{Z}^T)$ with $\alpha(p_1, p_2, r) \neq 0$, then $f(p_i) \geq -d_i$ and thus $f(r) \geq f(p_1) + f(p_2) \geq -d_1 - d_2$, so $r \in (d_1 + d_2)\Xi_f$. The second statement is obvious. ☐

Next we discuss a result comparing these convexity conditions on $\mathcal{A}^\vee_{\mathrm{prin}}$ and $\mathcal{A}^\vee$. From Proposition B.2(4) we have the natural map $\rho : \mathcal{A}^\vee_{\mathrm{prin}} \to \mathcal{A}^\vee$ such that $\rho^T : \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) \to \mathcal{A}^\vee(\mathbb{Z}^T)$ is the canonical projection $\widetilde{M}^\circ \to M^\circ$, the quotient by the N translation action.

Proposition 8.10. *Suppose $\Xi \subseteq \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ is a positive polytope defined over $\mathbb{Q}$ (i.e., all the functions ℓ_i of Definition* 8.8 *are rationally defined). Then $\Xi + N_{\mathbb{R}}$ is positive.*

Proof. Suppose $p_i \in d_i(\Xi + N_{\mathbb{R}})(\mathbb{Z})$, and $\alpha(p_1, p_2, r) \neq 0$. We can always write $p_i = p_i' + n_i$ with $p_i' \in (d_i\Xi)(\mathbb{Q})$ and $n_i \in N_{\mathbb{Q}}$ by the rationality assumption. Let k be a positive integer such that kp_i' and kn_i are all integral for $i = 1, 2$.

We first observe that because $\alpha(p_1, p_2, r) \neq 0$, $\alpha(kp_1, kp_2, kr) \neq 0$. This follows immediately from Definition-Lemma 6.2 and the argument given in the proof of saturatedness in Theorem 7.5. This latter argument shows that if there is a broken line γ with $I(\gamma) = p$, $F(\gamma) = r$, then there is a broken line γ' with $I(\gamma') = kp$, $F(\gamma') = kr$.

We next observe that this implies that $\alpha(kp'_1, kp'_2, k(r - n_1 - n_2)) \neq 0$. Indeed, to show this, we need to show a bijection between the following sets of broken lines and their decorations for $Q \in \mathcal{A}^{\vee}_{\text{prin}}(\mathbb{R}^T)$ general and $n \in N$:

$$\{\gamma \,|\, b(\gamma) = Q, I(\gamma) = q, F(\gamma) = s\},$$
$$\{\gamma \,|\, b(\gamma) = Q, I(\gamma) = q + n, F(\gamma) = s + n\},$$
$$\{\gamma \,|\, b(\gamma) = Q + n, I(\gamma) = q + n, F(\gamma) = s + n\}.$$

Once this is shown, if $\alpha(p, q, r) \neq 0$, then $\alpha(p+n, q, r+n) \neq 0$ and $\alpha(p, q+n, r+n) \neq 0$ by Definition-Lemma 6.2.

To get the bijections between the sets, we first recall that every wall of $\mathfrak{D}^{\mathcal{A}_{\text{prin}}}$ is invariant under the canonical N-translation and is contained in a hyperplane $(n, 0)^{\perp}$ for some $(n, 0) \in \widetilde{N}^{\circ}$. Thus N acts on broken lines, by translation on the underlying path, keeping the monomial decorations the same. This gives the bijection between the second and third sets.

For the bijection between the first and second sets, we need to translate the decorations on each straight segment of γ by n. This will change the slopes of each line segment. To do this precisely, take γ in the first set, say with straight decorated segments $L_1, \ldots, L_k$ taken in reverse order, with L_k the infinite segment. Suppose the monomial attached to L_i is $c_i z^{m_i}$ with $m_i \in \widetilde{M}^{\circ}$. Say the bends are at points $x_i \in L_{i-1} \cap L_i$ along a wall contained the hyperplane $(n_i, 0)^{\perp}$ so that L_i is parameterized (in the reverse direction to that of Definition 3.1) by $x_i + tm_i$, $0 \leq t \leq t_i$. Then we define

$$x'_i = Q + t_1(m_1 + (0, n)) + t_2(m_2 + (0, n)) + \cdots + t_{i-1}(m_{i-1} + (0, n)).$$

Observe that $x'_i \in (n_i, 0)^{\perp}$. Let L'_i be the segment $x'_i + t(m_i + (0, n))$, $0 \leq t \leq t_i$, with attached monomial $c_i z^{m_i + (0,n)}$. Then $L'_1, \ldots, L'_k$ form the straight pieces of a broken line γ' in the second set. This gives the desired bijection. We now conclude, as promised, that $\alpha(kp'_1, kp'_2, k(r - n_1 - n_2)) \neq 0$.

To complete the proof of the proposition, we note that as $kp'_i \in kd_i\Xi$, positivity of Ξ implies $k(r - n_1 - n_2) \in k(d_1 + d_2)\Xi$ and thus $r \in (d_1 + d_2)(\Xi + N_{\mathbb{R}})(\mathbb{Z})$. $\square$

The chief difficulty now lies in constructing min-convex functions or positive polytopes. We turn to this next.

8.2. **Convexity criteria.** The following would be a powerful tool for construction min-convex functions on cluster varieties:

Conjecture 8.11. *If $0 \neq f$ is a regular function on a log Calabi–Yau manifold V with maximal boundary, then $f^{\text{trop}} : V^{\text{trop}}(\mathbb{R}) \to \mathbb{R}$ is min-convex. Here $f^{\text{trop}}(v) = v(f)$ for the valuation f.*

Remark 8.12. To make sense of the conjecture, one needs a good theory of broken lines, currently constructed in [GHK11] in dimension two, and here for cluster varieties of all dimensions. In dimension two, the conjecture has been proven by Travis Mandel [M14]. Also, it is easy to see that in any case, for each seed $\mathbf{s}$ and regular function f, that $f^T : L_{\mathbb{R},\mathbf{s}} = V(\mathbb{R}^T) \to \mathbb{R}$ (see (2.5)) is min-convex in the sense of Definition-Lemma 8.1. Indeed this is the standard (min) tropicalization of a Laurent polynomial. We hope to eventually give a direct geometric description of broken lines (without reference to a scattering diagram) for any log Calabi-Yau, as tropicalizations of some algebraic analogue of holomorphic disks. We expect the conjecture to follow easily from such a description.

In fact, we can prove Conjecture 8.11 for global monomials, which gives our main tool for constructing min-convex functions (our inability to prove the conjecture in general is the main reason we use the condition EGM rather than the more natural condition of enough global functions):

Proposition 8.13. *For a global monomial f on $V^\vee$, the tropicalization f^T is min-convex, and in particular, by Lemma* 8.4*, decreasing. Further, if the global monomial f is of the form ϑ_p, for p to be an integral point in the interior of a maximal-dimensional cone $\mathcal{C}^+_{V^\vee,\mathbf{s}} \subset V(\mathbb{R}^T)$ (see Definition* 7.9*), then ϑ_p^T evaluated on monomial decorations strictly increases at any nontrivial bend of a broken line in $V^\vee(\mathbb{R}^T) = L_{\mathbb{R},\mathbf{s}}$.*

Proof. First consider the case $V = \mathcal{A}_{\mathrm{prin}}$. Suppose f is a global monomial which is a character on a chart indexed by $\mathbf{s}$. The integral points of the cluster chamber

$$\mathcal{C}^+_{\mathcal{A}^\vee_{\mathrm{prin}},\mathbf{s}} \cap \mathcal{A}_{\mathrm{prin}}(\mathbb{Z}^T) \subset T_{\widetilde{N}^\circ,\mathbf{s}}(\mathbb{Z}^T) = \widetilde{N}^\circ_{\mathbf{s}}$$

correspond to characters of $T_{\widetilde{M}^\circ,\mathbf{s}} \subset \mathcal{A}^\vee_{\mathrm{prin}}$ which extend to global regular functions on $\mathcal{A}^\vee_{\mathrm{prin}}$. Then by Lemma 7.8 such a character is regular on $\mathrm{TV}(\Sigma_{\mathbf{s},\mathcal{A}^\vee_{\mathrm{prin}}})$, i.e., it is a character whose geometric tropicalization (2.6) has nonnegative value on each ray in the fan $\Sigma_{\mathbf{s},\mathcal{A}^\vee_{\mathrm{prin}}}$. These rays are spanned by $-(v_i, e_i)$, $i \in I_{\mathrm{uf}}$, the negatives of the initial scattering monomials for $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$. Thus, because of the sign change between geometric and Fock–Goncharov tropicalization (see (2.5)), f^T is nonnegative on the initial scattering monomials. Further, if $f = \vartheta_p$ for p an interior point in $\mathcal{C}^+_{\mathcal{A}^\vee_{\mathrm{prin}},\mathbf{s}}$, f^T is positive on all initial scattering monomials. We now use this to show f^T is min-convex.

Indeed, still working in the seed $\mathbf{s}$, we consider a broken line γ, with two consecutive monomial decorations $cz^m, c'z^{m'}$. Let t, t' be points in the domain of γ in the two segments. Then

$$\begin{aligned}(df^T)_{\gamma(t')}(m') - (df^T)_{\gamma(t)}(m) &= f^T(m') - f^T(m)\\ &= f^T(m' - m).\end{aligned}$$

Now $m' - m$ is some positive multiple of the scattering monomial. Thus since f^T nonnegative on scattering monomials, $f^T(m' - m) \geq 0$. This gives min-convexity. Further, if p lies in the interior of $\mathcal{C}^+_{\mathcal{A}^\vee_{\mathrm{prin}},\mathbf{s}}$, then $f^T(m' - m) > 0$ and f^T is strictly decreasing on nontrivial bends.

The same argument then applies in the $V = \mathcal{X}$ case. Indeed, recall from §7.2 that for any choice of seed, the scattering monomials in $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$ lie in $w^{-1}(0) = N^\circ = \mathcal{X}^\vee(\mathbb{Z}^T)$. So it makes sense to evaluate functions defined only on $\mathcal{X}^\vee(\mathbb{R}^T)$ on scattering monomials for $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}$.

For the $V = \mathcal{A}$ case, a global monomial on $\mathcal{A}^\vee$ pulls back to a global monomial on $\mathcal{A}^\vee_{\mathrm{prin}}$ via the map $\rho : \mathcal{A}^\vee_{\mathrm{prin}} \to \mathcal{A}^\vee$. The result then follows from the $V = \mathcal{A}_{\mathrm{prin}}$ case, as a piecewise linear function f on $\mathcal{A}^\vee(\mathbb{R}^T)$ is min-convex if and only if $f \circ \rho^T$ on $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ is min-convex. Indeed, broken lines in $\mathcal{A}^\vee(\mathbb{R}^T)$ are by definition images of broken lines on $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ under ρ^T. $\square$

We now introduce our key assumption, which is necessary for proving strong results about theta functions and the algebras they generate.

Definition 8.14. We say that V has *EGM* if for any $x \in V(\mathbb{Z}^T)$, $x \neq 0$, there is a global monomial $\vartheta_p \in H^0(V, \mathcal{O}_V)$ such that $\vartheta_p^T(x) < 0$.

Lemma 8.15. *Under any of the identifications $V(\mathbb{R}^T) = L_{\mathbb{R},\mathbf{s}}$ induced by a choice of seed, the set*

$$\Xi_V := \bigcap_{p \in \Delta_V^+(\mathbb{Z}) \subset V^\vee(\mathbb{Z}^T)} \{x \in V(\mathbb{R}^T) \,|\, \vartheta_p^T(x) \geq -1\}$$

is a closed convex subset of $V(\mathbb{R}^T)$. The following are equivalent:

(1) *V has EGM.*
(2) *Ξ_V is bounded, or equivalently, the intersection of the sets $\{x \in V(\mathbb{R}^T) \,|\, \vartheta_p^T(x) \geq 0\}$ for $p \in \Delta_V^+(\mathbb{Z})$ equals $\{0\}$.*
(3) *There exists a finite number of points $p_1, \ldots, p_r \in \Delta_V^+(\mathbb{Z})$ such that*

$$\bigcap_{i=1}^{r} \{x \in V(\mathbb{R}^T) \,|\, \vartheta_{p_i}^T(x) \geq -1\}$$

is bounded, or equivalently, the intersection of the sets $\{x \in V(\mathbb{R}^T) \,|\, \vartheta_{p_i}^T(x) \geq 0\}$ for $1 \leq i \leq r$ equals $\{0\}$.
(4) *There is function $g \in \operatorname{ord}(V)$ whose associated polytope $\{x \in V(\mathbb{R}^T) \,|\, g^T(x) \geq -1\}$ is bounded.*

Proof. By Proposition 8.13, Ξ_V is the intersection of closed rational convex polytopes (with respect to any seed) and hence is a closed convex set.

The equivalence of (1) and (2) is immediate from the definitions, while (3) clearly implies (2). For the converse, let S be a sphere in $V(\mathbb{R}^T) = L_{\mathbb{R},\mathbf{s}}$ centered at the origin. For each $x \in S$ there is a global monomial ϑ_{p_x} such that $\vartheta_{p_x}^T(x) < 0$, and thus there is an open subset $U_x \subset S$ on which $\vartheta_{p_x}^T$ is negative. The $\{U_x\}$ form a cover of S, and hence by compactness there is a finite subcover $\{U_{x_i}\}$. Taking $p_i = p_{x_i}$ gives the desired collection of p_i.

Finally, we show the equivalence of (3) and (4). The ϑ_p, $p \in \Delta_V^+(\mathbb{Z})$ are exactly the global monomials on V, thus generators of $\operatorname{ord}(V)$. Now for any finite collection of functions g_i, $(\sum g_i)^T \geq \min_i g_i^T$, and for the g_i positive universal Laurent polynomials (for example for global monomials), we have equality. Thus given (3), we take $g = \sum_i \vartheta_{p_i}$. Conversely, an element g of $\operatorname{ord}(V)$ is a linear combination of some collection of ϑ_{p_i}. Then $\bigcap_i \{x \in V(\mathbb{R}^T) \,|\, \vartheta_{p_i}^T \geq -1\}$ is contained in $\{x \in V(\mathbb{R}^T) \,|\, g^T(x) \geq -1\}$, so if the latter is bounded, so is the former. $\square$

We note that the property of EGM is preserved by Fock–Goncharov duality in the principal coefficient case:

Proposition 8.16. *Let Γ be fixed data, and let $\Gamma^\vee$ be the Langlands dual data. We write, e.g., $N^\vee$ for the corresponding lattice for the data $\Gamma^\vee$ as in Appendix* A. *For each seed* $\mathbf{s}$, *the canonical inclusion*

$$M_{\mathbf{s}} = M \subset M^\circ = M_{\mathbf{s}^\vee}^\vee$$

commutes with the tropicalization of mutations and induces an isomorphism

$$\mathcal{X}_\Gamma(\mathbb{R}^T) = \mathcal{X}_{\Gamma^\vee}(\mathbb{R}^T).$$

For $n \in N_{\mathbf{s}}$, the monomial z^n on $T_{M,\mathbf{s}} \subset \mathcal{X}_\Gamma$ is a global monomial if and only if $z^{D\cdot n}$ on $T_{M^\vee,\mathbf{s}^\vee} \subset \mathcal{X}_{\Gamma^\vee}$ is a global monomial. Finally, $\mathcal{A}^\vee_{\mathrm{prin}}$ has EGM if and only if $\mathcal{A}_{\mathrm{prin}}$ has EGM.

Proof. The statement about tropical spaces is immediate from the definitions. (Note that a similar statement does not hold at the level of tori, so there is no isomorphism between $\mathcal{X}_\Gamma$ and $\mathcal{X}_{\Gamma^\vee}$.) The statement about global monomials is immediate from Lemma 7.8. Now the final statement follows from the definition of EGM, the isomorphism $\mathcal{A}_{\mathrm{prin}} \cong \mathcal{X}_{\mathrm{prin}}$ of Proposition B.2(1), and the equality $\mathcal{A}^\vee_{\mathrm{prin}} = \mathcal{X}_{\Gamma^\vee_{\mathrm{prin}}}$ of Proposition B.2(3). □

8.3. **The canonical algebra.** In (0.2) we introduced $\mathrm{can}(V)$ as a $\Bbbk$-vector space. In the presence of suitable convex objects on $V^\vee(\mathbb{R}^T)$, we can in fact put an algebra structure on $\mathrm{can}(V)$ using the structure constants given by α. Further, if the EGM condition holds, then $\mathrm{can}(V)$ contains $\mathrm{up}(V)$ and is a finitely generated algebra. This often makes it easier to work with $\mathrm{can}(V)$, as it is a more geometric object.

Precisely:

Proposition 8.17. *For $V = \mathcal{A}_{\mathrm{prin}}$ or $\mathcal{X}$, suppose there is a compact positive polytope $\Xi \subseteq V^\vee(\mathbb{R}^T)$. Assume further that Ξ is top dimensional, i.e., $\dim \Xi = \dim V$. Then for $p, q \in V^\vee(\mathbb{Z}^T)$, there are at most finitely many r with $\alpha(p,q,r) \neq 0$. These give structure constants for an associative multiplication on*

$$\mathrm{can}(V) := \bigoplus_{r \in V^\vee(\mathbb{Z}^T)} \Bbbk \cdot \vartheta_r.$$

If there is a compact positive polytope $\Xi \subseteq \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$, then the same conclusion holds for the structure constants (which are all finite) and multiplication rule of $\mathrm{can}(\mathcal{A}_t)$ for all t.

Proof. For $\mathcal{A}_{\mathrm{prin}}$ or $\mathcal{X}$, the structure constants are defined in terms of broken lines. The finiteness is then immediate from Lemma 8.9. Indeed, given $p, q \in V(\mathbb{Z}^T)$, we have $p \in d_1\Xi$, $q \in d_2\Xi$ for some d_1, d_2, by the fact that Ξ is top dimensional, and thus if $\alpha(p,q,r) \neq 0$, then r lies in the bounded polytope $(d_1 + d_2)\Xi$. The algebra structure is associative by Proposition 6.4(3). The $\mathcal{A}_t$ case follows from the $\mathcal{A}_{\mathrm{prin}}$ case and the definitions of the structure constants and multiplication rule for $\mathrm{can}(\mathcal{A}_t)$; see Theorem 7.16. □

Corollary 8.18. *For $V = \mathcal{A}_{\mathrm{prin}}$ or $\mathcal{X}$, assume $V^\vee$ has EGM. For $V = \mathcal{A}_t$, assume $\mathcal{A}^\vee_{\mathrm{prin}}$ has EGM. Then α defines a $\Bbbk$-algebra structure on $\mathrm{can}(V)$.*

Proof. The case of $V = \mathcal{A}_t$ follows from the case of $\mathcal{A}_{\mathrm{prin}}$, so we may assume V is either $\mathcal{X}$ or $\mathcal{A}_{\mathrm{prin}}$. Using the EGM hypothesis and Lemma 8.15, we can find a finite collection $p_1, \ldots, p_r \in \Delta^+_{V^\vee}(\mathbb{Z})$ such that the intersection of the finite collection of polytopes $\Xi_{\vartheta^T_{p_i}}$ is bounded. But since $\vartheta^T_{p_i}$ is min-convex by Proposition 8.13, each of these polytopes is positive by Lemma 8.9. Thus the result follows from Proposition 8.17. □

Finite generation of $\mathrm{can}(V)$ is a special case of a much more general result.

Theorem 8.19. *Let $V = \mathcal{A}_{\mathrm{prin}}$ or $\mathcal{X}$, assume $V^\vee$ has EGM, and let $\Xi \subseteq V^\vee(\mathbb{R}^T)$ be a positive polytope, which we assume is rationally defined and not necessarily compact. Then*

$$\widetilde{S}_\Xi := \bigoplus_{d\geq 0} \bigoplus_{q\in d\Xi(\mathbb{Z})} \Bbbk\vartheta_q x^d \subset \mathrm{can}(V)[x]$$

is a finitely generated $\Bbbk$-subalgebra.

Proof. Note that $\widetilde{S} := \widetilde{S}_\Xi$ is a subalgebra of $\mathrm{can}(V)[x]$ by the definition of positive polytope.

As in the proof of Corollary 8.18, we can choose $p_1, \ldots, p_r$ so that $\bigcap_i \Xi_{\vartheta^T_{p_i}}$ is a compact positive polytope. Moreover, because boundedness of the intersection is preserved by small perturbation of the functions, we can assume that each p_i is in the interior of some maximal dimensional cluster cone $\mathcal{C}^+_{\mathbf{s}_i}$. Note the seed $\mathbf{s}_i$ is then uniquely determined by p_i by Lemma 7.10. It follows that $\vartheta^T_{p_i}$ is strictly increasing on the monomial decorations at any nontrivial bend of any broken line in $L^*_{\mathbb{R},\mathbf{s}_i}$ by Proposition 8.13.

We define

$$\overline{S}_1 \subset \widetilde{S}[u]$$

to be the vector subspace spanned by all $\vartheta_q x^d u^s$, $s \geq 0$ where $\vartheta^T_{p_1}(q) \geq -s$. Then $\overline{S}_1$ is a graded subalgebra of $\widetilde{S}[u]$ by Proposition 8.13 (graded by u-degree).

The result then follows from the following claim, noting that there is a natural surjection $\overline{S}_1 \twoheadrightarrow \widetilde{S}$ by sending $u \mapsto 1$, $\vartheta_q \mapsto \vartheta_q$.

Claim 8.20. *$\overline{S}_1$ is a finitely generated $\Bbbk$-algebra.*

Proof. We argue first that $\overline{S}_1/u{\cdot}\overline{S}_1$ is finitely generated. We work on $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) = L^*_{\mathbb{R},\mathbf{s}_1}$, so that the multiplication rule is defined using broken lines for $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}_1}$, as described by Proposition 6.4(3) and Definition-Lemma 6.2. Note $\vartheta^T_{p_1}$ is linear on $L^*_{\mathbb{R},\mathbf{s}_1}$. If $\vartheta_q x^d u^s \in \overline{S}_1$, then modulo u, $\vartheta_q x^d u^s = 0$ unless $s = -\vartheta^T_{p_1}(q)$, for otherwise $\vartheta_q x^d u^{s-1} \in \overline{S}_1$. By Proposition 8.13, $\vartheta^T_{p_1}$ is strictly increasing on monomial decorations at any nontrivial bend of a broken line, and thus the only broken lines that will contribute (modulo u) to $\vartheta_{q_1} x^{d_1} u^{-\vartheta^T_{p_1}(q_1)} \cdot \vartheta_{q_2} x^{d_2} u^{-\vartheta^T_{p_1}(q_2)}$ are straight, thus modulo u,

$$\vartheta_{q_1} x^{d_1} u^{-\vartheta^T_{p_1}(q_1)} \cdot \vartheta_{q_2} x^{d_2} u^{-\vartheta^T_{p_1}(q_2)} = \vartheta_{q_1+q_2} x^{d_1+d_2} u^{-\vartheta^T_{p_1}(q_1+q_2)}$$

(addition here in $L^*_{\mathbf{s}_1}$). Thus $\overline{S}_1/u\cdot\overline{S}_1$ is the monoid ring associated to the rational convex cone

$$\mathbf{C}(\Xi) \subseteq L^*_{\mathbb{R},\mathbf{s}_1} \oplus \mathbb{R}$$

and is thus finitely generated.

The result now follows from the following general fact: If $S = \bigoplus_{d\geq 0} S_d$ is a graded $\Bbbk$-algebra, $u \in S_d$ a homogeneous element of degree $d > 0$, and S/uS is a finitely generated $\Bbbk$-algebra, then S is also a finitely generated $\Bbbk$-algebra. Indeed, if $f_1, \ldots, f_s \in S$ are homogeneous elements generating S/uS, we claim S is generated by $u, f_1, \ldots, f_s$. Let $f \in S$ be a homogeneous element of degree n. Then in S_n we can write $f = P(f_1, \ldots, f_s) + ug$ for some polynomial $P \in \Bbbk[X_1, \ldots, X_s]$ and $g \in S_{n-d}$. The assertion now follows by induction on n. $\square$

Corollary 8.21. *For $V = \mathcal{A}_{\mathrm{prin}}$ or $\mathcal{X}$, suppose that $V^\vee$ has EGM. For $V = \mathcal{A}_t$, assume $\mathcal{A}^\vee_{\mathrm{prin}}$ has EGM. Then* $\mathrm{can}(V)$ *is a finitely generated* $\Bbbk$*-algebra.*

Proof. In the $V = \mathcal{A}_{\mathrm{prin}}$ or $\mathcal{X}$ cases, apply the theorem with $\Xi = \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ or $\mathcal{X}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$, which are trivially positive. Then $\mathrm{can}(\mathcal{A}_{\mathrm{prin}})$ is the degree 0 part of the finitely generated ring $\widetilde{S}$ with respect to the x-grading.

Since $\mathrm{can}(\mathcal{A}_t)$ is a quotient of $\mathrm{can}(\mathcal{A}_{\mathrm{prin}})$ by construction of $\alpha_{\mathcal{A}_t}$ in Theorem 7.16(1), $\mathrm{can}(\mathcal{A}_t)$ is also finitely generated. □

Proposition 8.22. *Assume $\mathcal{A}^\vee_{\mathrm{prin}}$ has EGM. Then for each universal Laurent polynomial g on $\mathcal{A}_{\mathrm{prin}}$, the function $\alpha(g)$ of Theorem* 6.8 *has finite support* (*i.e.*, $\alpha(g)(q) = 0$ *for all but finitely many $q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$*), *and $g \mapsto \sum_q \alpha(g)(q)\vartheta_q$ gives inclusions of* $\Bbbk$*-algebras*

$$\mathrm{ord}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{can}(\mathcal{A}_{\mathrm{prin}}) \subset \widehat{\mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \otimes_{\Bbbk[N^+_{\mathbf{s}}]} \Bbbk[N].$$

Proof. Let g be a universal Laurent polynomial on $\mathcal{A}_{\mathrm{prin}}$. By Theorem 6.8 the sets $S_g := S_{g,\mathbf{s}}$ of Definition 6.6 are independent of the seed $\mathbf{s}$. We claim that for each global monomial ϑ_p on $\mathcal{A}^\vee_{\mathrm{prin}}$, there is a constant c_p such that

$$S_g \subset \{x \,|\, \vartheta_p^T(x) \geq c_p\} \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T).$$

To see that this is sufficient to prove the proposition, note that by Lemma 8.15, there are a finite number of p_i such that the intersection of the sets where $\vartheta^T_{p_i}(x) \geq 0$ is the origin in $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$. Thus, if the claim is true, S_g, the support of $\alpha(g)$, is a finite set. The inclusion of algebras follows by Proposition 6.4. So it is enough to establish the claim.

Let ϑ_p be a global monomial which is a character on the seed torus for $\mathbf{s}$. We follow the notation of Definition 6.6. Thus $S_g = S_{g,\mathbf{s}} \subset \overline{S}_{g,\mathbf{s}} + P_{\mathbf{s}}$, where $\overline{S}_{g,\mathbf{s}}$ itself depends on the seed $\mathbf{s}$ and g. The tropicalization ϑ^T_p of global monomials ϑ_p which restrict to a character on the seed torus $T_{\widetilde{M}^\circ,\mathbf{s}} \subset \mathcal{A}^\vee_{\mathrm{prin}}$ are identified with integer points of the dual cone $P^\vee_{\mathbf{s}}$ (i.e., elements nonnegative on each of the initial scattering monomials); see the proof of Proposition 8.13.

Note ϑ^T_p is linear on $\widetilde{M}^\circ_{\mathbb{R},\mathbf{s}}$. Since $\overline{S}_{g,\mathbf{s}}$ is a finite set, for any such $p \in P^\vee_{\mathbf{s}}$, there is constant c_p such that

$$S_g \subset \overline{S}_{g,\mathbf{s}} + P_{\mathbf{s}} \subset \{x \,|\, \vartheta^T_p(x) \geq c_p\}.$$

This completes the proof. □

8.4. **Conditions implying $\mathcal{A}_{\mathrm{prin}}$ has EGM and the full Fock–Goncharov conjecture.** We begin by showing that some standard conditions in cluster theory, namely acyclicity of the quiver or existence of a maximal green sequence, imply a weaker condition which in turn implies both the EGM condition and the full Fock–Goncharov conjecture. This suggests that this weaker condition is perhaps a more natural one in cluster theory. This point has been explored in [Mu15].

Definition 8.23. We say a cluster variety $\mathcal{A}$ has *large cluster complex* if for some seed $\mathbf{s}$, $\Delta^+(\mathbb{Z}) \subset \mathcal{A}^\vee(\mathbb{R}^T) = M^\circ_{\mathbb{R},\mathbf{s}}$ is not contained in a half-space.

Proposition 8.24. *Consider the following conditions on a skew-symmetric cluster algebra $\mathcal{A}$:*

(1) *$\mathcal{A}$ has an acyclic seed.*
(2) *$\mathcal{A}$ has a seed with a maximal green sequence* (*for the definition, see* [BDP, Def. 1.8]).
(3) *$\mathcal{A}$ has large cluster complex.*

Then (1) *implies* (2) *implies* (3).

Proof. (1) implies (2) is [BDP, Lemma 1.20]. For (2) implies (3), let $\mathbf{s}$ be an initial seed, and let $\mathbf{s}'$ be the seed obtained by mutations in a maximal green sequence. By definition the c-vectors for $\mathbf{s}'$ have nonpositive entries. By Lemma 5.12 the c-vectors are the equations for the walls of the cluster chamber $\mathcal{C}^+_{\mathbf{s}'}$, viewed as a chamber of $\mathfrak{D}_{\mathbf{s}}$. But then $\mathcal{C}^+_{\mathbf{s}'} \cap \mathcal{C}^-_{\mathbf{s}} \neq \{0\}$. Hence these two chambers coincide and, in particular, $\mathcal{A}$ has large cluster complex. $\square$

Proposition 8.25. *If $\mathcal{A}$ has large cluster complex, then $\mathcal{A}_{prin}$ has EGM, $\Theta = \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$, and the full Fock–Goncharov conjecture* (*see Definition* 0.6) *holds for $\mathcal{A}_{\text{prin}}$, $\mathcal{X}$, very general $\mathcal{A}_t$, and, if the convexity condition* (7) *of Theorem* 0.3 *holds, for $\mathcal{A}$.*

Proof. Assume EGM fails for $\mathcal{A}_{\text{prin}}$. Then we have some point $0 \neq x \in \mathcal{A}_{\text{prin}}(\mathbb{Z}^T)$ such that $\vartheta^T_p(x) \geq 0$ for all $p \in \Delta^+(\mathbb{Z}) \subset \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$. Take any seed $\mathbf{s}$. We can compute $\vartheta^T_p(x)$ by using the corresponding positive Laurent polynomial $\vartheta_{Q,p} \in \Bbbk[\widetilde{M}^\circ_{\mathbf{s}}]$, for Q a point in the distinguished chamber $\mathcal{C}^+_{\mathbf{s}}$ of $\mathfrak{D}^{\mathcal{A}_{\text{prin}}}_{\mathbf{s}}$. Thus using Lemma 2.8 (leaving the canonical isomorphism r out of the notation),

$$0 \leq \vartheta^T_{Q,p}(x) = \min_{\substack{I(\gamma)=p\\ b(\gamma)=Q}} \langle F(\gamma), -x\rangle \leq \langle p, -x\rangle.$$

Here the minimum is over all broken lines γ contributing to $\vartheta_{Q,p}$ and the final inequality comes from the fact that one of the broken lines is the obvious straight line. Thus $\Delta^+(\mathbb{Z})$ is contained in the half-space $\{\langle\cdot, -x\rangle \geq 0\} \subset \widetilde{M}^\circ_{\mathbf{s},\mathbb{R}}$. Since $\Delta^+(\mathbb{Z}) \subset \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ is the inverse image of $\Delta^+_{\mathcal{A}}(\mathbb{Z}) \subset \mathcal{A}^\vee(\mathbb{Z}^T)$ under the map $\rho^T : \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T) \to \mathcal{A}^\vee(\mathbb{Z}^T)$, the EGM statement follows. Now $\Theta = \mathcal{A}^\vee_{\text{prin}}(\mathbb{Z}^T)$ since $\Delta^+(\mathbb{Z}) \subset \Theta$ and Θ is saturated and intrinsically closed under addition; see Theorem 7.5. Since $\mathcal{A}_{\text{prin}}$ satisfies EGM, so does $\mathcal{A}^\vee_{\text{prin}}$ by Proposition 8.16, and the full Fock–Goncharov conjecture for $\mathcal{A}_{\text{prin}}$ then follows from Corollary 8.18 and Proposition 8.22. The $\mathcal{A}_t, \mathcal{X}$, and $\mathcal{A}$ cases then follow as in the proofs of Corollary 7.13 and Theorem 7.16. $\square$

We next give another condition for the EGM condition to hold. While this may appear very technical, it is in fact very important for group-theoretic examples; see Remark 8.28.

Proposition 8.26.

(1) *Let $U = \operatorname{Spec}(A)$ be an affine variety over a field $\Bbbk$, and let $f_1, \ldots, f_n$ generators of A be a $\Bbbk$-algebra. For each divisorial discrete valuation $v : Q(U)^* \to \mathbb{Z}$* (*where $Q(U)$ denotes the function field of U*) *which does not have center on U* (*or equivalently, for each boundary divisor $E \subset Y \setminus U$ in any partial compactification $U \subset Y$*), *$v(f_i) < 0$ for some i.*

(2) *Suppose* V *is a cluster variety,* $U = \operatorname{Spec}(\operatorname{up}(V))$ *is a smooth affine variety, and* $V \to U$ *is an open immersion. Let* $f_1, \ldots, f_n$ *generate* $\operatorname{up}(V)$ *as a* $\Bbbk$*-algebra. Then* $f = \min(f_1^T, \ldots, f_n^T)$ *is strictly negative on* $V(\mathbb{Z}^T) \setminus \{0\}$.

Proof. (1) Let $U \subset V$ be an open immersion with complement an irreducible divisor E. Suppose each f_i is regular along E. Then the inclusion $H^0(V, \mathcal{O}_V) \subset H^0(U, \mathcal{O}_U)$ is an equality. Thus the inverse birational map $V \dashrightarrow U$ is regular, which implies $U = V$. Thus (1) follows.

(2) Since the restriction $H^0(U, \mathcal{O}_U) \to H^0(V, \mathcal{O}_V)$ to the open subset $V \subset U$ is an isomorphism, it follows that $U \setminus V \subset U$ has codimension at least 2. Thus U itself is log Calabi–Yau by [GHK13, Lemma 1.4], and the restriction $(\omega_U)|_V$ of the holomorphic volume form is a scalar multiple of ω_V. In addition $V(\mathbb{Z}^T) = U(\mathbb{Z}^T)$. Now (2) follows from (1). ☐

Proposition 8.27. *If the canonical map*

$$p_2^*|_{N^\circ} : N^\circ \to N_{\mathrm{uf}}^*, \quad n \mapsto \{n, \cdot\}|_{N_{\mathrm{uf}}}$$

is surjective, then

(1) $\pi : \mathcal{A}_{\mathrm{prin}} \to T_M$ *is isomorphic to* $\mathcal{A} \times T_M$.
(2) *We can choose* $p^* : N \to M^\circ$ *so that the induced map* $p^* : N \otimes_{\mathbb{Z}} \mathbb{Q} \to M^\circ \otimes_{\mathbb{Z}} \mathbb{Q}$ *is an isomorphism.*
(3) *The map induced by the choice of* p^* *in* (2), $p : \mathcal{A} \to \mathcal{X}$, *is finite.*
(4) *If furthermore for each* $0 \neq x \in \mathcal{A}(\mathbb{Z}^T)$, *we can find a cluster variable* A *with* $A^T(x) < 0$, *then* $\mathcal{A}$ (*and* $\mathcal{A}_{\mathrm{prin}}$) *has EGM. This final condition holds if* $\operatorname{ord}(\mathcal{A}) = \operatorname{up}(\mathcal{A})$ *is finitely generated and* $\operatorname{Spec}(\operatorname{up}(\mathcal{A}))$ *is a smooth affine variety.*

Proof. (1) is Lemma B.7. (3) follows from (2). So we assume $p_2^*|_{N^\circ}$ is surjective, and show that we can choose p^* to have finite cokernel, or equivalently, so p^* is injective. We follow the notation of [GHK13, §2.1]. By the assumed surjectivity, p^* is injective iff the induced map $p^*|_K : K \to N_{\mathrm{uf}}^\perp \subset M^\circ$ is injective. We can replace p^* by $p^* + \alpha$ for any map $\alpha : N \to N_{\mathrm{uf}}^\perp \subset M^\circ$ which vanishes on N_{uf}, i.e., factors through a map $\alpha : N/N_{\mathrm{uf}} \to N_{\mathrm{uf}}^\perp$. Note by the assumed surjectivity that K and $N_{\mathrm{uf}}^\perp$ have the same rank, and moreover the restriction $p^*|_{N_{\mathrm{uf}}} = p_1^*$ (which is unaffected by the addition of α) is injective. In particular $p^*|_{K \cap N_{\mathrm{uf}}} : K \cap N_{\mathrm{uf}} \to N_{\mathrm{uf}}^\perp$ is injective. Thus we can choose $\beta : K \to N_{\mathrm{uf}}^\perp$, vanishing on $K \cap N_{\mathrm{uf}}$ (i.e., factoring through a map $\beta : K/K \cap N_{\mathrm{uf}} \to N_{\mathrm{uf}}^\perp$) so that $p^*|_K + \beta : K \to N_{\mathrm{uf}}^\perp$ is injective. By viewing the determinant of $p^*|_K + m \cdot \beta$ for m an integer as a polynomial in m, we see that $p^*|_K + m \cdot \beta$ is injective for all but a finite number of m. For sufficiently divisible m, $m \cdot \beta : K/K \cap N_{\mathrm{uf}} \to N_{\mathrm{uf}}^\perp$ extends to $\alpha : N/N_{\mathrm{uf}} \to N_{\mathrm{uf}}^\perp$ under the natural inclusion $K/K \cap N_{\mathrm{uf}} \subset N/N_{\mathrm{uf}}$. Now $p^* + \alpha : N \to M^\circ$ is injective as required. This shows (2).

For (4), when $\mathcal{A}_{\mathrm{prin}} \to T_M$ is a trivial bundle, it follows that

$$\mathcal{A}_{\mathrm{prin}}(\mathbb{Z}^T) = \mathcal{A}(\mathbb{Z}^T) \times M.$$

So we have EGM so long as we can find cluster variables on $\mathcal{A}$ with the given condition. The final statement of (4) follows from Proposition 8.26. ☐

Remark 8.28. Every double Bruhat cell is an affine variety by [BFZ05, Prop. 2.8] and smooth by [FZ99, Theorem 1.1]. The surjectivity condition in the statement of Proposition 8.27 holds for all double Bruhat cells by [BFZ05, Proposition 2.6] (the proposition states that the exchange matrix has full rank, but the proof shows the surjectivity). So by the proposition, $\mathcal{A}_{\mathrm{prin}}$ has EGM for double Bruhat cells for which the upper and ordinary cluster algebras are the same. This holds for the open double Bruhat cell of G and the base affine space G/N ($N \subset G$ maximal unipotent) for $G = \mathrm{SL}_n$ by [BFZ05, Remark 2.20], and is announced in [GY13] for all double Bruhat cells of all semisimple G.

8.5. **Compactifications from positive polytopes.** In this subsection, we will use positive polytopes in $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ to create partial compactifications of $\mathrm{Spec}(\mathrm{can}(\mathcal{A}_{\mathrm{prin}}))$ which fiber over an affine space $\mathbb{A}^n$. The fiber over 0 will be a toric variety, and the general fiber is log Calabi–Yau.

Fix seed data for a cluster variety, let $\mathbf{s} = (e_1, \ldots, e_n)$, and let $N^\oplus_{\mathbf{s}} \subset N$ be the monoid generated by the e_i. Similarly, let $N^\oplus_{\mathbf{s},\mathbb{R}} \subset N_{\mathbb{R}}$ be the cone generated by the e_i. The choice of seed gives an identification $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) = \widetilde{M}^\circ_{\mathbf{s}} = M^\circ \oplus N$ and in particular determines a second projection $\pi_N : \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) \to N$ (which depends on the choice of seed). We have the canonical inclusion $N \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ given in each seed by $N = 0 \oplus N \subset \widetilde{M}^\circ_{\mathbf{s}} = \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ and canonical translation action of N on $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$ making $\mathrm{can}(\mathcal{A}_{\mathrm{prin}})$ into a $\Bbbk[N]$-module.

Now assume we are given a compact, positive, rationally defined top-dimensional polytope $\Xi \subseteq \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$. We let $S = \mathrm{can}(\mathcal{A}_{\mathrm{prin}})$. By Proposition 8.17 and the compactness of Ξ, S is a $\Bbbk[N]$-algebra with $\Bbbk$-algebra structure constants $\alpha(p, q, r)$.

Lemma 8.29. *The set $\pi_N^{-1}(N^\oplus_{\mathbf{s},\mathbb{R}}) \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ is a positive polytope. Denote by $S_{N^\oplus_{\mathbf{s}}}$ the degree 0 part of the ring $\widetilde{S}_{\pi_N^{-1}(N^\oplus_{\mathbf{s},\mathbb{R}})}$ defined in Theorem 8.19. Then $S_{N^\oplus_{\mathbf{s}}}$ is a finitely generated $\Bbbk[N^\oplus_{\mathbf{s}}]$-algebra.*

Proof. Positivity follows from the fact that $\pi_N(m) \in N^\oplus_{\mathbf{s}}$ for each scattering monomial m in $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}}$. The finite generation statement then follows from Theorem 8.19. □

Let $\widetilde{\Xi} := \Xi + N_{\mathbb{R}}$ and

$$\Xi^+ := \widetilde{\Xi} \cap \pi_N^{-1}(N^\oplus_{\mathbf{s},\mathbb{R}}).$$

Then $\widetilde{\Xi}$ is positive by Proposition 8.10, and as the intersection of two positive sets is positive, Ξ^+ is positive. Hence the associated graded rings $\widetilde{S}_{\widetilde{\Xi}}$ and $\widetilde{S} := \widetilde{S}_{\Xi^+}$ (graded by x) defined via Theorem 8.19 are finitely generated. Note that $S_{N^\oplus_{\mathbf{s}}}$ is the set of homogeneous elements of degree 0 in the localization $\widetilde{S}_x$. Thus we have an inclusion $\mathrm{Spec}(S_{N^\oplus_{\mathbf{s}}}) \subset \mathrm{Proj}(\widetilde{S})$ of an open subset, with complement the zero locus of $x \in H^0(\mathrm{Proj}(\widetilde{S}), \mathcal{O}(1))$. The inclusion of $\Bbbk[N^\oplus_{\mathbf{s}}] = \vartheta_0 \Bbbk[N^\oplus_{\mathbf{s}}]$ in the degree 0 part of $\widetilde{S}$ induces a morphism $\mathrm{Proj}(\widetilde{S}) \to \mathrm{Spec}(\Bbbk[N^\oplus_{\mathbf{s}}]) = \mathbb{A}^n_{X_1,\ldots,X_n}$. This morphism is flat, since $\widetilde{S}$ is a free $\Bbbk[N^\oplus_{\mathbf{s}}]$-module.

Theorem 8.30. *The central fiber of*

$$\big(\mathrm{Spec}(S_{N^\oplus_{\mathbf{s}}}) \subset \mathrm{Proj}(\widetilde{S})\big) \to \mathbb{A}^n$$

is the polarized toric variety $T_{N^\circ} \subset \mathbb{P}_{\overline{\Xi}}$ given by[3] the polyhedron $\overline{\Xi} = \rho^T(\Xi)$ where $\rho : \mathcal{A}^\vee_{\mathrm{prin}} \to \mathcal{A}^\vee$ is the natural map of Proposition B.2(4).

[3]Although $\overline{\Xi} \subseteq M^\circ_{\mathbb{R}}$ is only a rationally defined polyhedron rather than a lattice polyhedron, we can still define $\mathbb{P}_{\overline{\Xi}} = \mathrm{Proj} \bigoplus_{d=0}^\infty \Bbbk^{d\overline{\Xi} \cap M^\circ}$.

Proof. This follows from the multiplication rule. Indeed, since all the scattering monomials project under π_N into the interior of $N_{\mathbf{s}}^{\oplus}$, $z^{F(\gamma)}$ vanishes modulo the maximal ideal of $\Bbbk[N_{\mathbf{s}}^{\oplus}]$ for any broken line γ that bends. Thus

$$\widetilde{S} \otimes_{\Bbbk[N_{\mathbf{s}}^{\oplus}]} (\Bbbk[N_{\mathbf{s}}^{\oplus}]/(X_1, \ldots, X_n)) = \bigoplus_{d \geq 0} \bigoplus_{q \in d \cdot \overline{\Xi}} \Bbbk \cdot \vartheta_q \cdot x^d,$$

with multiplication induced by $\vartheta_p \cdot \vartheta_q = \vartheta_{p+q}$ (addition in M°). This is the coordinate ring of $\mathbb{P}_{\overline{\Xi}}$. □

Example 8.31. Consider the fixed data and seed data given in Example 1.14. The scattering diagram for $\mathcal{A}_{\mathrm{prin}}$ in this case has three walls, pulled back from the walls of the scattering diagram for $\mathcal{A}$ as given in Example 1.14, with attached functions $1 + A_2X_1$, $1 + A_1^{-1}X_2$ and $1 + A_1^{-1}A_2X_1X_2$. Here, with basis e_1, e_2 of N and dual basis f_1, f_2 of M, we have $A_i = z^{(f_i,0)}$ and $X_i = z^{(0,e_i)}$.

Take $\overline{\Xi} \subseteq M_{\mathbb{R}}^\circ$ to be the pentagon with vertices (with respect to the basis f_1, f_2) $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(1, -1)$, which we write as $w_1, \ldots, w_5$. Then $\overline{\Xi}$ pulls back to $\widetilde{M}_{\mathbb{R}}^\circ$ to give a polytope Ξ. It is easy to see that Ξ is a positive polytope. Further, write $\vartheta_i := \vartheta_{(w_i,0)}$, $\vartheta_0 = \vartheta_{(0,0)}$. Then it is not difficult to describe the ring $\widetilde{S}$ determined by Ξ^+ as the graded ring generated in degree 1 by $\vartheta_0, \ldots, \vartheta_5$, with relations

$$\begin{aligned}
\vartheta_1 \cdot \vartheta_3 &= X_1\vartheta_2\vartheta_0 + \vartheta_0^2, \\
\vartheta_2 \cdot \vartheta_4 &= X_2\vartheta_3\vartheta_0 + \vartheta_0^2, \\
\vartheta_3 \cdot \vartheta_5 &= \vartheta_4\vartheta_0 + X_1\vartheta_0^2, \\
\vartheta_4 \cdot \vartheta_1 &= \vartheta_5\vartheta_0 + X_1X_2\vartheta_0^2, \\
\vartheta_5 \cdot \vartheta_2 &= \vartheta_1\vartheta_0 + X_2\vartheta_0^2.
\end{aligned}$$

These equations define a family of projective varieties in $\mathbb{P}^5$, parameterized by $(X_1, X_2) \in \mathbb{A}^2$. For $X_1X_2 \neq 0$, we obtain a smooth del Pezzo surface of degree 5. The boundary (where $\vartheta_0 = 0$) is a cycle of five projective lines. When $X_1 = X_2 = 0$, we obtain a toric surface with two ordinary double points.

Theorem 8.32. *Assume that $\mathcal{A}_{\mathrm{prin}}^\vee$ has EGM, Ξ is given as above, and that $\Bbbk$ is an algebraically closed field of characteristic zero. Let V be one of $\mathcal{X}, \mathcal{A}, \mathcal{A}_t$ or $\mathcal{A}_{\mathrm{prin}}$. We note* $\mathrm{can}(V)$ *has a finitely generated $\Bbbk$-algebra structure by Corollary* 8.18. *Define* $U := \mathrm{Spec}(\mathrm{can}(V))$.

Define $Y := \mathrm{Proj}(\widetilde{S}_{\overline{\Xi}}) \to T_M$ (constructed above) in case $V = \mathcal{A}_{\mathrm{prin}}$, and for $V := \mathcal{A}_t$, take instead its fiber over $t \in T_M$ (we are not defining Y in the $V = \mathcal{X}$ case), so by construction we have an open immersion $U \subset Y$. Define $B := Y \setminus U$. The following hold:

1. *In all cases U is a Gorenstein scheme with trivial dualizing sheaf.*
2. *For $V = \mathcal{A}_{\mathrm{prin}}$, $\mathcal{X}$, or $\mathcal{A}_t$ for t general, U is a K-trivial Gorenstein log canonical variety.*
3. *For $V = \mathcal{A}_{\mathrm{prin}}$ or $\mathcal{A}_t$ for t general, or all $\mathcal{A}_t$ assuming there exists a seed $(e_1, \ldots, e_n)$ and a strictly convex cone containing all of $v_i := \{e_i, \cdot\}$ for $i \in I_{\mathrm{uf}}$, we have $U \subset Y$ is a minimal model. In other words, Y is a (in the $\mathcal{A}_{\mathrm{prin}}$ case relative to T_M) projective normal variety, $B \subset Y$ is a reduced Weil divisor, $K_Y + B$ is trivial, and (Y, B) is log canonical.*

Proof. First we consider the theorem in the cases $V \neq \mathcal{X}$. Note that (3) implies (2) by restriction.

We consider the family $(\mathrm{Proj}(\widetilde{S}), B) \to \mathbb{A}^n$ constructed above, where B is the divisor given by $x = 0$ with its reduced structure. Using Lemma 8.33, the condition that on a fiber Z, $U \subset Z \setminus B_Z$ is a minimal model (in the sense of the statement) is open, and it holds for the central fiber as it is toric by Theorem 8.30. Thus the condition holds for fibers over some nonempty Zariski open subset $0 \in W \subset \mathbb{A}^n$. This gives (3) for $\mathcal{A}_t$ with t general. The convexity condition (on the v_i) implies there is a one-parameter subgroup of $T_{\widetilde{K}^\circ}$ which pushes a general point of $\mathbb{A}^n$ to 0 (see the proof of Theorem 7.16), and now (3) for $\mathcal{A}_t$ for all t follows by the $T_{\widetilde{K}^\circ}$-equivariance.

Now note given seed data Γ the convexity assumption holds for the seed data Γ_{prin}. Thus the final paragraph applies with $\mathcal{A} = \mathcal{A}_{\Gamma_{\mathrm{prin}}}$ and so in particular $\mathrm{Spec}(\mathrm{can}(\mathcal{A}_{\mathrm{prin}}))$ is Gorenstein with trivial dualizing sheaf. The same then holds for the fibers of the flat map $\mathrm{Spec}(\mathrm{can}(\mathcal{A}_{\mathrm{prin}})) \to T_M$, which are $U = \mathrm{Spec}(\mathrm{can}(\mathcal{A}_t))$ (for arbitrary $t \in T_M$). This gives (1).

Finally, we consider the case $V = \mathcal{X}$. The graded ring construction above applied with seed data Γ_{prin} gives a degeneration of a compactification of $\mathrm{Spec}(\mathrm{can}(\mathcal{A}_{\mathrm{prin}})) \subset Y$ (which is now a fiber of the family) to a toric compactification of $T_{\widetilde{N}^\circ}$. The torus T_{N° acts on the family, trivially on the base, and the quotient gives an isotrivial degeneration of an analogously defined compactification of $\mathrm{Spec}(\mathrm{can}(\mathcal{X}))$ to a toric compactification of T_M. We leave the details of the construction (which is exactly analogous to the construction of $\mathrm{Proj}(\widetilde{S})$ above) to the reader. Now exactly the same openness argument applies. $\square$

We learned of the following result, and its proof, from J. Kollár.

Lemma 8.33 (Kollár)**.** *Let $\Bbbk$ be an algebraically closed field of characteristic* 0*. Let $p\colon X \to S$ be a proper flat morphism of schemes of finite type over $\Bbbk$, and let $B \subset X$ be a closed subscheme which is flat over S. Let (X_0, B_0) denote the fiber of $(X, B)/S$ over a closed point $0 \in S$. Assume that S is regular and for $s = 0 \in S$ the following hold:*

1. *X_s is normal and Cohen–Macaulay.*
2. *$B_s \subset X_s$ is a reduced divisor.*
3. *The pair (X_s, B_s) is log canonical.*
4. *$\omega_{X_s}(B_s) \simeq \mathcal{O}_{X_s}$.*
5. *$H^1(X_s, \mathcal{O}_{X_s}) = 0$.*

Then the natural morphism $\omega_{X/S}(B)|_{X_0} \to \omega_{X_0}(B_0)$ is an isomorphism, and there exists a Zariski open neighborhood $0 \in V \subset S$ such that the conditions (1)–(5) *hold for all $s \in V$. In particular, $X_s \setminus B_s$ is a K-trivial Gorenstein log canonical variety for all $s \in V$.*

Proof. We are free to replace S by an open neighborhood of $0 \in S$ and will do so during the proof without further comment.

By assumption $\omega_{X_0}(B_0) \simeq \mathcal{O}_{X_0}$ and X_0 is Cohen–Macaulay. So

$$\mathcal{O}_{X_0}(-B_0) = \mathcal{H}om_{\mathcal{O}_{X_0}}(\omega_{X_0}(B_0), \omega_{X_0})$$

is Cohen–Macaulay by [K13, Corollary 2.71, p. 82]. It follows that B_0 is Cohen–Macaulay by [K13, Corollary 2.63, p. 80].

The base S is regular by assumption, so $0 \in S$ is cut out by a regular sequence. Since X_0 and B_0 are Cohen–Macaulay and $(X, B) \to S$ is proper and flat, we may assume that X and B are Cohen–Macaulay. Now $\mathcal{O}_X(-B)$ is Cohen–Macaulay by [K13, Corollary 2.63], and $\omega_X(B) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-B), \omega_X)$ is Cohen–Macaulay by [K13, Corollary 2.71]. The relative dualizing sheaf $\omega_{X/S}$ is identified with $\omega_X \otimes (p^*\omega_S)^\vee$, so $\omega_{X/S}(B)$ is also Cohen–Macaulay. It follows that $\omega_{X/S}(B)|_{X_0}$ is Cohen–Macaulay, and so in particular it satisfies Serre's condition S_2. The natural map $\omega_{X/S}(B)|_{X_0} \to \omega_{X_0}(B_0)$ is an isomorphism in codimension 1 (because X_0 is smooth in codimension 1) and both sheaves are S_2, hence the map is an isomorphism. Now $\omega_{X_0}(B_0) \simeq \mathcal{O}_{X_0}$ implies that we may assume $\omega_{X/S}(B) \simeq \mathcal{O}_X$ using $H^1(X_0, \mathcal{O}_{X_0}) = 0$.

The conditions (1), (2), and (5) are open conditions on $s \in S$ because $(X, B) \to S$ is proper and flat. So we may assume they hold for all $s \in S$. We established above that $\omega_{X/S}(B)$ is invertible. It follows that condition (3) is also open on $s \in S$ by [K13, Corollary 4.10, p. 159], and that condition (4) is open on S (using (5)). $\square$

Remark 8.34. Note that directly from its definition, with the multiplication rule counting broken lines, it is difficult to prove anything about $\mathrm{can}(V)$, e.g., that it is an integral domain or to determine its dimension. But the convexity, i.e., existence of a convex polytope in the intrinsic sense, gives this very simple degeneration from which we get many properties, at least for very general $\mathcal{A}_t$, for free.

There have been many constructions of degenerations of flag varieties and the like to toric varieties; see [AB] and references therein. We expect these are all instances of Theorem 8.30.

Many authors have looked for a nice compactification of the moduli space $\mathcal{M}$ of (say) rank 2 vector bundles with algebraic connection on an algebraic curve X. We know of no satisfactory solution. For example, in [IIS] the case of X the complement of four points in $\mathbb{P}^1$ is considered, a compactification is constructed, but the boundary is rather nasty (it lies in $|-K|$, but this anticanonical divisor is not reduced). This can be explained as follows: $\mathcal{M}$ has a different algebraic structure, the $\mathrm{SL}_2(\mathbb{C})$ character variety, V (as complex manifolds they are the same). Note $\mathcal{M}$ is covered by affine lines (the space of connections on a fixed bundle is an affine space), thus it is not log Calabi–Yau. Rather, it is the log version of uniruled, and there is no Mori theoretic reason to expect a natural compactification. V however is log Calabi–Yau, and then by Mori theory one expects (infinitely many) nice compactifications, the minimal models; see [GHK13, §1], for an introduction to these ideas. When X has punctures, V is a cluster variety; see [FST] and [FG06]. In the case of S^2 with four punctures, V is the universal family of affine cubic surfaces (the complement of a triangle of lines on a cubic surface in $\mathbb{P}^3$); see [GHK11, Example 6.12]. Each affine cubic has an obvious normal crossing minimal model, the cubic surface. This compactification is an instance of the above, for a natural choice of polygon Ξ. The same procedure will give a minimal model compactification for any SL_2 character variety (of a punctured Riemann surface) by the above simple procedure that has nothing to do with Teichmüller theory.

For the remainder of this section we will assume that $\mathcal{A}^\vee_{\mathrm{prin}}$ has EGM. By Lemma 8.15, there are global monomials $\vartheta_{p_1}, \ldots, \vartheta_{p_r}$ with $p_1, \ldots, p_r \in \mathcal{A}_{\mathrm{prin}}(\mathbb{Z}^T)$ such that $w := \min\{\vartheta^T_{p_i}\}$ is min-convex with

$$\Xi := \{x \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) \mid w(x) \geq -1\}$$

being compact. Thus we have seeds $\mathbf{s}_1, \dots, \mathbf{s}_r$ (possibly repeated) such that ϑ_{p_i} is a character on $T_{\widetilde{M}^\circ,\mathbf{s}_i}$, so that $\vartheta^T_{p_i}$ is linear after making the identification $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) \cong \widetilde{M}^\circ_{\mathbf{s}_i}$. Furthermore, as in the proof of Theorem 8.19, we can assume p_i is in the interior of the cone $\mathcal{C}^+_{\mathbf{s}_i}$. We will now observe that with these assumptions the irreducible components of the boundary in the compactification of $\mathcal{A}_{\mathrm{prin}}$ induced by Ξ are toric.

Note for each p_i there is at least one seed where $\vartheta^T_{p_i}$ is linear. We assume the collection of p_i is minimal for defining Ξ, and thus $\{\vartheta^T_{p_i} = -1\} \cap \Xi$ is a union of maximal faces of Ξ, a nonempty closed subset of codimension 1.

Writing $S = \mathrm{can}(\mathcal{A}_{\mathrm{prin}})$, let $\widetilde{S}_\Xi$ be the graded algebra of Theorem 8.19, again a finitely generated algebra. Then $Y = \mathrm{Proj}(\widetilde{S}_\Xi) \supset \mathrm{Spec}(S)$ is a projective variety and $x = 0$ gives a Cartier (but not necessarily reduced) boundary $D \subset Y$.

Theorem 8.35. *In the above situation, the irreducible components of D are projective toric varieties. More precisely, for each p_i we have a seed $\mathbf{s}_i$ such that ϑ_{p_i} is a character on $T_{\widetilde{N}^\circ,\mathbf{s}_i}$. Then*

$$\{\vartheta^T_{p_i} = -1\} \cap \Xi \subset \widetilde{M}^\circ_{\mathbb{R},\mathbf{s}_i}$$

is a bounded polytope. The associated projective toric variety is an irreducible component of D, and all irreducible components of D occur in this way.

Proof. For each i consider the vector subspace $I_i \subset \widetilde{S} := \widetilde{S}_\Xi$ with basis $\vartheta_q \cdot x^s$ with $\vartheta^T_{p_i}(q) > -s$ and $\vartheta^T_{p_j}(q) \geq -s$ for $j \neq i$.

Note that I_i is an ideal of $\widetilde{S}$. Indeed, the fact that p_i lies in the interior of its cone of the cluster complex for $\mathcal{A}^\vee_{\mathrm{prin}}$ implies by Proposition 8.13 that $\vartheta^T_{p_i}$ is strictly increasing at bends on monomial decorations of broken lines. Now if $\vartheta_p x^s \in I_i$, $\vartheta_q x^w \in \widetilde{S}$, and ϑ_r appears in $\vartheta_p \cdot \vartheta_q$, then $\vartheta^T_{p_i}(r) > -s-w$, and thus $\vartheta_p x^s \cdot \vartheta_q x^w \in I_i$.

Now the definitions imply $\bigcap_i I_i = (x)$. So it is enough to show that $\mathrm{Proj}(\widetilde{S}/I_i)$ is the projective toric variety given by the polytope $\Xi_i := \{\vartheta^T_{p_i} = -1\} \cap \Xi \subset \widetilde{M}^\circ_{\mathbb{R},\mathbf{s}_i}$. Now $\widetilde{S}/I_i$ has basis $\vartheta_q x^s$, $q \in s\Xi_i$. By the multiplication rule, and the fact again that $\vartheta^T_{p_i}$ is strictly increasing at bends on monomial decorations of broken lines, the only broken line that contributes to $\vartheta_q x^s \cdot \vartheta_p x^w$ is the straight broken line, and the multiplication rule on

$$\widetilde{S}/I_i = \bigoplus_{s \geq 0} \Bbbk \cdot (s\Xi_i \cap \widetilde{M}^\circ_{\mathbf{s}_i})$$

is given by lattice addition, i.e., $\mathrm{Proj}(\widetilde{S}/I_i)$ is the projective toric variety given by the polytope Ξ_i. $\square$

Remark 8.36. The result is (at least to us) surprising in that many cluster varieties come with a natural compactification, where the boundary is not at all toric. For example, order the columns of a $k \times n$ matrix and consider the open subset $\mathrm{Gr}^o(k,n) \subset \mathrm{Gr}(k,n)$, where the n consecutive Plücker coordinates (the determinant of the first k columns, columns $2, \dots, k+1$, etc.) are nonzero. This is a cluster variety. Its boundary in the given compactification $\mathrm{Gr}(k,n)$ is a union of Schubert cells (which are not toric). This has EGM by Proposition 8.27. Then generic compactifications given by bounded polytopes Ξ gives an alternative compactification in which we replace all these Schubert cells by toric varieties. We do not know, e.g., how to produce such a compactification by birational geometric operations beginning with $\mathrm{Gr}(k, n)$.

9. Partial compactifications and representation-theoretic results

9.1. **Partial minimal models.** As discussed in the introduction, many basic objects in representation theory, e.g., a semisimple group G, are not log Calabi–Yau, and we cannot expect that they have a canonical basis of regular functions. However, in many cases the basic object is a *partial minimal model* of a log Calabi–Yau variety, i.e., contains a Zariski open log Calabi–Yau subset whose volume form has a pole along all components of the complement. For example, the group G will be a partial compactification of an open double Bruhat cell, and this is a partial minimal model. We have a canonical basis of functions on the cluster variety, and from this, we conjecture one can get a canonical basis on the partial compactification (the thing we really care about) in the most naive possible way, by taking those elements in the basis of functions for the open set which extend to regular functions on the compactification. We are only able to prove the conjecture under rather strong assumptions; see Corollary 9.17. Happily these conditions hold in many important examples.

Note that a frozen variable for $\mathcal{A}$ (or $\mathcal{A}_{\mathrm{prin}}$) canonically determines a valuation, a point of $\mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$, namely the boundary divisor where that variable becomes zero. See Construction B.9.

While we have myriad (and near optimal) sufficient conditions guaranteeing a canonical basis Θ for $\mathrm{up}(\mathcal{A})$, we can only prove our conjecture that $\Theta \cap \mathrm{up}(\overline{\mathcal{A}}) \subset \mathrm{up}(\mathcal{A})$ is a basis of $\mathrm{up}(\overline{\mathcal{A}})$ for $\mathcal{A} \subset \overline{\mathcal{A}}$ a partial minimal model under a much stronger condition (which happily holds in the most important representation theoretic examples):

Definition 9.1. We say a seed $\mathbf{s} = (e_i)_{i \in I}$ is *optimized* for $n \in \mathcal{A}(\mathbb{Z}^T)$ if

$$\{e_k, (r \circ i)(n)\} \geq 0 \text{ for all } k \in I_{\mathrm{uf}},$$

where

$$r \circ i : \mathcal{A}(\mathbb{Z}^T) \xrightarrow{i} \mathcal{A}(\mathbb{Z}^t) = \mathcal{A}^{\mathrm{trop}}(\mathbb{Z}) \xrightarrow{r} N^{\circ}$$

is the composition of canonical identifications defined in §2. If instead $n \in \mathcal{A}(\mathbb{Z}^t) = \mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$, we say $\mathbf{s}$ is *optimized* for n if $\{e_k, r(n)\} \geq 0$ for all $k \in I_{\mathrm{uf}}$.

We say $\mathbf{s}$ is optimized for a frozen index if it is optimized for the corresponding point of $\mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$.

The reason for defining this notion for both $n \in \mathcal{A}(\mathbb{Z}^T)$ and $n \in \mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$ despite the canonical identification between these two sets is that it is sometimes convenient to think of n as specified by a boundary divisor.

For the connection between optimal seeds and our conjecture on $\Theta \cap \mathrm{up}(\overline{\mathcal{A}}) \subset \mathrm{up}(\mathcal{A})$, see Proposition 9.7 and Conjecture 9.8.

Lemma 9.2. *In the skew-symmetric case, a seed is optimized for a frozen index if and only if in the quiver for this seed all arrows between unfrozen vertices and the given frozen vertex point toward the given frozen vertex.*

Proof. Under the identification $r : \mathcal{A}^{\mathrm{trop}}(\mathbb{Z}) \to N^{\circ}$ (which is just N in the skew-symmetric case), the valuation corresponding to the divisor given by the frozen variable indexed by $i \in I \setminus I_{\rm uf}$ is simply e_i. Thus the seed is optimized for this frozen variable if $\{e_k, e_i\} \geq 0$ for all $k \in I_{\rm uf}$; this is the number of arrows from k to i in the quiver, with sign telling us that they are incoming arrows. $\square$

Lemma 9.3.

(1) *The seed* $\mathbf{s}$ *is optimized for* $n \in \mathcal{A}(\mathbb{Z}^T)$ *if and only if the monomial* $z^{r(n)}$ *on* $T_{M^\circ,\mathbf{s}} \subset \mathcal{A}^\vee$ *is a global monomial. In this case*

$$n \in \mathcal{C}_{\mathbf{s}}^+(\mathbb{Z}) \subset \Delta_{\mathcal{A}^\vee}^+(\mathbb{Z}) \subset \Theta(\mathcal{A}^\vee)$$

and the global monomial $z^{r(n)}$ *is the restriction to* $T_{M^\circ,\mathbf{s}} \subset \mathcal{A}^\vee$ *of* ϑ_n. *In the* $\mathcal{A}_{\rm prin}$ *case, for* $n \in \mathcal{A}_{\rm prin}(\mathbb{Z}^T)$ *primitive, this holds if and only if each of the initial scattering monomials* $z^{(v_i,e_i)}$ *in* $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\rm prin}}$ *is regular along the boundary divisor of* $\mathcal{A}_{\rm prin}$ *corresponding to* n *under the identification* $i : \mathcal{A}_{\rm prin}(\mathbb{Z}^T) \to \mathcal{A}_{\rm prin}^{\rm trop}(\mathbb{Z})$.

(2) $n \in \mathcal{A}(\mathbb{Z}^T)$ *has an optimized seed if and only if* n *lies in* $\Delta_{\mathcal{A}^\vee}^+(\mathbb{Z})$.

Proof. For (1), the rays for the fan $\Sigma_{\mathbf{s}}$ giving the toric model for $\mathcal{A}^\vee$ are $-\mathbb{R}_{\geq 0} v_k$ for $k \in I_{\rm uf}$. Note that $r(i(n)) = -r(n)$; see (2.6). Now the statement concerning $\mathcal{A}$ follows from Lemmas 7.8 and 7.10. The additional statement in the $\mathcal{A}_{\rm prin}$ case is clear from the definitions. For (2), one notes that the forward implication is given by (1), while for the converse, if $n \in \Delta_{\mathcal{A}^\vee}^+(\mathbb{Z})$, then $n \in \mathcal{C}_{\mathbf{s}}^+(\mathbb{Z})$ for some seed $\mathbf{s}$, and then n is optimized for that seed. $\square$

Proposition 9.4. *For the standard cluster algebra structure on* $\mathrm{CGr}(k,n)$ (*the affine cone over* $\mathrm{Gr}(k,n)$ *in its Plücker embedding*) *every frozen variable has an optimized seed.*

Proof. As was pointed out to us by Lauren Willams, for $\mathrm{Gr}(k,n)$, the initial seed in [GSV, Figure 4.4], is optimized for one frozen variable (the *special* upper right hand vertex for the initial quiver). The result follows from the cyclic symmetry of this cluster structure. $\square$

Remark 9.5. B. LeClerc, and independently L. Shen, gave us an explicit sequence of mutations that shows the proposition holds as well for the cluster structure of [BFZ05], [GLS] on the maximal unipotent subgroup $N \subset \mathrm{SL}_{r+1}$, and the same argument applies to the Fock–Goncharov cluster structure on $(G/N \times G/N \times G/N)^G$, $G = \mathrm{SL}_{r+1}$. The argument appears in [Ma17].

Lemma 9.6. *Let* L *be a lattice, and let* $P \subset L$ *be a submonoid with* $P^\times = 0$. *For any subset* $S \subseteq L$ *and collection of elements* $\{Z_q \,|\, q \in S\}$ *such that* $Z_q \in \Bbbk[q + (P \setminus \{0\})]$, *the subset* $\{z^q + Z_q \,|\, q \in S\} \subset \Bbbk[L]$ *is linearly independent over* $\Bbbk$.

Proof. Suppose

$$\sum_{q \in S'} \alpha_q (z^q + Z_q) = 0$$

for α_q all nonzero, and suppose $S' \subseteq S$ is a nonempty finite set. Let $q' \in S'$ be minimal with respect to the partial ordering on L given by P (where $n_1 \leq n_2$ means $n_2 = n_1 + p$ for some $p \in P$). The coefficient of $z^{q'}$ in the sum, expressed in the basis of monomials, must be zero. But the minimality of q' implies the monomial $z^{q'}$ does not appear in any of the Z_q, $q \in S'$. Thus the coefficient of $z^{q'}$ is just $\alpha_{q'}$, a contradiction. $\square$

Proposition 9.7. *Suppose a valuation $v \in \mathcal{A}^{\mathrm{trop}}_{\mathrm{prin}}(\mathbb{Z})$ has an optimized seed. If $v(\sum_{q\in\Theta}\alpha_q\vartheta_q) \geq 0$, then $v(\vartheta_q) \geq 0$ for all q with $\alpha_q \neq 0$.*

Proof. Let $\mathbf{s} = (e_1, \ldots, e_n)$ be optimized for v. Let C be the strictly convex cone spanned by the exponents of the initial scattering monomials $(v_i, e_i) \in \widetilde{M}^\circ$. Let $P = C \cap \widetilde{M}^\circ$. Take Q a basepoint in the positive chamber of $\mathfrak{D}_{\mathbf{s}}$. For each $q \in \Theta$, by definition $\vartheta_{Q,q} = z^q + Z_q$ where $Z_q = \sum_{m\in q+P\setminus\{0\}}\beta_{m,q}z^m$ is a finite sum of monomials. By Lemma 9.3(1) we have $v(z^m) \geq v(z^q)$, and thus by Lemma 2.8(2), $v(\vartheta_q) = v(z^q)$.

Let r be the minimum of $v(\vartheta_q)$ over all q with $\alpha_q \neq 0$, and suppose $r < 0$. Since $v(\sum \alpha_q\vartheta_q) \geq 0$, necessarily

$$\sum_{v(z^q)=r}\alpha_q\left(z^q + \sum_{m:v(z^m)=r}\beta_{m,q}z^m\right) = 0 \in \Bbbk[\widetilde{M}^\circ].$$

Note this is the sum of all the monomial terms in $\sum\alpha_q\vartheta_q$ which have the maximal order of pole, $|r|$, along v. This contradicts Lemma 9.6. $\square$

We believe the assumption of an optimized seed is not necessary:

Conjecture 9.8. *The proposition holds for any $v \in \mathcal{A}^{\mathrm{trop}}_{\mathrm{prin}}(\mathbb{Z})$.*

Any finite set $S \subset \mathcal{A}^{\mathrm{trop}}_{\mathrm{prin}}(\mathbb{Z})\setminus\{0\}$ of primitive elements gives a partial compactification (defined canonically up to codimension 2) $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^S_{\mathrm{prin}}$, with the boundary divisors of this partial compactification in one-to-one correspondence with the elements of S. (This is true for any finite collection S of divisorial discrete valuations on the function field of a normal variety $\mathcal{A}$: there is always an open immersion $\mathcal{A} \subset \overline{\mathcal{A}}^S$, with divisorial boundary $\overline{\mathcal{A}}^S \setminus \mathcal{A}$ corresponding to S, and $\mathcal{A} \subset \overline{\mathcal{A}}^S$ is unique up to changes in codimension greater than or equal to 2.)

We then define

$$\Theta(\overline{\mathcal{A}}^S_{\mathrm{prin}}) := \{q \in \Theta(\mathcal{A}_{\mathrm{prin}}) \,|\, v(\vartheta_q) \geq 0 \text{ for all } v \in S\}$$

and $\mathrm{mid}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) \subset \mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$ the vector subspace with basis $\Theta(\overline{\mathcal{A}}^S_{\mathrm{prin}})$. Similarly, we define $\mathrm{ord}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$ to be the subalgebra of $\mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$ generated by those cluster variables that are regular (generically) along all $v \in S$.

Definition 9.9. Each choice of seed $\mathbf{s}$ gives a pairing

$$\langle\cdot,\cdot\rangle_{\mathbf{s}} : \mathcal{A}_{\mathrm{prin}}(\mathbb{Z}^T) \times \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) \to \mathbb{Z},$$

which is just the dual pairing composed with the identifications

$$\mathcal{A}_{\mathrm{prin}}(\mathbb{Z}^T) = T_{\widetilde{N}^\circ,\mathbf{s}}(\mathbb{Z}^T) \stackrel{r}{=} \widetilde{N}^\circ_{\mathbf{s}},$$
$$\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) = T_{\widetilde{M}^\circ,\mathbf{s}}(\mathbb{Z}^T) \stackrel{r}{=} \widetilde{M}^\circ_{\mathbf{s}}.$$

Lemma 9.10.

(1) *The subspace* $\mathrm{mid}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) \subset \mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$ *is a subalgebra containing* $\mathrm{ord}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$. *If* $\mathrm{ord}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$, *then*

$$\mathrm{ord}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) = \mathrm{mid}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}}).$$

(2) *Assume each $v \in S$ has an optimized seed. Then*

$$\mathrm{mid}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) = \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) \cap \mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}}).$$

If $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) = \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, *then* $\mathrm{mid}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$.

(3) *If each $v \in S$ has an optimized seed and $\mathbf{s}$ is optimized for $v \in S$, the piecewise linear function*

$$\vartheta^T_{i(v)} = \langle \cdot, r(v)\rangle_{\mathbf{s}} : (\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) = \widetilde{M}^\circ_{\mathbb{R},\mathbf{s}}) \to \mathbb{R}$$

is min-convex, and for all $q \in \Theta(\mathcal{A}_{\mathrm{prin}}) \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$,

$$\vartheta^T_q(v) = \langle r(q), r(v)\rangle_{\mathbf{s}} = \vartheta^T_{i(v)}(q),$$

where $\vartheta_{i(v)}$ is the global monomial on $\mathcal{A}^\vee_{\mathrm{prin}}$ corresponding to $i(v)$ (which exists by Lemma 9.3*).*

Remark 9.11. There are pairings

$$\Theta(V) \times \Theta(V^\vee) \to \mathbb{Z},$$

which are much more natural than Definition 9.9. Indeed, $v \in \Theta(V)$ gives a canonical function $\vartheta_v \in \mathrm{up}(V)$ and, since $\Theta(V) \subset V^\vee(\mathbb{Z}^t)$, a valuation on $\mathrm{up}(V^\vee)$. The analogous statements apply to $w \in \Theta(V^\vee)$. So we could define a pairing by either

$$\langle v, w\rangle \mapsto w(\vartheta_v) \quad \text{or} \quad \langle v, w\rangle \mapsto v(\vartheta_w).$$

We conjecture that these two pairings are equal. Lemma 9.10(3) gives the result when one of v, w lies in the cluster complex. One can pose the same symmetry conjecture for mirror pairs of affine log Calabi–Yau varieties (with maximal boundary) in general, the two-dimensional case having been shown in [M14]. Suppose the symmetry conjecture holds, and furthermore $\Theta(\mathcal{A}_{\mathrm{prin}}) = \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$. Then (see the proof of Lemma 9.10) the cone of (0.18) cut out by the tropicalization of the potential function is

$$\Xi := \{x \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) \mid W^T(x) \geq 0\} = \Theta(\mathcal{A}_{\mathrm{prin},S}).$$

If furthermore Conjecture 9.8 holds and $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) = \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, then Ξ gives a basis of $\mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$, canonically determined by the open set $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^S_{\mathrm{prin}}$ (together with its cluster structure, though we conjecture the basis is independent of the cluster structure); see Corollary 9.17.

Proof of Lemma 9.10. The subalgebra statement of (1) follows from the positivity (both of structure constants and the Laurent polynomials $\vartheta_{Q,q}$) just as in the proof of Theorem 7.5. Every cluster variable is a theta function, so the inclusion $\mathrm{ord} \subset \mathrm{mid}$ is clear. Now obviously if $\mathrm{ord}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$, then both are equal to mid.

The intersection expression of (2) for the middle algebra follows from Proposition 9.7. Now obviously if $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) = \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, then $\mathrm{mid}(\overline{\mathcal{A}}^S_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}^S_{\mathrm{prin}})$.

For (3) we work with the scattering diagram $\mathfrak{D}_{\mathbf{s}}$. Then $i(v)$ is the g-vector of the global monomial $\vartheta_{i(v)}$, with $(\vartheta_{i(v)})|_{T_{\widetilde{M}^\circ,\mathbf{s}} \subset \mathcal{A}^\vee_{\mathrm{prin}}} = z^{r(i(v))}$, by Lemma 9.3. Using $r(v) = -r(i(v))$, one sees that $\vartheta^T_{i(v)} = \langle \cdot, r(v)\rangle$ is linear on $\widetilde{M}^\circ$, so obviously min-convex in the sense of Definition-Lemma 8.1. Since it is the tropicalization of a global monomial, it is also min-convex in the sense of Definition 8.2 by Proposition 8.13.

Now fix a basepoint $Q \in \mathcal{C}^+_{\mathbf{s}} \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$, and consider $\vartheta_{Q,q}$, $q \in \Theta$. By Lemma 9.3(1) each scattering function is regular along the boundary divisor corresponding to $v \in \mathcal{A}^{\mathrm{trop}}_{\mathrm{prin}}(\mathbb{Z})$. By definition $\vartheta_{Q,q} = z^{r(q)} + Z_{r(q)}$, where $Z_{r(q)}$ is a linear combination of monomials $z^{r(q)+q'}$ with $z^{q'}$ regular along the boundary divisor corresponding to v. Thus

$$\vartheta^T_q(i(v)) = v(\vartheta_{Q,q}) = \langle r(q), r(v)\rangle$$

by Lemma 2.8. Since $\vartheta_{i(v)}$ is the monomial $z^{r(i(v))}$ on $T_{\widetilde{M^\circ},\mathbf{s}}$,

$$\vartheta^T_{i(v)}(q) = -\langle r(q), r(i(v))\rangle = \langle r(q), r(v)\rangle.$$

This completes the proof of (3). $\square$

9.2. **Cones cut out by the tropicalized potential.** Recall a choice of seed gives a partial compactification $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ and a map $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \to \mathbb{A}^n_{X_1,\dots,X_n}$. The boundary $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \setminus \mathcal{A}_{\mathrm{prin}}$ has n irreducible components, primitive elements of $\mathcal{A}^{\mathrm{trop}}_{\mathrm{prin}}(\mathbb{Z})$, the vanishing loci of the X_i.

Lemma 9.12. *The seed $\tilde{\mathbf{s}}$ is optimized for each of the boundary divisors of $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$.*

Proof. If $\mathbf{s} = (e_1, \dots, e_n)$, the corresponding seed for $\mathcal{A}_{\mathrm{prin}}$ is

$$\tilde{\mathbf{s}} = \big((e_1,0),\dots,(e_n,0),(0,f_1),\dots,(0,f_n)\big),$$

and the boundary divisors correspond to the $(0, f_i)$. But $\{(e_i, 0), (0, f_j)\} = \langle e_i, f_j\rangle = \delta_{ij} \geq 0$, hence the claim. $\square$

We adjust slightly the notation $\overline{\mathcal{A}}^{S}_{\mathrm{prin}}$ of the previous subsection to this case:

Definition 9.13. Let

$$\Theta(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) \subset \Theta \subset \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$$

be the subset of points q such that ϑ_q remains regular on the partial compactification $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}} \supset \mathcal{A}_{\mathrm{prin}}$, i.e., such that

$$\vartheta_q \in \mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}}).$$

Lemma 9.14. *Under the identification $\mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T) = M^\circ \oplus N$, we have $\Theta = \Theta(\mathcal{A}) \times N$ and $\Theta(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) = \Theta(\mathcal{A}) \times N^+_{\mathbf{s}}$.*

Proof. Θ is invariant under translation by $0 \oplus N$, and thus $\Theta = \Theta(\mathcal{A}) \times N$.

By Lemma 5.2 we construct $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ from the atlas of toric compactifications

$$T_{N^\circ} \times T_M \subset T_{N^\circ} \times \mathbb{A}^n_{X_i}$$

parameterized by the cluster chambers in $\Delta^+_{\mathbf{s}}$. Now take $q \in \Theta$, and consider $\vartheta_{Q,q}$ for some basepoint in the cluster complex. This is a positive sum of monomials, so it will be regular on the boundary of $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$ iff each summand is. One summand is z^q, so if ϑ_q is regular on $\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}$, then $\pi_N(q) \in N^+_{\mathbf{s}}$. Thus $\Theta(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) \subset \Theta(\mathcal{A}) \times N^+_{\mathbf{s}}$. But now suppose $q = (m, n)$, some $m \in \Theta(\mathcal{A})$ and $n \in N^+_{\mathbf{s}}$. Then z^q is regular on the boundary. Since the initial scattering monomials are (v_i, e_i), any bend in a broken line multiplies the decorating monomial by a monomial regular on the boundary. Thus $q \in \Theta(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})$. This completes the proof. $\square$

We define

$$\mathrm{mid}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) := \bigoplus_{q\in\Theta(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})} \Bbbk\vartheta_q \subset \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}).$$

Recall $\mathrm{ord}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ are the cluster and upper cluster algebras with principal coefficients, respectively, with the frozen variables inverted. On the other hand, $\mathrm{ord}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) \subset \mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})$ are the cluster and upper cluster algebras with principal coefficients, respectively, with the frozen variables not inverted. By Lemma 9.10, $\mathrm{mid}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) \subset \mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$ is a subalgebra, and $\mathrm{ord}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) \subset \mathrm{mid}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) \subset \mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})$.

By Lemmas 9.10 and 9.12, we have

Corollary 9.15. *If* $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) = \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$, *then* $\mathrm{mid}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}^{\mathbf{s}}_{\mathrm{prin}})$.

Here is another sufficient condition for the full Fock–Goncharov conjecture to hold, which will prove immediately useful below:

Proposition 9.16. *Suppose there is a min-convex function* $w : \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{R}^T) \to \mathbb{R}$, *such that* $w(p) > 0$ *implies* $p \in \Theta$ *for* p *integral, and such that* $w(p) > 0$ *for some* p. *Suppose also that there is a bounded positive polytope in* $\mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$ (*which holds, for example, if* $\mathcal{A}^{\vee}_{\mathrm{prin}}$ *has EGM*). *Then* $\Theta = \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$.

Proof. Take any $p \in \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$ and $q \in \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$ with $w(q) > 0$. Then consider any ϑ_r appearing in $\vartheta_p \cdot \vartheta_{mq}$, for $m \geq 1$. By Lemma 8.4

$$w(r) \geq w(p) + w(mq) = w(p) + mw(q) > 0$$

for m sufficiently large. In particular $r \in \Theta$, so for each ϑ_r that appears, $\vartheta_{Q,r}$ is a universal positive Laurent polynomial, for any basepoint Q in the cluster complex. The existence of the bounded positive polytope implies $\vartheta_p \cdot \vartheta_{mq}$ is a finite sum of ϑ_r. Thus the product $\vartheta_p \cdot \vartheta_{mq}$ is also a universal positive Laurent polynomial, and thus by the positivity of the scattering diagram, $\vartheta_{Q,p}$ must be a finite positive Laurent polynomial. Thus $p \in \Theta$. $\square$

If there are frozen variables, there is a canonical candidate for w in the proposition. When we have frozen variables, this gives a partial compactification $\mathcal{A} \subset \overline{\mathcal{A}}$. In this case, let us change notation slightly and write a seed $\mathbf{s}$ as

$$\mathbf{s} = (e_1, \ldots, e_{n_u}, h_1, \ldots, h_{n_f}),$$

with $n_u = \#I_{\mathrm{uf}}$ and $n_f = \#(I \setminus I_{\mathrm{uf}})$, and the h_i are frozen. In this case the elements $d_i h_i \in N^{\circ}_{\mathbf{s}} = \mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$ give n_f canonical boundary divisors for a partial compactification $\mathcal{A} \subset \overline{\mathcal{A}}$, and an analogous $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}_{\mathrm{prin}}$. An atlas for $\mathcal{A}_{\mathrm{prin}} \subset \overline{\mathcal{A}}_{\mathrm{prin}}$ is given by gluing the partial compactification $T_{\widetilde{N}^{\circ}} \subset \mathrm{TV}(\Sigma_{\mathbf{s}})$, where $\Sigma_{\mathbf{s}}$ is the fan consisting of the rays $\mathbb{R}_{\geq 0}(d_i h_i, 0)$.

Corollary 9.17. *Assume that for each* $1 \leq j \leq n_f$, $(d_j h_j, 0) \in \mathcal{A}^{\mathrm{trop}}_{\mathrm{prin}}(\mathbb{Z})$ *has an optimized seed,* $\mathbf{s}_j$. *Let* $W := \sum \vartheta_{i(d_j h_j, 0)}$ *be the* (Landau–Ginzburg) potential, *the sum of the corresponding global monomials on* $\mathcal{A}^{\vee}_{\mathrm{prin}}$ *given by Lemma* 9.3. *Then:*

(1) *The piecewise linear function*

$$W^T : \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{R}^T) \to \mathbb{R}$$

is min-convex and

$$\Xi := \{x \in \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{R}^T) \,|\, W^T(x) \geq 0\}$$

is a positive polytope.

(2) Ξ *has the alternative description*

$$\Xi = \{x \in \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{R}^T) \mid \langle x, (d_j h_j, 0)\rangle_{\mathbf{s}_j} \geq 0 \textit{ for all } j\}.$$

(3) *The set*

$$\Xi \cap \Theta(\mathcal{A}_{\mathrm{prin}}) = \{p \in \Theta(\mathcal{A}_{\mathrm{prin}}) \mid \vartheta_p \in \mathrm{up}(\overline{\mathcal{A}}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}})\}$$

parameterizes a canonical basis of

$$\mathrm{mid}(\overline{\mathcal{A}}_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}_{\mathrm{prin}}) \cap \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}}).$$

Proof. This is immediate from Lemma 9.10. □

Corollary 9.18. *Assume we have EGM on* $\mathcal{A}^{\vee}_{\mathrm{prin}}$*, and every frozen variable has an optimized seed. Let* W *and* Ξ *be as in Corollary* 9.17*. If for some seed* $\mathbf{s}$*,* Ξ *is contained in the convex hull* $\mathrm{Conv}(\Theta)$ *of* Θ (*which itself contains the integral points of the cluster complex* $\Delta^+(\mathbb{Z})$)*, then* $\Theta = \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$*,* $\mathrm{mid}(\overline{\mathcal{A}}_{\mathrm{prin}}) = \mathrm{up}(\overline{\mathcal{A}}_{\mathrm{prin}})$ *is finitely generated, and the integer points* $\Xi \cap \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$ *parameterize a canonical basis of* $\mathrm{up}(\overline{\mathcal{A}}_{\mathrm{prin}})$*.*

Proof. By definition $\Xi := \{W^T \geq 0\}$, and W^T is min-convex by Lemma 9.3. Thus $\Theta = \mathcal{A}^{\vee}_{\mathrm{prin}}(\mathbb{Z}^T)$ by Proposition 9.16. Now the result follows from the inclusions

$$\mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin}}) \subset \mathrm{can}(\mathcal{A}_{\mathrm{prin}})$$

of Proposition 8.22. □

The corollary applies in important representation-theoretic examples:

Proof of Corollary 0.21*.* We check the conditions of Theorem 0.19: The existence of an optimized seed is proven in [Ma15], following suggestions of L. Shen and B. LeClerc. The cluster variety has large cluster complex (Definition 8.23) by [GS16, Theorems 1.12 and 1.17]. This gives the hypotheses of Theorem 0.19(3). The equality of our W with the Goncharov–Shen potential is given in [Ma17, Theorem 2]. It is shown in [Ma17, Proposition 22] that the exchange matrix has full rank, in the sense of Lemma B.7. Now the $\mathcal{A}_{\mathrm{prin}}$ results imply the analogous statements for $\mathcal{A}$ as in the proof directly above. Magee identifies the $H^{\times 3}$ action with the T_{K°-action in [Ma17, §4.c], which gives the weight statements as above. □

Proof of Corollary 0.20*.* The hyptheses of Theorem 0.19 are proven in [Ma15], using Proposition 9.16 applied to the tropicalization of our potential W. The agreement of W with the Berenstein–Kazhdan potential is given in [Ma17, §5]. Theorem 0.19 is stated for $\mathcal{A}_{\mathrm{prin}}$. But in this case it is shown in [Ma15] that the exchange matrix has full rank, i.e., the equivalent conditions of Lemma B.7 hold. Now the results for $\mathcal{A}_{\mathrm{prin}}$ imply the analogous result for $\mathcal{A}$ using $T_{\widetilde{K}^\circ}$ equivariance, as in the proof of Theorem 7.16(7). Magee identifies the H-action with the action of T_{K° on $\mathcal{A}^{\vee}$, and the various statements about H-weights now follow immediately from the equivariance, Proposition 7.7. □

10. Links with quiver representations and work of Reineke

We briefly make a connection with work of Reineke [R10] in the acyclic skew-symmetric case. In this case $\mathfrak{D}_{\mathbf{s}} = \mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\text{prin}}}$ has a natural interpretation in terms of moduli of quiver representations. Consider skew-symmetric fixed and initial data with no frozen variables. Set $N^{\oplus} = \{\sum a_i e_i \,|\, a_i \geq 0\}$, and let $\widehat{\Bbbk[N^{\oplus}]}$ be the completion of the polynomial ring $\Bbbk[N^{\oplus}]$ with respect to the maximal monomial ideal. Let $P \subseteq \widetilde{M} = M \oplus N$ be a monoid as in §1 containing all (v_i, e_i), so that G, the pronilpotent group of §1.1 (in the $\mathcal{A}_{\text{prin}}$ case), acts by automorphisms of $\widehat{\Bbbk[P]}$ as usual. Note there is an embedding $\widehat{\Bbbk[N^{\oplus}]} \hookrightarrow \widehat{\Bbbk[P]}$ given by $z^n \mapsto z^{(p^*(n),n)}$. The action of G on $\widehat{\Bbbk[P]}$ then induces an action on $\widehat{\Bbbk[N^{\oplus}]}$. Indeed, one checks immediately that an automorphism (for $d \in N^+$)

$$z^{(m,n)} \mapsto z^{(m,n)} f\big(z^{(p^*(d),d)}\big)^{\langle (d,0),(m,n)\rangle}$$

induces the automorphism on $\widehat{\Bbbk[N^{\oplus}]}$ given by

$$z^n \mapsto z^n f(z^d)^{-\{d,n\}}.$$

Proposition 10.1. *Suppose we are given fixed skew-symmetric data with no frozen variables along with an acyclic seed $\mathbf{s} = (e_1, \ldots, e_n)$. Let Q be the associated quiver.*[4] *Each $x \in M_{\mathbb{R}}$ gives a stability in the sense of* [R10]*. Assume there is a unique primitive $d \in N_{\mathbf{s}}^+$ with $x \in d^{\perp}$. For each $i \in I$, let*

$$Q^i(z^d) = \sum_{k \geq 0} \chi(\mathcal{M}^x_{kd,i}(Q)) z^{kd},$$

where $\mathcal{M}^x_{d,i}(Q)$ is the framed moduli space (framed by the vector spaces V_j with $\dim V_j = 0$ *unless $j = i$, in which case* $\dim V_j = 1$) *of semistable representations of Q with dimension vector d and x-slope* 0 (*see* [R10, §5.1]), *and χ denotes topological Euler characteristic. Let $d = \sum d_i e_i$ for some $d_i \in \mathbb{N}$. Then*

$$f(z^d) := (Q^i)^{\frac{1}{d_i}} \text{ for } i \in I, d_i \neq 0,$$

depends only on Q and x (i.e., is independent of the vertex $i \in I$). Furthermore, for arbitrary $y \in N_{\mathbb{R}}$, $g_{(x,y)}(\mathfrak{D}_{\mathbf{s}})$ (see Lemma 1.9) acts on $\widehat{\Bbbk[N^{\oplus}]}$ by

$$z^n \mapsto z^n \cdot f^{-\{d,n\}}$$

and on $\widehat{\Bbbk[P]}$ by

$$z^{(m,n)} \mapsto z^{(m,n)} \cdot f\big(z^{(p^*(d),d)}\big)^{\langle d,m\rangle}.$$

Proof. The equality of the $(Q^i)^{\frac{1}{d_i}}$ follows from the argument in the proof of [R10, Lemma 3.6].

If $d = e_i$ for some i, then one checks easily that $\mathcal{M}^x_{e_i,i}$ is a point and $\mathcal{M}^x_{ke_i,j} = \emptyset$ for $i \neq j$ or $k > 1$. Thus $f(z^d) = 1 + z^{e_i}$ and the formula for $g_{x,y}(\mathfrak{D}_{\mathbf{s}})$ holds by Remark 1.29.

Let G be the pronilpotent group of §1.1 (in the $\mathcal{A}_{\text{prin}}$ case) associated to the completion of the Lie algebra

$$\mathfrak{g} = \bigoplus_{n \in N^+} \mathfrak{g}_n = \bigoplus_{n \in N^+} \Bbbk \cdot z^{(p^*(n),n)} \partial_{(n,0)}.$$

[4]Note that because of the assumption made in Appendix A that $v_i \neq 0$ for any $i \in I_{\text{uf}}$, the quiver Q has no isolated vertex.

The group G acts faithfully on $\widehat{\Bbbk[P]}$ but need not act faithfully via restriction on $\widehat{\Bbbk[N^{\oplus}]}$. It turns out, however, that there is a subgroup $G' \subseteq G$ which does act faithfully on $\widehat{\Bbbk[N^{\oplus}]}$ and that all automorphisms attached to walls in $\mathfrak{D}_{\mathbf{s}}$ are elements of G'. We see this as follows.

Consider the subspace

$$\mathfrak{g}' := \bigoplus_{\substack{n \in N^+ \\ p^*(n) \neq 0}} \mathfrak{g}_n \subset \mathfrak{g}.$$

By the commutator formula (1.1) we have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}'$, and in particular $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$. Let $G' \subset G$ denote the associated pronilpotent subgroup. Because of the assumption that $p^*(e_i) \neq 0$ for any unfrozen i, all automorphisms associated to initial walls of $\mathfrak{D}_{\mathbf{s}}$ lie in G', and thus by the inductive construction of the scattering diagram in §C.1, all automorphisms associated to outgoing walls of $\mathfrak{D}_{\mathbf{s}}$ also lie in G'.

Let G'' denote the pronilpotent group acting faithfully on $\widehat{\Bbbk[N^{\oplus}]}$ associated to the completion of the Lie algebra

$$\mathfrak{g}'' := \bigoplus_{\substack{n \in N^+ \\ p^*(n) \neq 0}} \Bbbk \cdot z^n \partial_{p^*(n)}.$$

Then the restriction of the action of G on $\widehat{\Bbbk[P]}$ to $\widehat{\Bbbk[N^{\oplus}]}$ is given by the group homomorphism $G \to G''$ associated to the Lie algebra homomorphism

$$\mathfrak{g} \to \mathfrak{g}'', \quad z^{(p^*(n),n)} \partial_{(n,0)} \mapsto -z^n \partial_{p^*(n)}.$$

This homomorphism restricts to an isomorphism $G' \to G''$, and in particular the restriction of the G' action on $\widehat{\Bbbk[P]}$ to $\widehat{\Bbbk[N^{\oplus}]}$ is faithful.

Assume now that the indices are ordered so that Q has arrows from the vertex with index i to the vertex with index j only if $i > j$. We compute $\mathfrak{p}_{+,-} \in G'$, the automorphism associated to a path from the positive to the negative chamber, in two different ways.

First, there is a sequence of chambers connecting $\mathcal{C}^+_{\mathbf{s}}$ to $\mathcal{C}^-_{\mathbf{s}}$ via the mutations $\mu_n, \mu_{n-1}, \ldots, \mu_1$. Indeed, it is easy to check that the c-vectors obtained by mutating $\mu_n, \mu_{n-1}, \ldots, \mu_i$ are precisely $e_1, \ldots, e_{i-1}, -e_i, \ldots, -e_n$, and the chamber corresponding to this sequence of mutations is precisely the dual of the cone generated by the c-vectors; see Lemma 5.12. Thus in particular, we can find a path γ from $\mathcal{C}^+_{\mathbf{s}}$ to $\mathcal{C}^-_{\mathbf{s}}$ which only crosses the walls $e_n^{\perp}, \ldots, e_1^{\perp}$ in order. Note that the element of G' attached to the wall $e_j^{\perp}$ acts on $\widehat{\Bbbk[N^{\oplus}]}$ by $z^n \mapsto z^n(1 + z^{e_j})^{-\{e_j, n\}}$, which agrees with the automorphism in [R10] written as T_{i_j} (noting that [R10] uses the opposite sign convention for the skew form $\{\cdot,\cdot\}$ associated to the quiver). From this we conclude that $\mathfrak{p}_{+,-} = T_{i_1} \circ \cdots \circ T_{i_n}$, the left-hand side of the equality of Theorem 2.1 of [R10].

On the other hand, choose a stability condition x and consider the path γ from $\mathcal{C}^+_{\mathbf{s}}$ to $\mathcal{C}^-_{\mathbf{s}}$ parameterized by μ, with $\gamma(\mu) = x - \mu \sum_i e_i^*$, with domain sufficiently large so the initial and final endpoints lie in the positive and negative orthants, respectively. Then a dimension vector has $\gamma(\mu)$-slope 0 if and only if it has x-slope μ. Thus if the description in the statement of the theorem of $g_{x,y}(\mathfrak{D}_{\mathbf{s}})$ is correct, then $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}}}$ coincides with the right-hand side of the equality of Theorem 2.1 of

[R10]. By the uniqueness of the factorization of $\mathfrak{p}_{+,-}$ from the proof of Theorem 1.17, and the faithful action of G' on $\widehat{\Bbbk[N^\oplus]}$ shown above, we obtain the result. $\square$

Because nonnegativity of Euler characteristics for the quiver moduli spaces appearing in the above statement is known (see [R14]), this gives an alternate proof of positivity of the scattering diagram in this case.

Remark 10.2. Since the initial version of this paper was released, Bridgeland [Bri] developed a Hall algebra version of scattering diagrams in the context of quiver representation theory, and the above result follows conceptually from his results.

Example 10.3. Let Q be a quiver given by an orientation of the Dynkin diagram of a simply laced finite-dimensional simple Lie algebra. Then the dimension vectors of the indecomposable complex representations of Q are the positive roots of the associated root system Δ (Gabriel's theorem). Moreover, for each positive root d, there is a unique indecomposable representation V with dimension vector d, and $\operatorname{Hom}(V,V)=\mathbb{C}$; see, e.g., [BGP73].

The $\mathcal{A}$ cluster variety associated to Q is the cluster variety of finite type associated to the root system Δ [FZ03a]. Using Proposition 10.1, we can give an explicit description of the scattering diagram $\mathfrak{D}$ for $\mathcal{A}_{\mathrm{prin}}$ as follows.

First we observe that a representation of Q that contributes to $\mathfrak{D}$ is a direct sum of copies of an indecomposable representation. Let $d \in N^+$ be a primitive vector, and let $x \in M_{\mathbb{R}}$ be such that $x^\perp \cap N = \mathbb{Z}\cdot d$. Suppose W is an x-semistable representation of Q with dimension vector a multiple of d, and consider the decomposition of W into indecomposable representations. By x-semistability and our assumption $x^\perp \cap N = \mathbb{Z}\cdot d$, each factor must have dimension vector a multiple of d. By Gabriel's theorem, we see that d is a positive root and W is a direct sum of copies of the associated indecomposable representation.

We see that the walls of $\mathfrak{D}$ are in bijection with the positive roots of Δ. Let $d \in \Delta_+$ be a positive root, and let V be the indecomposable representation with dimension vector d. Let $\mathfrak{d} \subset d^\perp \subset M_{\mathbb{R}}$ be the locus of $x \in M_{\mathbb{R}}$ such that V is x-semistable of x-slope zero; that is, $\langle x, d\rangle = 0$ and $\langle x, d'\rangle \le 0$ for d' the dimension vector of any subrepresentation of V. Then $\mathfrak{d}$ is a rational polyhedral cone in $M_{\mathbb{R}}$, and is nonempty of real codimension 1. Indeed, there exists $x \in d^\perp$ such that V is x-stable by [K94, Remark 4.5] and [S92, Theorem 6.1], and this is an open condition on $x \in d^\perp$. Now let $x \in \mathfrak{d}$ be a point such that $x^\perp \cap N = \mathbb{Z}\cdot d$. Then the x-semistable representations of x-slope zero are the direct sums of copies of V.

Let us now examine the moduli space $\mathcal{M}^x_{kd,i}$. An object in this moduli space is a direct sum $V^{\oplus k} = \mathbb{C}^k \otimes V$ of k copies of the unique indecomposable representation of dimension vector d, along with the framing, a choice of a vector $v=(v_1,\dots,v_k) \in \mathbb{C}^k \otimes V_i$. Such an object is stable if and only if v is not contained in a proper subrepresentation of $V^{\oplus k}$ of the form $W \otimes V$ for some subspace $W \subseteq \mathbb{C}^k$. In order for this to be the case, the $v_1,\dots,v_k$ must be linearly independent elements of V_i, and hence span a k-dimensional subspace of V_i. The automorphism group of $V^{\oplus k}$ is GL_k, which has the effect of changing the basis of the subspace spanned by $v_1,\dots,v_k$. Now it follows easily from the definitions that, for each $k \in \mathbb{Z}_{\ge 0}$ and $i \in I$ such that $d_i \neq 0$, the moduli space $\mathcal{M}^x_{kd,i}$ of x-semistable representations with framing at vertex i is isomorphic to the Grassmannian $\mathrm{Gr}(k, d_i)$.

So, in the notation of Proposition 10.1,

$$Q^i(z^d) = \sum_{k\geq 0} \chi(\mathrm{Gr}(k,d_i))z^{kd} = \sum_{k\geq 0} \binom{d_i}{k} z^{kd} = (1+z^d)^{d_i}$$

and

$$f(z^d) = Q^i(z^d)^{1/d_i} = 1+z^d.$$

Thus the wall of $\mathfrak{D}$ associated to $d \in \Delta_+$ is

$$(\mathfrak{d} \times N_{\mathbb{R}}, 1 + z^{(p^*(d),d)}).$$

For example, suppose Q is the quiver with vertices $1,2,3$, and arrows from 1 to 2 and 2 to 3. This is an orientation of the Dynkin diagram A_3. We have the following isomorphism types of indecomposable representations:

$$1 \to 0 \to 0, \quad 0 \to 1 \to 0, \quad 0 \to 0 \to 1,$$
$$1 \overset{\sim}{\to} 1 \to 0, \quad 0 \to 1 \overset{\sim}{\to} 1, \quad 1 \overset{\sim}{\to} 1 \overset{\sim}{\to} 1.$$

(Here the numbers denote the dimension of the vector space at the vertex, and the symbol $\sim$ over an arrow indicates that the corresponding linear transformation is an isomorphism.) We write $A_i = z^{e_i^*}$ and $X_i = z^{e_i}$. Then the walls of $\mathfrak{D}$ are

$$(e_1^\perp, 1 + A_2X_1), \quad (e_2^\perp, 1 + A_1^{-1}A_3X_2), \quad (e_3^\perp, 1 + A_2^{-1}X_3)$$
$$(\mathbb{R}e_3^* + \mathbb{R}_{\geq 0}(e_1^* - e_2^*) + N_{\mathbb{R}}, 1 + A_1^{-1}A_2A_3X_1X_2),$$
$$(\mathbb{R}e_1^* + \mathbb{R}_{\geq 0}(e_2^* - e_3^*) + N_{\mathbb{R}}, 1 + A_1^{-1}A_2^{-1}A_3X_2X_3),$$
$$(\mathbb{R}_{\geq 0}(e_1^* - e_2^*) + \mathbb{R}_{\geq 0}(e_2^* - e_3^*) + N_{\mathbb{R}}, 1 + A_1^{-1}A_3X_1X_2X_3).$$

For example, the indecomposable representation with dimension vector $(1,1,1)$ has subrepresentations with dimension vectors $(0,1,1)$ and $(0,0,1)$. So the associated wall has support $\mathfrak{d} \subset (e_1+e_2+e_3)^\perp$ defined by the inequalities $e_2 + e_3 \leq 0$ and $e_3 \leq 0$. This gives the last wall in the list.

Example 10.4. Kac generalized Gabriel's theorem to the case of an arbitrary quiver Q (without edge loops) as follows (see [K80], [K82]): Let $\mathfrak{g}$ be the Kac–Moody algebra associated to the underlying graph of Q. Then the dimension vectors of indecomposable complex representations of Q are the positive roots of $\mathfrak{g}$.

The roots Δ of $\mathfrak{g}$ are divided into real and imaginary roots. The real roots are the translates of the simple roots $e_1, \ldots, e_n$ under the action of the Weyl group. Let $\chi\colon N \times N \to \mathbb{Z}$ be the asymmetric bilinear form defined by

$$\chi(d,d') = \sum_{i\in I} d_i d'_i - \sum_{a\colon i\to j} d_i d'_j.$$

Then for representations V and V' of Q with dimension vectors d and d',

$$\chi(d,d') = \chi(V,V') := \dim \mathrm{Hom}(V,V') - \dim \mathrm{Ext}^1(V,V').$$

Let $d \in \Delta^+$ be a positive root. We have $\chi(d,d) = 1$ if d is real and $\chi(d,d) \leq 0$ if d is imaginary. The indecomposable representations of dimension vector d depend on $1 - \chi(d,d)$ parameters. We say d is a *Schur* root if there exists a representation V of Q with dimension vector d such that $\mathrm{Hom}(V,V) = \mathbb{C}$.

Assume that Q is acyclic. Let $\mathfrak{D}$ be the scattering diagram for the associated $\mathcal{A}_{\mathrm{prin}}$ cluster variety. We study $\mathfrak{D}$ using Proposition 10.1.

We show that each wall of $\mathfrak{D}$ is contained in $d^\perp$ for d a primitive Schur root. First, as in Example 10.3, each wall is contained in $d^\perp$ for d a primitive positive root (note that the set of roots is saturated by [K80, Proposition 1.2]). It remains to show that d is necessarily a Schur root. Otherwise, a representation V with dimension vector d deforms to a decomposable representation V' [K82, Proposition 1(b)]. Then, for $x \in M_{\mathbb{R}}$ such that $x^{\perp} \cap N = \mathbb{Z} \cdot d$, V' is x-unstable, and so V is x-unstable (as x-semistability is an open condition). A representation with dimension vector a multiple of d is x-unstable for the same reason, using [S92, Theorem 3.8]. Now by Proposition 10.1 we see that there does not exist a wall of $\mathfrak{D}$ contained in $d^{\perp}$.

For a real Schur root d there is a unique wall contained in $d^{\perp}$ which can be described explicitly as in Example 10.3. We remark that d is a real Schur root iff there is an indecomposable representation V of Q with dimension vector d such that $\mathrm{Hom}(V, V) = \mathbb{C}$ and $\mathrm{Ext}^1(V, V) = 0$. (Moreover, V is uniquely determined by d.) Such a representation V is an exceptional object in the category of representations of Q in the sense of [B90].

For an imaginary Schur root the associated walls involve contributions from positive-dimensional moduli spaces of semistable representations of Q. The case of the imaginary root $d = (1, 1)$ for the quiver Q with vertices $1, 2$ and two arrows from 1 to 2 is described in [R10, §6.1]. (This is the case $b = c = 2$ of Example 1.15.)

Appendix A. Review of notation and Langlands duality

We first review basic cluster variety notation as adopted in [GHK13]. None of this is original to [GHK13], but we follow that source for consistency of notation.

As in [GHK13, §2], *fixed data* Γ means

- A lattice N with a skew-symmetric bilinear form
$$\{\cdot, \cdot\} : N \times N \to \mathbb{Q}.$$
- An *unfrozen sublattice* $N_{\mathrm{uf}} \subseteq N$, a saturated sublattice of N. If $N_{\mathrm{uf}} = N$, we say the fixed data has no frozen variables.
- An index set I with $|I| = \mathrm{rank}\, N$ and a subset $I_{\mathrm{uf}} \subseteq I$ with $|I_{\mathrm{uf}}| = \mathrm{rank}\, N_{\mathrm{uf}}$.
- Positive integers d_i for $i \in I$ with greatest common divisor 1.
- A sublattice $N^{\circ} \subseteq N$ of finite index such that $\{N_{\mathrm{uf}}, N^{\circ}\} \subseteq \mathbb{Z}$ and $\{N, N_{\mathrm{uf}} \cap N^{\circ}\} \subseteq \mathbb{Z}$.
- $M = \mathrm{Hom}(N, \mathbb{Z})$, $M^{\circ} = \mathrm{Hom}(N^{\circ}, \mathbb{Z})$.

Here we modify the definition slightly, and include in the *fixed data* $[\mathbf{s}]$ a mutation class of seed. Recall a seed $\mathbf{s} = (e_1, \ldots, e_n)$ is a basis of N satisfying certain properties; see [GHK13, §2], for the precise definitions, including that of mutation. In particular, we write $e_1^*, \ldots, e_n^*$ for the dual basis and $f_i = d_i^{-1} e_i^*$. We write

$$\epsilon_{ij} := \{e_i, e_j\} d_j. \tag{A.1}$$

We shall also assume that if $i \in I_{\mathrm{uf}}$, then the linear functional $\{e_i, \cdot\}$ is nonzero. (If this happens, then one can view e_i as frozen.)

We have two natural maps defined by $\{\cdot, \cdot\}$:

$$\begin{array}{cc} p_1^* : N_{\mathrm{uf}} \to M^{\circ} & p_2^* : N \to M^{\circ}/N_{\mathrm{uf}}^{\perp} \\ N_{\mathrm{uf}} \ni n \mapsto (N^{\circ} \ni n' \mapsto \{n, n'\}) & N \ni n \mapsto (N_{\mathrm{uf}} \cap N^{\circ} \ni n' \mapsto \{n, n'\}) \end{array}$$

We also choose a map

$$p^* : N \to M^{\circ} \tag{A.2}$$

such that (a) $p^*|_{N_{\mathrm{uf}}} = p_1^*$ and (b) the composed map $N \to M^\circ/N_{\mathrm{uf}}^\perp$ agrees with p_2^*. Different choices of p^* differ by a choice of map $N/N_{\mathrm{uf}} \to N_{\mathrm{uf}}^\perp$. Further, if there are no frozen variables, i.e., $I_{\mathrm{uf}} = I$, then $p^* = p_1^* = p_2^*$ is canonically defined.

We also define

$$K = \ker p_2^*, \quad K^\circ = K \cap N^\circ.$$

Following our conventions in [GHK13], let $\mathfrak{T}$ be the infinite oriented rooted tree with $|I_{\mathrm{uf}}|$ outgoing edges from each vertex, labeled by the elements of I_{uf}. Let v be the root of the tree. Attach some choice of initial seed $\mathbf{s} \in [\mathbf{s}]$ to the vertex v. (We write $\mathfrak{T}_{\mathbf{s}}$ if we want to record this choice of initial seed.) Now each simple path starting at v determines a sequence of mutations, just mutating at the label attached to the edge. In this way we attach a seed to each vertex of $\mathfrak{T}$. We write the seed attached to a vertex w as $\mathbf{s}_w$, and write $T_{N^\circ,\mathbf{s}_w}, T_{M,\mathbf{s}_w}$, etc., for the corresponding tori. Mutations define birational maps between these tori, and the associated Fock–Goncharov $\mathcal{A}$, $\mathcal{X}$ cluster varieties are defined by

$$\mathcal{A}_\Gamma = \bigcup_{w\in\mathfrak{T}} T_{N^\circ,\mathbf{s}_w}, \qquad \mathcal{X}_\Gamma = \bigcup_{w\in\mathfrak{T}} T_{M,\mathbf{s}_w}. \tag{A.3}$$

This parameterization of torus charts is very redundant, with infinitely many copies of the same chart appearing. In particular, given a vertex w of $\mathfrak{T}$, one can consider the subtree $\mathfrak{T}_w$ rooted at w, with initial seed $\mathbf{s}_w$. This tree can similarly be used to define $\mathcal{A}_\Gamma$, and the obvious inclusion between these two versions of $\mathcal{A}_\Gamma$ is in fact an isomorphism, as can be easily checked.

As one expects the mirror of a variety obtained by gluing charts of the form T_{M° to be obtained by gluing charts of the form T_{N°, the mirror of $\mathcal{A}$ is not $\mathcal{X}$, as the latter is obtained by gluing charts of the form T_N. To get the correct mirrors of $\mathcal{A}$ and $\mathcal{X}$, one follows [FG09] in defining the *Langlands dual* cluster varieties. This is done by, given fixed data Γ, defining fixed data $\Gamma^\vee$ to be the fixed data

$$I^\vee := I, \quad I_{\mathrm{uf}}^\vee := I_{\mathrm{uf}}, \quad d_i^\vee := d_i^{-1} D,$$

where

$$D := \mathrm{lcm}(d_1, \ldots, d_n).$$

The lattice, with its finite index sublattice, is

$$D \cdot N =: (N^\vee)^\circ \subset N^\vee := N^\circ,$$

and the $\mathbb{Q}$-valued skew-symmetric form on $N^\vee = N^\circ$ is

$$\{\cdot,\cdot\}^\vee := D^{-1}\{\cdot,\cdot\}.$$

For each $\mathbf{s} = (e_1, \ldots, e_n) \in [\mathbf{s}]$, we define

$$\mathbf{s}^\vee := (d_1 e_1, \ldots, d_n e_n).$$

One checks easily that $\mathbf{s} \mapsto \mathbf{s}^\vee$ gives a bijection between $[\mathbf{s}]$ and $[\mathbf{s}^\vee]$.

Note that for skew-symmetric cluster algebras, i.e., when all the multipliers $d_i = 1$, Langlands duality is the identity, $\Gamma^\vee = \Gamma$.

Definition A.4 (Fock–Goncharov dual)**.** We write $\mathcal{A}_\Gamma^\vee := \mathcal{X}_{\Gamma^\vee}$ and $\mathcal{X}_\Gamma^\vee := \mathcal{A}_{\Gamma^\vee}$.

Note in the skew-symmetric case, that $\mathcal{A}^\vee = \mathcal{X}$.

One observes the elementary

Proposition A.5. *Given fixed data Γ, the double Langlands dual data $\Gamma^{\vee\vee}$ is canonically isomorphic to the data Γ via the map $D \cdot N \to N$ given by $n \mapsto D^{-1}n$.*

Appendix B. The $\mathcal{A}$ and $\mathcal{X}$-varieties with principal coefficients

We recall briefly the construction of principal fixed data from [GHK13, Construction 2.11]. For fixed data Γ, the data for the cluster variety with principal coefficients Γ_{prin} is defined by:

- $\widetilde{N} := N \oplus M^\circ$ with the skew-symmetric bilinear form
$$\{(n_1, m_1), (n_2, m_2)\} = \{n_1, n_2\} + \langle n_1, m_2\rangle - \langle n_2, m_1\rangle.$$
- $\widetilde{N}_{\text{uf}} := N_{\text{uf}} \oplus 0 \subset \widetilde{N}$.
- The sublattice $\widetilde{N}^\circ$ is $N^\circ \oplus M$.
- The index set I is now the disjoint union of two copies of I, with the d_i taken to be as in Γ. The set of unfrozen indices I_{uf} is just the original I_{uf} thought of as a subset of the first copy of I.
- Given an initial seed $\mathbf{s} = (e_1, \ldots, e_n)$, we define

$$\tilde{\mathbf{s}} = \big((e_1, 0), \ldots, (e_n, 0), (0, f_1), \ldots, (0, f_n)\big). \tag{B.1}$$

  We then take the mutation class $[\tilde{\mathbf{s}}]$.

Note that $[\tilde{\mathbf{s}}]$ depends on the choice of $\mathbf{s}$: it is not true that if $\mathbf{s}'$ is obtained by mutation from $\mathbf{s}$, then $\tilde{\mathbf{s}}'$ is obtained from the same set of mutations applied to $\tilde{\mathbf{s}}$. Nevertheless, the cluster varieties

$$\mathcal{X}_{\text{prin}} := \mathcal{X}_{\Gamma_{\text{prin}}}, \quad \mathcal{A}_{\text{prin}} := \mathcal{A}_{\Gamma_{\text{prin}}}$$

are defined independently of the seed $\mathbf{s}$. This is a very important point, which we shall revisit in Remark B.8.

The following summarizes all of the important relationships between the various varieties which will be made use of in this paper.

Proposition B.2. *Giving fixed data Γ, we have:*

(1) *There is a commutative diagram where the dotted arrows are only present if there are no frozen variables (i.e., $N_{\text{uf}} = N$) :*

$$\begin{tikzcd}
\mathcal{A}_t \arrow[r] \arrow[d] & \mathcal{A}_{\text{prin}} \arrow[r, "\tilde{p}"] \arrow[d, "\pi"] \arrow[rr, bend left, "p"] & \mathcal{X} \arrow[d, "\lambda"] & \mathcal{X}_{\text{prin}} \arrow[l, "\rho"'] \arrow[d, "w"] & \mathcal{A} \arrow[l, "\xi"'] \arrow[d] \\
t \arrow[r] & T_M \arrow[r, dashed] & T_{K^*} & T_M \arrow[l, dashed] & e \arrow[l]
\end{tikzcd}$$

with $t \in T_M$ any point, $e \in T_M$ the identity, and with the left- and right-hand squares cartesian and p an isomorphism, canonical if there are no frozen variables.

(2) *There are torus actions*

$$T_{N^\circ} \text{ on } \mathcal{A}_{\text{prin}}, \qquad T_{K^\circ} \text{ on } \mathcal{A}, \qquad T_{N_{\text{uf}}^\perp} \text{ on } \mathcal{X}, \qquad T_{\widetilde{K}^\circ} \text{ on } \mathcal{A}_{\text{prin}}.$$

Here $\widetilde{K}^\circ$ is the kernel of the map

$$\begin{aligned} N^\circ \oplus M &\to N_{\text{uf}}^* \\ (n, m) &\mapsto p_2^*(n) - m. \end{aligned}$$

Furthermore T_{N° and $T_{\widetilde{K}^\circ}$ act on T_M so that the map $\pi : \mathcal{A}_{\mathrm{prin}} \to T_M$ is T_{N°- and $T_{\widetilde{K}^\circ}$-equivariant. The map $\tilde{p} : \mathcal{A}_{\mathrm{prin}} \to \mathcal{X} = \mathcal{A}_{\mathrm{prin}}/T_{N^\circ}$ is a T_{N°-torsor. There is a map $T_{\widetilde{K}^\circ} \to T_{N_{\mathrm{uf}}^\perp}$ such that the map $\tilde{p}$ is also compatible with the actions of these two tori on $\mathcal{A}_{\mathrm{prin}}$ and $\mathcal{X}$, respectively, so that

$$\tau : \mathcal{A}_{\mathrm{prin}} \to \mathcal{X}/T_{N_{\mathrm{uf}}^\perp}$$

is a $T_{\widetilde{K}^\circ}$-torsor.

(3) $(\Gamma_{\mathrm{prin}})^\vee$ *and* $(\Gamma^\vee)_{\mathrm{prin}}$ *are isomorphic data, so we can define*

$$\mathcal{A}^\vee_{\mathrm{prin}} := \mathcal{X}_{(\Gamma^\vee)_{\mathrm{prin}}}, \quad \mathcal{X}^\vee_{\mathrm{prin}} := \mathcal{A}_{(\Gamma^\vee)_{\mathrm{prin}}},$$

(4) *There is a commutative diagram:*

$$\begin{tikzcd}
\mathcal{X}^\vee \arrow[r] \arrow[d] & \mathcal{X}^\vee_{\mathrm{prin}} \arrow[r, "\tilde{p}"] \arrow[d, "\pi"'] \arrow[rr, bend left, "p"] & \mathcal{A}^\vee \arrow[d, "\lambda"'] & \mathcal{A}^\vee_{\mathrm{prin}} \arrow[l, "\rho"'] \arrow[d, "w"'] & \mathcal{X}^\vee \arrow[l, "\xi"'] \arrow[d] \\
e \arrow[r] & T_{M^\circ} \arrow[r, dashed] & T_{(K^\circ)^*} & T_{M^\circ} \arrow[l, dashed] & e \arrow[l]
\end{tikzcd}$$

Proof. We consider the diagram of (1). The maps with names are given as follows on cocharacter lattices:

$$\begin{aligned}
\pi : N^\circ \oplus M \to M, &\quad (n, m) \mapsto m, \\
\tilde{p} : N^\circ \oplus M \to M, &\quad (n, m) \mapsto m - p^*(n), \\
\rho : M \oplus N^\circ \to M, &\quad (m, n) \mapsto m, \\
\lambda : M \to K^*, &\quad m \mapsto m|_K, \\
w : M \oplus N^\circ \to M, &\quad (m, n) \mapsto m - p^*(n), \\
\xi : N^\circ \to M \oplus N^\circ, &\quad n \mapsto (-p^*(n), -n), \\
p : N^\circ \oplus M \to M \oplus N^\circ, &\quad (n, m) \mapsto (m - p^*(n), n).
\end{aligned} \tag{B.3}$$

Note λ is the transpose of the inclusion $K \to N$. In the case there are no frozen variables, the two dotted horizontal lines are just given on cocharacter lattices by λ again. One checks commutativity from these formulas at the level of individual tori, and one checks the maps are compatible with mutations. Note the left-hand diagram defines $\mathcal{A}_t$; see [GHK13, Definition 2.12]. The statements that $\tilde{p}$, π and λ are compatible with mutations are in [GHK13, §2], as well as the commutativity of the second square in case of no frozen variables. It is clear that p induces an isomorphism of lattices, hence an isomorphism of the relevant tori. This isomorphism is canonical in the no frozen variable case because p^* is well-defined in this case. The fact the right-hand square is cartesian follows from the fact that $\operatorname{Im} \xi = \ker w$. Note the signs in the definition of ξ are necessary to be compatible with mutations. This gives (1).

For (2), the first action is specified on the level of cocharacter lattices by

$$N^\circ \to N^\circ \oplus M, \quad n \mapsto (n, p^*(n)),$$

while the last three are given by the inclusions

$$K^\circ \subset N^\circ, \quad N_{\mathrm{uf}}^\perp \subset M, \quad \widetilde{K}^\circ \subset N^\circ \oplus M.$$

One checks easily that the induced actions are compatible with mutations. The action of T_{N° and $T_{\widetilde{K}^\circ}$ on T_M are induced by the maps $n \mapsto p^*(n)$ and $(n, m) \mapsto m$, respectively, in order to achieve the desired equivariance. The map $T_{\widetilde{K}^\circ} \to T_{N_{\mathrm{uf}}^\perp}$ is given by

$$\widetilde{K}^\circ \ni (m, n) \mapsto m - p^*(n) \in N_{\mathrm{uf}}^\perp.$$

The other statements are easily checked.

For (3), from the definitions, the lattices playing the role of $N^\circ \subseteq N$ are

$$\begin{aligned}(\Gamma_{\mathrm{prin}})^\vee: &\quad D \cdot \widetilde{N} = D \cdot N \oplus D \cdot M^\circ \subseteq \widetilde{N}^\circ = N^\circ \oplus M,\\ (\Gamma^\vee)_{\mathrm{prin}}: &\quad D \cdot N \oplus M^\circ \subseteq N^\circ \oplus D^{-1} \cdot M.\end{aligned}$$

These are isomorphic under the map $(n, m) \mapsto (n, D^{-1}m)$. Furthermore, the pairings in the two cases are given by

$$\{(n_1, m_1), (n_2, m_2)\} = \begin{cases} D^{-1}(\{n_1, n_2\} + \langle n_1, m_2\rangle - \langle n_2, m_1\rangle) & \text{in the } (\Gamma_{\mathrm{prin}})^\vee \text{ case},\\ D^{-1}\{n_1, n_2\} + \langle n_1, m_2\rangle - \langle n_2, m_1\rangle & \text{in the } (\Gamma^\vee)_{\mathrm{prin}} \text{ case},\end{cases}$$

respectively. The isomorphism given preserves the pairings, hence the isomorphism.

(4) is the same as (1), but for the Langlands dual data $\Gamma^\vee$. For reference, the maps are given as follows:

$$\begin{aligned} \pi: D \cdot N \oplus M^\circ \to M^\circ, &\quad (n, m) \mapsto m,\\ \tilde{p}: D \cdot N \oplus M^\circ \to M^\circ, &\quad (Dn, m) \mapsto m - p^*(n),\\ \rho: M^\circ \oplus D \cdot N \to M^\circ, &\quad (m, Dn) \mapsto m,\\ \lambda: M^\circ \to (K^\circ)^*, &\quad m \mapsto m|_K,\\ w: M^\circ \oplus D \cdot N \to M^\circ, &\quad (m, Dn) \mapsto m - p^*(n),\\ \xi: D \cdot N \to M^\circ \oplus D \cdot N, &\quad Dn \mapsto (-p^*(n), -Dn),\\ p: D \cdot N \oplus M^\circ \to M^\circ \oplus D \cdot N, &\quad (Dn, m) \mapsto (m - p^*(n), Dn).\end{aligned} \tag{B.4}$$

□

Remark B.5. Whenever the lattice $D \cdot N$ appears in dealing with the Langlands dual data, we will always identify this with N in the obvious way.

Simple linear algebra gives:

Lemma B.6. *The choice of the map p^* gives an inclusion $N^\circ \subset \widetilde{K}^\circ$ (see Proposition B.2(2)) given by $n \mapsto (n, p^*(n))$. We also have $N_{\mathrm{uf}}^\perp$ (a sublattice of M) included in $\widetilde{K}^\circ$ via $m \mapsto (0, m)$. These inclusions induce an isomorphism $N^\circ \oplus N_{\mathrm{uf}}^\perp \to \widetilde{K}^\circ$.*

Lemma B.7. *The map $T_{\widetilde{K}^\circ} \to T_M$ induced by the composition of the inclusion and projection $\widetilde{K}^\circ \subset \widetilde{N}^\circ \to M$ is a split surjection if and only if the map*

$$p_2^*|_{N^\circ}: N^\circ \to N_{\mathrm{uf}}^*, \quad n \mapsto \{n, \cdot\}|_{N_{\mathrm{uf}}}$$

is surjective. This holds if and only if in some seed $\mathbf{s} = (e_i)_{i \in I}$, the $\#I_{\mathrm{uf}} \times \#I$ matrix with entries for $i \in I_{\mathrm{uf}}$, $j \in I$, $\epsilon_{ij} = \{e_i, d_j e_j\}$ gives a surjective map $\mathbb{Z}^{\#I} \to \mathbb{Z}^{\#I_{\mathrm{uf}}}$. In this case $\pi: \mathcal{A}_{\mathrm{prin}} \to T_M$ is isomorphic to the trivial bundle $\mathcal{A} \times T_M \to T_M$.

Proof. For the first statement, note using Lemma B.6 that the map $\widetilde{K}^\circ \to M$ is surjective if and only if the map $N^\circ \oplus N_{\mathrm{uf}}^\perp \to M$ given by $(n, m) \mapsto m + p^*(n)$ is surjective, and this is the case if and only if the induced map $N^\circ \to M/N_{\mathrm{uf}}^\perp = N_{\mathrm{uf}}^*$

is surjective. The given matrix is the matrix for $N^\circ \to N_{\mathrm{uf}}^*$ in the given bases, so the second equivalence is clear. The final statement follows from the $T_{\widetilde{K}^\circ}$-equivariance of π (the trivialization then comes by choosing a splitting of $\widetilde{K}^\circ \twoheadrightarrow M$). $\square$

Remark B.8. In general, a seed is defined to be a basis of the lattice N (or $\widetilde{N}$), but to define the seed mutations [GHK13, (2.2)] and the union of tori (A.3), all one needs are elements $e_i \in N$, $i \in I_{\mathrm{uf}}$ (the definitions as given make sense even if the e_i are dependent or fail to span). If one makes the construction in this greater generality, the characters $X_i := z^{e_i}$ on $T_{M,\mathbf{s}} \subset \mathcal{X}$ will not be independent (if the e_i are not), and unless we take a full basis, we cannot define the cluster variables $A_i := z^{f_i}$ on $T_{N^\circ,\mathbf{s}}$, as the f_i are defined as the dual basis to the basis $(d_1e_1, \ldots, d_ne_n)$ for N°.

In the case of the principal data, given a seed $\mathbf{s} = (e_1, \ldots, e_n)$ for Γ, we get a seed $\big((e_1, 0), \ldots, (e_n, 0)\big)$ in this modified sense for the data Γ_{prin}. We also write this seed as $\mathbf{s}$. On the other hand, in [GHK13], the seed $\tilde{\mathbf{s}}$ for Γ_{prin} is defined in the more traditional sense to be the basis $\big((e_1, 0), \ldots, (e_n, 0), (0, f_1), \ldots, (0, f_n)\big)$. It is not the case that if $\mathbf{s}'$ is obtained from $\mathbf{s}$ via a sequence of mutations, then $\tilde{\mathbf{s}}'$ is obtained from $\tilde{\mathbf{s}}$ by the same sequence of mutations. In particular, the set $[\tilde{\mathbf{s}}]$ of seeds mutation equivalent to $\tilde{\mathbf{s}}$ depends not just on the mutation equivalence class of $\mathbf{s}$, but on the original seed $\mathbf{s}$. However, using the seed $\mathbf{s}$ as a seed for Γ_{prin} in this modified sense, we can build $\mathcal{A}_{\mathrm{prin}}$, and this depends only on the mutation class of $\mathbf{s}$. Thus $\mathcal{A}_{\mathrm{prin}}$ does not depend on the initial choice of seed, but only on its mutation equivalence class.

However, as we shall now see, the choice of initial seed does give a partial compactification. This is a more general phenomenon when there are frozen variables.

Construction B.9 (Partial compactifications from frozen variables)**.** When the cluster data Γ includes frozen variables, $\mathcal{A}$ comes with a canonical partial compactification $\mathcal{A} \subset \overline{\mathcal{A}}$, given by partially compactifying each torus chart via $T_{N^\circ,\mathbf{s}} \subset \mathrm{TV}(\Sigma^{\mathbf{s}})$, where for $\mathbf{s} = (e_i)$, $\Sigma^{\mathbf{s}} = \sum_{i \notin I_{\mathrm{uf}}} \mathbb{R}_{\geq 0} e_i \subset N^\circ_{\mathbb{R},\mathbf{s}}$. Thus the dual cone $(\Sigma^{\mathbf{s}})^\vee \subset M^\circ_{\mathbb{R},\mathbf{s}}$ is cut out by the half-spaces $e_i \geq 0$, $i \notin I_{\mathrm{uf}}$. Note that the monomials $A_i := z^{f_i}$, $i \notin I_{\mathrm{uf}}$ are invariant under mutation. These give a canonical map $\overline{\mathcal{A}} \to \mathbb{A}^{\operatorname{rank} N - u}$, where u is the number of unfrozen variables. Note that the basis elements e_i for $i \notin I_{\mathrm{uf}}$, though they have frozen indices, can change under mutation. What is invariant is the associated boundary divisor with valuation given by $e_i \in N^\circ_{\mathbf{s}} = \mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$. These are the boundary divisors of $\mathcal{A} \subset \overline{\mathcal{A}}$. We remark that like $\mathcal{A}$, $\overline{\mathcal{A}}$ is also separated, with the argument given in [GHK13, Theorem 3.14] working equally well for $\overline{\mathcal{A}}$.

Here is another way of seeing the same thing. Given any cluster variety $V = \bigcup_{\mathbf{s} \in S} T_{L,\mathbf{s}}$ and a single fan $\Sigma \subset L_{\mathbb{R}}$ for a toric partial compactification $T_{L,\mathbf{s}'} \subset \mathrm{TV}(\Sigma)$ for some $\mathbf{s}' \in S$, there is a canonical way to build a partial compactification

$$V \subset \overline{V} = \bigcup_{\mathbf{s} \in S} \mathrm{TV}(\Sigma^{\mathbf{s}}).$$

We let $\Sigma^{\mathbf{s}'} := \Sigma$ and $\Sigma^{\mathbf{s}} := (\mu^t_{\mathbf{s},\mathbf{s}'})^{-1}(\Sigma^{\mathbf{s}'})$, where $\mu_{\mathbf{s},\mathbf{s}'}$ is the birational map given by the composition

$$\mu_{\mathbf{s},\mathbf{s}'} : T_{L,\mathbf{s}} \subset V \supset T_{L,\mathbf{s}'}$$

and $\mu^t_{\mathbf{s},\mathbf{s}'}$ is the geometric tropicalization; see §2.

Remark B.10. We now return to the discussion of $\mathcal{A}_{\text{prin}}$. Note that the frozen variables for $\mathcal{A}_{\text{prin}}$ are indexed by $I \setminus I_{\text{uf}}$ in the first copy of I, along with all indices in the second copy of I. However, we can apply Construction B.9 taking only the second copy of I as the set of frozen indices, with the initial choice of seed $\mathbf{s}$ determining a partial compactification of $\mathcal{A}_{\text{prin}}$. In this case, we indicate the partial compactification by $\mathcal{A}_{\text{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}}$. It is important to keep in mind the dependence on $\mathbf{s}$. Fixing $\mathbf{s}$ fixes $\tilde{\mathbf{s}}$, and hence cluster variables $A_i = z^{(f_i,0)}$, $X_i = z^{(0,e_i)}$. The variables X_i can then take the value 0 in the compactification. In particular, we obtain an extension of $\pi : \mathcal{A}_{\text{prin}} \to T_M$ to $\pi : \overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}} \to \mathbb{A}^n_{X_1,\ldots,X_n}$, $X_i := z^{e_i}$ pulling back to $X_i = z^{(0,e_i)}$.

Note that the seeds in $[\mathbf{s}]$ and $[\tilde{\mathbf{s}}]$ are in one-to-one correspondence. Given any seed $\mathbf{s}' = (e'_i)_{i\in I} \in [\mathbf{s}]$ and seed $\tilde{\mathbf{s}}' \in [\tilde{\mathbf{s}}]$ obtained via the same sequence of mutations, we have $\tilde{\mathbf{s}}' = \big((e'_i, 0)_{i\in I}, (g_i)_{i\in I}\big)$ for some $g_i \in \widetilde{N}$. These two seeds give rise to coordinates A'_i on the chart of $\mathcal{A}$ indexed by $\mathbf{s}'$ and coordinates A'_i, X_i on the chart of $\mathcal{A}_{\text{prin}}$ indexed by $\tilde{\mathbf{s}}'$. As $\mathcal{A}$ is the fiber of π over the point of $\mathbb{A}^n$ with all coordinates 1, the coordinate A'_i on the chart of $\mathcal{A}_{\text{prin}}$ restricts to the coordinate A'_i on the chart of $\mathcal{A}$. This gives a one-to-one correspondence between cluster variables on $\mathcal{A}$ and A-type cluster variables on $\mathcal{A}_{\text{prin}}$. To summarize:

Proposition B.11. *The cluster variety* $\mathcal{A}_{\text{prin}} := \bigcup_{w\in\mathfrak{T}_{\mathbf{s}}} T_{\widetilde{N}^\circ,\mathbf{s}_w}$ *depends only on the mutation class* $[\mathbf{s}]$. *But the choice of a seed* $\mathbf{s}$ *determines:*

1. *a partial compactification* $\mathcal{A}_{\text{prin}} \subset \overline{\mathcal{A}}^{\mathbf{s}}_{\text{prin}}$;
2. *the canonical extension of each cluster variable on any chart of* $\mathcal{A}$ *to a cluster variable on the corresponding chart of* $\mathcal{A}_{\text{prin}} \supset \mathcal{A}$.

Appendix C. Construction of scattering diagrams

This appendix is devoted to giving proofs of Theorems 1.28 and 1.13. The proof of 1.28 is essentially given in [GS11], but the special case here is considerably simpler than the general case covered there, and it is likely to be very difficult for the reader to extract the needed results from [GS11]. In addition, the details of the proof of Theorem 1.28 will be helpful in proving Theorem 1.13.

C.1. An algorithmic construction of scattering diagrams.

Construction C.1. There is a simple order by order algorithm, introduced in [KS06] in the two-dimensional case and in [GS11] in the higher-dimensional case, for producing the diagram $\mathfrak{D} \supset \mathfrak{D}_{\text{in}}$ of Theorem 1.21, which we will describe shortly after a bit of preparation. This is useful both from a computational point of view and because a more complicated version of this will be necessary in the remainder of this appendix.

We continue with fixed data Γ, yielding the Lie algebra $\mathfrak{g}$ in §1.1.

We first introduce some additional terminology. For any scattering diagram $\mathfrak{D}$ for $N^+, \mathfrak{g}$, and any $k > 0$, we let $\mathfrak{D}_k \subset \mathfrak{D}$ be the (by definition, finite) set of $(\mathfrak{d}, g_{\mathfrak{d}})$ with $g_{\mathfrak{d}}$ nontrivial in $G^{\le k}$. A scattering diagram for N^+, $\mathfrak{g}$ induces a scattering diagram for N^+, $\mathfrak{g}^{\le k}$ in the obvious way, viewing $g_{\mathfrak{d}} \in G^{\le k}$ for a wall $(\mathfrak{d}, g_{\mathfrak{d}})$. We say two scattering diagrams $\mathfrak{D}$, $\mathfrak{D}'$ are equivalent to order k if they are equivalent as scattering diagrams for $\mathfrak{g}^{\le k}$.

Definition-Lemma C.2. *Let* $\mathfrak{j}$ *be a joint of the scattering diagram* $\mathfrak{D}_k$. *Either every wall containing* $\mathfrak{j}$ *has direction tangent to* $\mathfrak{j}$ (*where the direction of a wall* *contained in* $n^\perp$ *is* $-p^*(n) = -\{n,\cdot\}$*), or every wall containing* $\mathfrak{j}$ *has direction* not *tangent to* $\mathfrak{j}$*. In the first case we call the joint parallel; in the second case, perpendicular.*

Proof. Suppose $\mathfrak{j}$ spans the subspace $n_1^\perp \cap n_2^\perp$. Then the direction of any wall containing $\mathfrak{j}$ is of the form $-p^*(a_1n_1+a_2n_2)$ for some $a_1, a_2 \in \mathbb{Q}$. If this is tangent to $\mathfrak{j}$, then $\langle p^*(a_1n_1+a_2n_2), n_i\rangle = 0$ for $i = 1, 2$, and hence $0 = \langle p^*(n_1), n_2\rangle = \{n_1, n_2\}$. From this it follows that $\langle p^*(a_1'n_1 + a_2'n_2), n_i\rangle = 0$ for all a_1', a_2', and hence the direction of any wall containing $\mathfrak{j}$ is tangent to $\mathfrak{j}$. $\square$

A joint $\mathfrak{j}$ is a codimension 2 convex rational polyhedral cone. Let $\Lambda_{\mathfrak{j}} \subseteq M^\circ$ be the set of integral tangent vectors to $\mathfrak{j}$. This is a saturated sublattice of M°. Then we set

$$\mathfrak{g}_{\mathfrak{j}} := \bigoplus_{n\in N^+\cap\Lambda_{\mathfrak{j}}^\perp} \mathfrak{g}_n. \tag{C.3}$$

This is closed under Lie bracket. If $\mathfrak{j}$ is a parallel joint, then $\mathfrak{g}_{\mathfrak{j}}$ is abelian, since if $n_1, n_2 \in \Lambda_{\mathfrak{j}}^\perp$ with $p^*(n_1), p^*(n_2) \in \Lambda_{\mathfrak{j}}$, $\{n_1, n_2\} = \langle p^*(n_1), n_2\rangle = 0$, so $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] = 0$. We denote by $G_{\mathfrak{j}}$ the corresponding group.

We will build a sequence of finite scattering diagrams $\tilde{\mathfrak{D}}_1 \subset \tilde{\mathfrak{D}}_2 \subset \cdots$, with the property that $\tilde{\mathfrak{D}}_k$ is equivalent to $\mathfrak{D}$ to order k. Taking $\tilde{\mathfrak{D}} = \bigcup_{k=1}^\infty \tilde{\mathfrak{D}}_k$, we obtain $\tilde{\mathfrak{D}}$ equivalent to $\mathfrak{D}$. Let $(\mathfrak{D}_{\text{in}})_k$ denote the subset of $\mathfrak{D}_{\text{in}}$ consisting of walls which are nontrivial in $G^{\leq k}$. We start with

$$\tilde{\mathfrak{D}}_1 = (\mathfrak{D}_{\text{in}})_1.$$

If $\mathfrak{j}$ is a joint of a finite scattering diagram, we write $\gamma_{\mathfrak{j}}$ for a simple loop around $\mathfrak{j}$ small enough so that it only intersects walls containing $\mathfrak{j}$. In particular, for each joint $\mathfrak{j}$ of $\tilde{\mathfrak{D}}_1$, $\mathfrak{p}_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}_1} = \text{id} \in G^{\leq 1}$. Indeed, $G^{\leq 1}$ is abelian and by the form given for $\mathfrak{D}_{\text{in}}$ in the statement of Theorem 1.21, all walls containing $\mathfrak{j}$ are hyperplanes. Thus the automorphism associated to crossing each wall and its inverse occur once in $\mathfrak{p}_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}_1}$, and hence cancel.

Now suppose we have constructed $\tilde{\mathfrak{D}}_k$. For every perpendicular joint $\mathfrak{j}$ of $\tilde{\mathfrak{D}}_k$, we can write uniquely in $G_{\mathfrak{j}}^{\leq k+1}$

$$\mathfrak{p}_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}_k} = \exp\left(\sum_{\alpha\in S} g_\alpha\right), \tag{C.4}$$

where $S \subseteq \{\alpha \in N^+ \cap \Lambda_{\mathfrak{j}}^\perp \,|\, d(\alpha) = k+1\}$ and $g_\alpha \in \mathfrak{g}_\alpha$. Such an expression holds because all wall-crossing automorphisms for walls containing $\mathfrak{j}$ lie in $G_{\mathfrak{j}}$, so that $\mathfrak{p}_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}_k}$ can be viewed as an element of $G_{\mathfrak{j}}^{\leq k+1}$. Furthermore, by the inductive hypothesis, this element is trivial in $G_{\mathfrak{j}}^{\leq k}$. Because $\mathfrak{j}$ is perpendicular, we never have $p^*(\alpha) \in \Lambda_{\mathfrak{j}}$. Now define

$$\mathfrak{D}[\mathfrak{j}] := \{(\mathfrak{j} - \mathbb{R}_{\geq 0}p^*(\alpha), \exp(\pm g_\alpha)) \,|\, \alpha \in S\},$$

where the sign is chosen so that the contribution to crossing the wall indexed by α in $\mathfrak{p}_{\gamma_{\mathfrak{j}},\mathfrak{D}[\mathfrak{j}]}$ is $\exp(-g_\alpha)$. Note the latter element is central in $G^{\leq k+1}$. Thus $\mathfrak{p}_{\gamma_{\mathfrak{j}},\mathfrak{D}[\mathfrak{j}]} = \mathfrak{p}^{-1}_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}_k}$ and

$$\mathfrak{p}_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}_k\cup\mathfrak{D}[\mathfrak{j}]} = \mathfrak{p}_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}_k} \circ \mathfrak{p}_{\gamma_{\mathfrak{j}},\mathfrak{D}[\mathfrak{j}]} = \text{id} \tag{C.5}$$

in $G^{\leq k+1}$.

We define
$$\tilde{\mathfrak{D}}_{k+1} = \tilde{\mathfrak{D}}_k \cup ((\mathfrak{D}_{\text{in}})_{k+1} \setminus (\mathfrak{D}_{\text{in}})_k) \cup \bigcup_{\mathfrak{j}} \mathfrak{D}[\mathfrak{j}],$$
where the union is over all perpendicular joints of $\tilde{\mathfrak{D}}_k$.

Lemma C.6. $\tilde{\mathfrak{D}}_{k+1}$ *is equivalent to* $\mathfrak{D}$ *to order* $k+1$.

Proof. Consider a perpendicular joint $\mathfrak{j}$ of $\tilde{\mathfrak{D}}_{k+1}$. If $\mathfrak{j}$ is contained in a joint $\mathfrak{j}'$ of $\tilde{\mathfrak{D}}_k$, $\mathfrak{j}'$ is the unique such joint, and we constructed $\mathfrak{D}[\mathfrak{j}']$ above. If $\mathfrak{j}$ is not contained in a joint of $\tilde{\mathfrak{D}}_k$, we define $\mathfrak{D}[\mathfrak{j}']$ to be the empty set. There are three types of walls $\mathfrak{d}$ in $\tilde{\mathfrak{D}}_{k+1}$ containing $\mathfrak{j}$:

(1) $\mathfrak{d} \in \tilde{\mathfrak{D}}_k \cup \mathfrak{D}[\mathfrak{j}']$.
(2) $\mathfrak{d} \in \tilde{\mathfrak{D}}_{k+1} \setminus (\tilde{\mathfrak{D}}_k \cup \mathfrak{D}[\mathfrak{j}'])$, but $\mathfrak{j} \not\subseteq \partial\mathfrak{d}$. This type of wall does not contribute to $\mathfrak{p}_{\gamma_\mathfrak{j},\tilde{\mathfrak{D}}_{k+1}} \in G^{\leq k+1}$, as the associated automorphism is central in $G^{\leq k+1}$, and in addition this wall contributes twice to $\mathfrak{p}_{\gamma_\mathfrak{j},\tilde{\mathfrak{D}}_{k+1}}$, with the two contributions inverse to each other.
(3) $\mathfrak{d} \in \tilde{\mathfrak{D}}_{k+1} \setminus (\tilde{\mathfrak{D}}_k \cup \mathfrak{D}[\mathfrak{j}'])$ and $\mathfrak{j} \subseteq \partial\mathfrak{d}$. Since each added wall is of the form $\mathfrak{j}'' - \mathbb{R}_{\geq 0} m$ for some joint $\mathfrak{j}''$ of $\tilde{\mathfrak{D}}_k$, where $-m$ is the direction of the wall, the direction of the wall is parallel to $\mathfrak{j}$, contradicting $\mathfrak{j}$ being a perpendicular joint. Thus this does not occur.

From this, it is clear that $\mathfrak{p}_{\gamma_\mathfrak{j},\tilde{\mathfrak{D}}_{k+1}} = \mathfrak{p}_{\gamma_\mathfrak{j},\tilde{\mathfrak{D}}_k \cup \mathfrak{D}[\mathfrak{j}']}$, which is the identity in $G^{\leq k+1}$ by (C.5). This holds for every perpendicular joint of $\tilde{\mathfrak{D}}_{k+1}$.

The result then follows from Lemma C.7. $\square$

Lemma C.7. *Let* $\mathfrak{D}$ *and* $\tilde{\mathfrak{D}}$ *be two scattering diagrams for* $N^+, \mathfrak{g}$ *such that*

(1) $\mathfrak{D}$ *and* $\tilde{\mathfrak{D}}$ *are equivalent to order* k.
(2) $\mathfrak{D}$ *is consistent to order* $k+1$.
(3) $\mathfrak{p}_{\gamma_\mathfrak{j},\tilde{\mathfrak{D}}}$ *is the identity for every perpendicular joint* $\mathfrak{j}$ *of* $\tilde{\mathfrak{D}}$ *to order* $k+1$.
(4) $\mathfrak{D}$ *and* $\tilde{\mathfrak{D}}$ *have the same set of incoming walls.*

Then $\mathfrak{D}$ *and* $\tilde{\mathfrak{D}}$ *are equivalent to order* $k+1$, *and in particular* $\tilde{\mathfrak{D}}$ *is consistent to order* $k+1$.

Proof. We work with scattering diagrams in the group $G^{\leq k+1}$. There is a finite scattering diagram $\mathfrak{D}'$ with the following properties:

(1) $\tilde{\mathfrak{D}} \cup \mathfrak{D}'$ is equivalent to $\mathfrak{D}$;

(2) $\mathfrak{D}'$ consists only of walls trivial to order k but nontrivial to order $k+1$. Indeed, $\mathfrak{D}'$ can be chosen so that $g_x(\mathfrak{D}') = g_x(\tilde{\mathfrak{D}})^{-1} g_x(\mathfrak{D})$ for any general point x in any $n^\perp$, $n \in N^+$. Note that $\mathfrak{D}'$ is finite because the same is true of $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$.

Thus to show $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ are equivalent, it is sufficient to show that $\mathfrak{D}'$ is equivalent to the empty scattering diagram. To do so, replace $\mathfrak{D}'$ with an equivalent scattering diagram with minimal support. Let $\mathfrak{j}$ be a perpendicular joint of $\tilde{\mathfrak{D}} \cup \mathfrak{D}'$. Then in $G^{\leq k+1}$, $\mathrm{id} = \mathfrak{p}_{\mathfrak{D},\gamma_\mathfrak{j}} = \mathfrak{p}_{\mathfrak{D}',\gamma_\mathfrak{j}}$, since $\mathfrak{p}_{\tilde{\mathfrak{D}},\gamma_\mathfrak{j}} = \mathrm{id}$ and automorphisms in $\mathfrak{D}'$ are central in $G^{\leq k+1}$. However, this implies that for each $n_0 \in N^+$ with $\mathfrak{j} \subseteq n_0^\perp$ and x, x' two points in $n_0^\perp$ on either side of $\mathfrak{j}$, the automorphisms associated with crossing $n_0^\perp$ in $\mathfrak{D}'$ through either x or x' must be the same in order for these two automorphisms to cancel in $\mathfrak{p}_{\mathfrak{D}',\gamma_\mathfrak{j}}$. From this it is easy to see that $\mathfrak{D}'$ is equivalent to a scattering diagram such that for every wall $\mathfrak{d} \in \mathfrak{D}'$, each facet of $\mathfrak{d}$ is a parallel joint of $\mathfrak{D}'$, i.e., the direction $-p^*(n)$ is tangent to every facet of $\mathfrak{d}$. However, such a wall must be incoming, contradicting, if $\mathfrak{D}'$ is nonempty, the fact that $\tilde{\mathfrak{D}}$ and $\mathfrak{D}$ have the same set of incoming walls by assumption. $\square$

C.2. **The proof of Theorem 1.28.** We fix the notation of Theorem 1.28, and in addition we make use of the notation $\mathcal{H}_{k,\pm}$ of Definition 1.22 and $\mathfrak{p}_{\mathfrak{d}_k}$ as in (1.26), the map associated to crossing the slab $\mathfrak{d}_k = (e_k^\perp, 1 + z^{v_k})$ from $\mathcal{H}_{k,-}$ to $\mathcal{H}_{k,+}$.

We define the Lie algebra

$$\bar{\mathfrak{g}} := \bigoplus_{n \in N^{+,k}} \Bbbk z^{p^*(n)} \partial_n,$$

and set $\bar{G}^{\leq j} := \exp(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}^{>j})$, $\bar{G} = \varprojlim \bar{G}^{\leq j}$ as usual, with the degree function $\bar{d} : N^{+,k} \to \mathbb{N}$ given by $\bar{d}(\sum_i a_i e_i) = \sum_{i \neq k} a_i$. We note that $\bar{G}$ acts on $\widehat{\Bbbk[\bar{P}]}$ as usual, and if $\mathfrak{D}$ is a scattering diagram in the sense of Definition 1.27, then all automorphisms associated to crossing walls (rather than slabs) lie in $\bar{G}$.

Besides the Lie algebra $\bar{\mathfrak{g}}$ just defined, recall we also have $\mathfrak{g} = \bigoplus_{n \in N^+} \Bbbk z^{p^*(n)} \partial_n$ as usual. We have the degree map $d : N^+ \to \mathbb{N}$ given by $d(\sum_i a_i e_i) = \sum a_i$, but we also have $\bar{d} : N^+ \to \mathbb{N}$ given by the restriction of $\bar{d} : N^{+,k} \to \mathbb{N}$. We use the notation $\mathfrak{g}^{d>l}$ and $\mathfrak{g}^{\bar{d}>l}$ to distinguish between the two possibilities for $\mathfrak{g}^{>l}$ determined by the two choices of degree map. Then $G = \varprojlim \exp(\mathfrak{g}/\mathfrak{g}^{d>j})$, and we define $\tilde{G} = \varprojlim \exp(\mathfrak{g}^{\bar{d}>0}/\mathfrak{g}^{\bar{d}>j})$. Note that $G, \tilde{G}$ both act faithfully on $\widehat{\Bbbk[P]}$, where the completion is with respect to the maximal monomial ideal $P \setminus \{0\}$, and $\tilde{G}, \bar{G}$ act faithfully on $\widehat{\Bbbk[\bar{P}]}$. There are inclusions $\tilde{G} \subset G$ and $\tilde{G} \subset \bar{G}$. Only the second inclusion holds at finite order, i.e., $\tilde{G}^{\leq j} \subset \bar{G}^{\leq j}$.

For each of the above Lie algebras $\mathfrak{g}'$, we can now also talk about scattering diagrams for $\mathfrak{g}'$ using Definitions 1.4 and 1.6, replacing $\mathfrak{g}$ with $\mathfrak{g}'$ in those definitions.

For a joint $\mathfrak{j}$, we define $\bar{G}_{\mathfrak{j}}$, $\tilde{G}_{\mathfrak{j}}$ as subgroups of $\bar{G}$, $\tilde{G}$ defined analogously to (C.3).

Finally, we will need one other group. We define, for a fixed j,

$$\hat{G}^{\leq j} := \varprojlim_{j'} \exp(\mathfrak{g}/(\mathfrak{g}^{d>j'} + \mathfrak{g}^{\bar{d}>j})).$$

There is an inclusion $\tilde{G}^{\leq j} = \exp(\mathfrak{g}^{\bar{d}>0}/\mathfrak{g}^{\bar{d}>j}) \subset \hat{G}^{\leq j}$ and a surjection $G \to \hat{G}^{\leq j}$.

We need to understand the interaction between elements of G and the automorphism associated to crossing the slab (see [GS11, Lemma 2.15]). Recall the notation $G_{\mathfrak{j}}$ from Construction C.1; this is applied also to the various assorted groups above.

Lemma C.8. *Let $n \in N^{+,k}$ (resp. N^+), and let $\mathfrak{p} \in \bar{G}$ (resp. $\mathfrak{p} \in \tilde{G}$) be an automorphism of the form $\exp(f\partial_n)$ for $f = 1 + \sum_{\ell \geq 1} c_\ell z^{\ell p^*(n)}$. Let $\mathfrak{j} = n^\perp \cap e_k^\perp$. If $\{n, e_k\} > 0$, then*

$$\mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ \mathfrak{p} \circ \mathfrak{p}_{\mathfrak{d}_k} \in \bar{G}_{\mathfrak{j}} \ (\textit{resp. } \tilde{G}_{\mathfrak{j}})$$

while if $\{n, e_k\} < 0$, then

$$\mathfrak{p}_{\mathfrak{d}_k} \circ \mathfrak{p} \circ \mathfrak{p}_{\mathfrak{d}_k}^{-1} \in \bar{G}_{\mathfrak{j}} \ (\textit{resp. } \tilde{G}_{\mathfrak{j}}).$$

Here, we view $\mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ \mathfrak{p} \circ \mathfrak{p}_{\mathfrak{d}_k}$ or $\mathfrak{p}_{\mathfrak{d}_k} \circ \mathfrak{p} \circ \mathfrak{p}_{\mathfrak{d}_k}^{-1}$ as automorphisms of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$ and $\bar{G}$ or $\tilde{G}$ as subgroups of the group of automorphisms of this ring.

Proof. Let us prove the first statement, the second being similar. It is enough to check that

$$\mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ (z^{p^*(n)} \partial_n) \circ \mathfrak{p}_{\mathfrak{d}_k} \in \bar{\mathfrak{g}}_{\mathfrak{j}} \ (\text{resp. } \tilde{\mathfrak{g}}_{\mathfrak{j}}).$$

But, with $h = 1 + z^{v_k}$,

$$\begin{aligned}
&(\mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ (z^{p^*(n)}\partial_n) \circ \mathfrak{p}_{\mathfrak{d}_k})(z^m) \\
&\quad = (\mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ (z^{p^*(n)}\partial_n))(z^m h^{-\langle d_k e_k, m\rangle}) \\
&\quad = \mathfrak{p}_{\mathfrak{d}_k}^{-1}\big(\langle n, m\rangle z^{m+p^*(n)} h^{-\langle d_k e_k, m\rangle}\big) \\
&\qquad\qquad - \mathfrak{p}_{\mathfrak{d}_k}^{-1}\big(\langle d_k e_k, m\rangle\langle v_k, n\rangle z^{m+p^*(n)+v_k} h^{-\langle d_k e_k, m\rangle - 1}\big) \\
&\quad = z^m\big(\langle n, m\rangle z^{p^*(n)} h^{\langle d_k e_k, p^*(n)\rangle} - \langle d_k e_k, m\rangle\langle v_k, n\rangle z^{p^*(n)+v_k} h^{\langle d_k e_k, p^*(n)+v_k\rangle - 1}\big).
\end{aligned}$$

Noting that $\langle v_k, n\rangle = \{e_k, n\} = -\{n, e_k\} = -\langle e_k, p^*(n)\rangle = -d_k^{-1}\langle d_k e_k, p^*(n)\rangle$ and in addition $\langle d_k e_k, v_k\rangle = 0$, we see that as a derivation, writing $\alpha = \langle d_k e_k, p^*(n)\rangle > 0$,

$$\begin{aligned}
&\mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ (z^{p^*(n)}\partial_n) \circ \mathfrak{p}_{\mathfrak{d}_k} \\
&\quad = z^{p^*(n)} h^{\langle d_k e_k, p^*(n)\rangle}\partial_n + z^{p^*(n)+v_k}\langle d_k e_k, p^*(n)\rangle h^{\langle d_k e_k, p^*(n)\rangle - 1}\partial_{e_k} \\
&\quad = \sum_{\beta=0}^{\alpha} z^{p^*(n)+\beta v_k}\binom{\alpha}{\beta}\partial_n + \alpha\sum_{\beta=1}^{\alpha} z^{p^*(n)+\beta v_k}\binom{\alpha-1}{\beta-1}\partial_{e_k} \qquad \text{(C.9)} \\
&\quad = \sum_{\beta=0}^{\alpha} z^{p^*(n+\beta e_k)}\binom{\alpha}{\beta}\partial_{n+\beta e_k}.
\end{aligned}$$

Of course $n + \beta e_k \in \Lambda_{\mathfrak{j}}^{\perp}$ by definition of $\mathfrak{j}$, so the derivation $z^{p^*(n+\beta e_k)}\partial_{n+\beta e_k}$ lives in $\bar{\mathfrak{g}}_{\mathfrak{j}}$ (resp. $\tilde{\mathfrak{g}}_{\mathfrak{j}}$). □

We now proceed with the proof of Theorem 1.28.

Step I. Strategy of the proof. We will first construct $\overline{\mathfrak{D}}_{\mathbf{s}}$ using essentially the same algorithm as the one given in Construction C.1, but working with the group $\tilde{G}$. The algorithm is slightly more complex because of the slab, and it needs to be carried out in two steps. To show that the diagram constructed is consistent at each step, we compare it with the scattering diagram $\mathfrak{D}_{\mathbf{s}}$ for the group G, which we know exists, using $\tilde{G}^{\leq j}$ as an intermediary group. Because $\tilde{G} \subset \bar{G}, G$, we obtain a consistent scattering diagram for $\bar{G}$ and G. While $\overline{\mathfrak{D}}_{\mathbf{s}}$ is equivalent to $\mathfrak{D}_{\mathbf{s}}$ as a scattering diagram for G by construction, this does not show uniqueness of $\overline{\mathfrak{D}}_{\mathbf{s}}$, as there may be a different choice with wall-crossing automorphisms in $\bar{G}$ but not in $\tilde{G}$, so it cannot be compared with $\mathfrak{D}_{\mathbf{s}}$. Thus, the final step involves showing uniqueness directly for the group $\bar{G}$, again as part of the inductive proof.

We will proceed by induction on j, constructing for each j a finite scattering diagram $\overline{\mathfrak{D}}_j$ for $\tilde{G}$ containing $\mathfrak{D}_{\mathrm{in},\mathbf{s}}$ such that the following induction hypotheses hold:

(1) For every joint $\mathfrak{j}$ of $\overline{\mathfrak{D}}_j$, there is a simple loop $\gamma_{\mathfrak{j}}$ around $\mathfrak{j}$ small enough so that it only intersects walls and slabs containing $\mathfrak{j}$ and such that $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_j}$, as an automorphism of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$, lies in $\tilde{G}$ and is trivial in $\tilde{G}^{\leq j}$, or equivalently, by the inclusion $\tilde{G}^{\leq j} \subset \bar{G}^{\leq j}$, trivial in $\bar{G}^{\leq j}$.

(2) If $\overline{\mathfrak{D}}'_j$ is a scattering diagram for $\bar{G}$ which has the same incoming walls as $\overline{\mathfrak{D}}_j$ and satisfies (1) (with $\tilde{G}$ replaced by $\bar{G}$ everywhere), then $\overline{\mathfrak{D}}'_j$ is equivalent to $\overline{\mathfrak{D}}_j$ in $\bar{G}^{\leq j}$.

Recall that joints of $\overline{\mathfrak{D}}_j$ are either parallel or perpendicular; see Definition-Lemma C.2.

Step II. The base case. For $j=0$, $\overline{\mathfrak{D}}_0 = \mathfrak{D}_{\mathrm{in},\mathbf{s}}$ does the job. Indeed, all walls are trivial in $\tilde{G}^{\leq 0} = \{\mathrm{id}\}$, leaving just the single initial slab, and thus there are no joints.

Step III. From $\overline{\mathfrak{D}}_j$ to $\overline{\mathfrak{D}}_{j+1}$: adding walls associated to joints not contained in $e_k^{\perp}$. Now assume we have found $\overline{\mathfrak{D}}_j$ satisfying the induction hypotheses. We need to add a finite number of walls to get $\overline{\mathfrak{D}}_{j+1}$. We will carry out the construction of $\overline{\mathfrak{D}}_{j+1}$ in two steps, following Construction C.1.

First, let $\mathfrak{j}$ be a perpendicular joint of $\overline{\mathfrak{D}}_j$ with $\mathfrak{j} \not\subseteq e_k^{\perp}$. Let $\Lambda_{\mathfrak{j}} \subseteq M^{\circ}$ be the set of integral tangent vectors to $\mathfrak{j}$. If $\gamma_{\mathfrak{j}}$ is a simple loop around $\mathfrak{j}$ small enough so that it only intersects walls containing $\mathfrak{j}$, we note that every wall-crossing automorphism $\mathfrak{p}_{\gamma_{\mathfrak{j}},\mathfrak{d}}$ contributing to $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_j}$ lies in $\tilde{G}_{\mathfrak{j}}$. Thus as in (C.4), in $\tilde{G}^{\leq j+1}$ we can write

$$\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_j} = \exp\left(\sum_{i=1}^{s} c_i z^{p^*(n_i)} \partial_{n_i}\right) \tag{C.10}$$

with $c_i \in \Bbbk$, and $n_i \in \Lambda_{\mathfrak{j}}^{\perp}$ with $\bar{d}(n_i) = j+1$ as $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_j}$ is the identity in $\tilde{G}^{\leq j}$ by the induction hypothesis. Finally, $p^*(n_i) \notin \Lambda_{\mathfrak{j}}$ because the joint is perpendicular. Let

$$\mathfrak{D}[\mathfrak{j}] := \{(\mathfrak{j} - \mathbb{R}_{\geq 0} p^*(n_i), (1+z^{p^*(n_i)})^{\pm c_i}) \mid i = 1, \ldots, s\}.$$

Here $(1+z^{p^*(n_i)})^{\pm c_i} = \exp(\pm c_i \log(1+z^{p^*(n_i)}))$ makes sense as a power series. The sign is chosen in each wall so that its contribution to $\mathfrak{p}_{\gamma_{\mathfrak{j}},\mathfrak{D}[\mathfrak{j}]}$ is $\exp(-c_i z^{p^*(n_i)} \partial_{n_i})$ to $\bar{d}$-order $j+1$.

We now take

$$\overline{\mathfrak{D}}'_j := \overline{\mathfrak{D}}_j \cup \bigcup_{\mathfrak{j}} \mathfrak{D}[\mathfrak{j}],$$

where the union is over all perpendicular joints not contained in $e_k^{\perp}$. We have only added a finite number of walls.

Step IV. From $\overline{\mathfrak{D}}_j$ to $\overline{\mathfrak{D}}_{j+1}$: adding walls associated to joints contained in $e_k^{\perp}$. If we did not have a slab, $\overline{\mathfrak{D}}'_j$ constructed above would now do the job as in the proof of Lemma C.6. However, the elements of $\tilde{G}$ trivial in $\tilde{G}^{\leq j}$ do not commute with $\mathfrak{p}_{\mathfrak{d}_k}$ to order $j+1$ as automorphisms of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$ in any reasonable sense. As a consequence, we will need to add some additional walls coming from joints in $e_k^{\perp}$, some of which have arisen as the intersection of $e_k^{\perp}$ with walls added in Step III.

Consider a perpendicular joint $\mathfrak{j} \subseteq e_k^{\perp}$ of $\overline{\mathfrak{D}}'_j$. Necessarily, the linear span of $\mathfrak{j}$ is $e_k^{\perp} \cap n^{\perp}$ for some $n \in N^+$. Furthermore, we can choose n so that any wall containing $\mathfrak{j}$ then has linear span $(ae_k + bn)^{\perp}$ for some a, b nonnegative rational numbers. The direction of such a wall is positively proportional to $-p^*(ae_k + bn)$. We now distinguish between two cases. Note that $\langle e_k, p^*(n)\rangle \neq 0$ as the joint is not parallel, so we call the joint $\mathfrak{j}$ *positive* or *negative* depending on the sign of $\langle e_k, p^*(n)\rangle = \{n, e_k\}$. Note that if the joint is positive (negative), then $\langle e_k, p^*(ae_k + bn)\rangle$ is positive (negative) for all $b > 0$.

If the joint is positive, then choose $\gamma_{\mathfrak{j}}$ so that the first wall crossed is $\mathfrak{d}_k$, passing from $\mathcal{H}_{k,-}$ to $\mathcal{H}_{k,+}$. We can write

$$\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}'_j} = \mathfrak{p}_2 \circ \mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ \mathfrak{p}_1 \circ \mathfrak{p}_{\mathfrak{d}_k}, \tag{C.11}$$

where $\mathfrak{p}_i \in \tilde{G}_{\mathfrak{j}}$ are compositions of wall-crossing automorphisms. It then follows from $\langle e_k, p^*(ae_k + bn)\rangle > 0$ for all $a \geq 0$, $b > 0$ and Lemma C.8 that $\mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ \mathfrak{p}_1 \circ \mathfrak{p}_{\mathfrak{d}_k} \in \tilde{G}_{\mathfrak{j}}$, hence $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}'_j} \in \tilde{G}_{\mathfrak{j}}$. If the joint is negative, then we use a slightly different loop: without changing the orientation of the loop $\gamma_{\mathfrak{j}}$, change the endpoints so that $\gamma_{\mathfrak{j}}$ now starts and ends in $\mathcal{H}_{k,+}$, crossing $\mathfrak{d}_k$ just before its endpoint. Then

$$\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}'_j} = \mathfrak{p}_{\mathfrak{d}_k} \circ \mathfrak{p}_2 \circ \mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ \mathfrak{p}_1,$$

and again by Lemma C.8, $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}'_j} \in \tilde{G}_{\mathfrak{j}}$.

Thus in both cases, $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}'_j} \in \tilde{G}_{\mathfrak{j}}$ and is the identity in $\tilde{G}^{\leq j}$. Thus we still have (C.10), and we can produce a scattering diagram $\mathfrak{D}[\mathfrak{j}]$ in the same way as for the joints $\mathfrak{j}$ not contained in $e_k^{\perp}$. We then set

$$\overline{\mathfrak{D}}_{j+1} = \mathfrak{D}'_j \cup \bigcup_{\mathfrak{j}} \mathfrak{D}[\mathfrak{j}],$$

where the union is over perpendicular joints of $\overline{\mathfrak{D}}'_j$ contained in $e_k^{\perp}$.

Step V. Part (1) *of the induction hypothesis is satisfied.* Consider a perpendicular joint $\mathfrak{j}$ of $\overline{\mathfrak{D}}_{j+1}$. First suppose $\mathfrak{j} \not\subseteq e_k^{\perp}$. We proceed as in the proof of Lemma C.6. If $\mathfrak{j}$ is contained in a joint of $\overline{\mathfrak{D}}_j$, there is a unique such joint, say $\mathfrak{j}'$, and we constructed $\mathfrak{D}[\mathfrak{j}']$ above. If $\mathfrak{j}$ is not contained in a joint of $\overline{\mathfrak{D}}_j$, we define $\mathfrak{D}[\mathfrak{j}']$ to be the empty set. There are three types of walls $\mathfrak{d}$ in $\overline{\mathfrak{D}}_{j+1}$ containing $\mathfrak{j}$:

(1) $\mathfrak{d} \in \overline{\mathfrak{D}}_j \cup \mathfrak{D}[\mathfrak{j}']$.
(2) $\mathfrak{d} \in \overline{\mathfrak{D}}_{j+1} \setminus (\overline{\mathfrak{D}}_j \cup \mathfrak{D}[\mathfrak{j}'])$, but $\mathfrak{j} \not\subseteq \partial\mathfrak{d}$. This type of wall does not contribute to $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}_{j+1}}$ in $\tilde{G}^{\leq j+1}$. Indeed, the associated automorphism is in the center of $\tilde{G}^{\leq j+1}$, and this wall contributes twice to $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}_{j+1}}$, with the two contributions inverse to each other, so the contribution cancels.
(3) $\mathfrak{d} \in \overline{\mathfrak{D}}_{j+1} \setminus (\overline{\mathfrak{D}}_j \cup \mathfrak{D}[\mathfrak{j}'])$ and $\mathfrak{j} \subseteq \partial\mathfrak{d}$. Since each added wall is of the form $\mathfrak{j}'' - \mathbb{R}_{\geq 0} m$ for some joint $\mathfrak{j}''$ of $\overline{\mathfrak{D}}_j$, where $-m$ is the direction of the wall, the direction of the wall is parallel to $\mathfrak{j}$, contradicting $\mathfrak{j}$ being a perpendicular joint. Thus this does not occur.

From this we see by construction of $\mathfrak{D}[\mathfrak{j}']$ that $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}_{j+1}}$ is the identity in $\tilde{G}^{\leq j+1}$.

On the other hand, suppose $\mathfrak{j}$ is a perpendicular joint of $\overline{\mathfrak{D}}_{j+1}$ contained in $e_k^{\perp}$. Then since no wall of $\overline{\mathfrak{D}}_{j+1} \setminus \overline{\mathfrak{D}}'_j$ is contained in $e_k^{\perp}$, by definition of $N^{+,k}$, in fact $\mathfrak{j}$ is a joint of $\overline{\mathfrak{D}}'_j$. Thus we see again by construction of $\mathfrak{D}[\mathfrak{j}]$ that to order $j+1$, $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}_{j+1}}$ is the identity for $\gamma_{\mathfrak{j}}$ the loop around $\mathfrak{j}$ described in Step IV. Recall the choice of loop depends on whether the joint is positive or negative.

Now we show that $\overline{\mathfrak{D}}_{j+1}$ satisfies the induction hypothesis (1), using the above existence of a $\gamma_{\mathfrak{j}}$ such that $\mathfrak{p}_{\gamma_{\mathfrak{j}}, \overline{\mathfrak{D}}_{j+1}} = \mathrm{id}$ for each perpendicular joint $\mathfrak{j}$. Note that there is a map $\tilde{G}^{\leq j+1} \to \exp(\mathfrak{g}/(\mathfrak{g}^{d>j'} + \mathfrak{g}^{\bar{d}>j+1})) =: \hat{G}_{j'}$ for any j'. The slab automorphism $\mathfrak{p}_{\mathfrak{d}_k}$ can be viewed as an element of $\hat{G}_{j'}$ for any j', and hence $\overline{\mathfrak{D}}_{j+1}$ can be viewed as a scattering diagram for $\hat{G}_{j'}$ in the sense of Definition 1.6. We will first show that $\overline{\mathfrak{D}}_{j+1}$ is consistent as a diagram for $\hat{G}_{j'}$ inductively on j'.

The base case is $j' = j$. All walls of $\overline{\mathfrak{D}}_{j+1} \setminus \overline{\mathfrak{D}}_j$ are trivial to $\bar{d}$-order j and hence to d-order j. Now $\overline{\mathfrak{D}}_j$ satisfies the main induction hypothesis (1) at order j, which implies via the natural map $\tilde{G}^{\leq j} \to \hat{G}_j = G^{\leq j}$ that $\overline{\mathfrak{D}}_{j+1}$ is consistent as a diagram for $\hat{G}_j$. Indeed, as $\overline{\mathfrak{D}}_{j+1}$ is a finite scattering diagram, it is enough to check that $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_{j+1}}$ is the identity in $G^{\leq j}$ for any small loop $\gamma_{\mathfrak{j}}$ around any joint $\mathfrak{j}$. By the hypothesis (1), this is the case for some loop $\gamma_{\mathfrak{j}}$, and hence for all loops. Note that by uniqueness of consistent scattering diagrams with the same incoming walls, we also record for future use:

$$\overline{\mathfrak{D}}_{j+1} \text{ is equivalent to } \mathfrak{D}_{\mathbf{s}} \text{ as diagrams for } G^{\leq j}. \tag{C.12}$$

The induction step follows from Lemma C.7, applied to $\tilde{\mathfrak{D}} = \overline{\mathfrak{D}}_{j+1}$, $\mathfrak{D} = \mathfrak{D}_{\mathbf{s}}$, and the group being $\hat{G}_{j'}$ (a quotient of G, so the argument of Lemma C.7 still applies). Indeed, if we assume $\overline{\mathfrak{D}}_{j+1}$ is consistent in $\hat{G}_{j'}$, then it is equivalent to $\mathfrak{D}_{\mathbf{s}}$ as a scattering diagram in $\hat{G}_{j'}$. Furthermore, $\mathfrak{D}_{\mathbf{s}}$ is consistent to all orders by Theorem 1.12, and has the same set of incoming walls as $\overline{\mathfrak{D}}_{j+1}$ by construction. Finally, $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_{j+1}}$ is the identity in $\hat{G}_{j'+1}$ for any perpendicular joint $\mathfrak{j}$, as shown above. Thus $\overline{\mathfrak{D}}_{j+1}$ and $\mathfrak{D}_{\mathbf{s}}$ are equivalent in $\hat{G}_{j'+1}$, and in particular $\overline{\mathfrak{D}}_{j+1}$ is consistent in $\hat{G}_{j'+1}$.

Thus taking the inverse limit, we see that $\overline{\mathfrak{D}}_{j+1}$ is consistent as a scattering diagram for $\hat{G}^{\leq j+1}$. This almost completes the proof of the induction hypothesis (1) in degree $j+1$. Indeed, as $\tilde{G}^{\leq j+1}$ is a subgroup of $\hat{G}^{\leq j+1}$, certainly $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_{j+1}}$ is the identity for any joint not contained in $e_k^{\perp}$, including the parallel joints. For a perpendicular joint contained in $e_k^{\perp}$, if we choose $\gamma_{\mathfrak{j}}$ as given in Step IV, $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_{j+1}}$ lies in $\tilde{G}$ and is the identity in $\tilde{G}^{\leq j+1}$ by the construction of $\mathfrak{D}[\mathfrak{j}]$ in Step IV. Finally, for a parallel joint $\mathfrak{j}$ contained in $e_k^{\perp}$, note that all wall and slab-crossing automorphisms associated to walls containing $\mathfrak{j}$ commute, and in particular the contribution of $\mathfrak{p}_{\mathfrak{d}_k}$ and $\mathfrak{p}_{\mathfrak{d}_k}^{-1}$ in $\mathfrak{p}_{\gamma_{\mathfrak{j}},\overline{\mathfrak{D}}_{j+1}}$ as an automorphism of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$ cancel, so that the latter automorphism lies in $\tilde{G}$. Hence the image of this automorphism in $\tilde{G}^{\leq j+1} \subset \hat{G}^{\leq j+1}$ must also be trivial. This gives the induction hypothesis (1).

Step VI. Uniqueness. Suppose we have constructed two scattering diagrams $\overline{\mathfrak{D}}_{j+1}, \overline{\mathfrak{D}}'_{j+1}$ for $\bar{G}$ from $\overline{\mathfrak{D}}_j$ which satisfy the inductive hypothesis (1) to $\bar{d}$-order $j+1$, but with the group $\tilde{G}$ replaced with $\bar{G}$. By the induction hypothesis (2), these two scattering diagrams are equivalent to $\bar{d}$-order j, and we wish to show they are equivalent to $\bar{d}$-order $j+1$. One first constructs a finite scattering diagram $\mathfrak{D}$ consisting only of outgoing walls whose attached functions are of the form $1+cz^{p^*(n)}$ with $c \in \Bbbk$ and $\bar{d}(n) = j+1$, with the property that $\overline{\mathfrak{D}}_{j+1} \cup \mathfrak{D}$ is equivalent to $\overline{\mathfrak{D}}'_{j+1}$ to $\bar{d}$-order $j+1$. This is done precisely as in the proof of Lemma C.7. We need to show $\mathfrak{D}$ is equivalent to the empty scattering diagram to $\bar{d}$-order $j+1$.

To show this, first note that for any loop γ which does not cross the slab $\mathfrak{d}_k$, $\mathfrak{p}_{\gamma,\overline{\mathfrak{D}}_{j+1}} = \mathfrak{p}_{\gamma,\overline{\mathfrak{D}}'_{j+1}} = \mathrm{id}$ to $\bar{d}$-order $j+1$ implies that $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathrm{id}$ to $\bar{d}$-order $j+1$. Indeed, all wall-crossing automorphisms of $\mathfrak{D}$ are central in $\bar{G}^{\leq j+1}$. Now if $n \in N^{+,k}$ with $\bar{d}(n) = j+1$, let $\mathfrak{D}_n \subseteq \mathfrak{D}$ be the set of walls in $\mathfrak{D}$ with attached functions of the form $1+cz^{p^*(n)}$. Note all wall-crossing automorphisms of $\mathfrak{D}$, viewed as elements of $\bar{G}^{\leq j+1}$, lie in $\exp(\bar{\mathfrak{g}}^{>j}/\bar{\mathfrak{g}}^{>j+1})$, which as a group coincides with the additive group structure on $\bar{\mathfrak{g}}^{>j}/\bar{\mathfrak{g}}^{>j+1}$. Thus for any path γ not crossing $\mathfrak{d}_k$, we obtain a unique decomposition $\mathfrak{p}_{\gamma,\mathfrak{D}} = \prod_n \mathfrak{p}_{\gamma,\mathfrak{D}_n}$ from the $N^{+,k}$-grading on $\bar{\mathfrak{g}}^{>j}/\bar{\mathfrak{g}}^{>j+1}$, and if $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is the identity, so is each $\mathfrak{p}_{\gamma,\mathfrak{D}_n}$.

Fixing n as above, replace $\mathfrak{D}_n$ with an equivalent scattering diagram with smallest possible support, and let $C_n = \mathrm{Supp}(\mathfrak{D}_n)$. So if $x \in n^\perp$ is a general point, $x \in C_n$ if and only if $g_x(\mathfrak{D}_n)$ is not the identity. Assume first that $\langle e_k, p^*(n)\rangle \geq 0$. We shall show $C_n \subseteq \mathcal{H}_{k,-}$. Assume not. Taking a general point $x \in C_n \setminus \mathcal{H}_{k,-}$, it is not possible for the ray $L := x + \mathbb{R}_{\geq 0} p^*(n)$ to be contained in C_n. This is because $\mathfrak{D}$ consists of only a finite number of walls, none of which are incoming. Let $\lambda = \max\{t \in \mathbb{R}_{\geq 0} \,|\, x + tp^*(n) \in C_n\}$, and $y = x + \lambda p^*(n)$. This makes sense as $t = 0$ is in the set over which we are taking the maximum, as we are assuming $x \in C_n$. Then necessarily y is in a joint $\mathfrak{j}$ of $\mathfrak{D}_n$, and every wall of $\mathfrak{D}_n$ containing $\mathfrak{j}$ is contained in $\mathbb{R}\mathfrak{j} - \mathbb{R}_{\geq 0} p^*(n)$. Furthermore, since $\langle e_k, x\rangle > 0$, $\langle e_k, p^*(n)\rangle \geq 0$, it follows that $y \notin e_k^\perp$ and $\mathfrak{j}$ is not contained in $e_k^\perp$. Thus given a loop $\gamma_\mathfrak{j}$ around $\mathfrak{j}$, $\mathfrak{p}_{\gamma_\mathfrak{j},\mathfrak{D}_n}$ is the identity. This implies that in fact to $\bar{d}$-order $j+1$,

$$\prod_{\substack{\mathfrak{d}\in\mathfrak{D}_n\\ \mathfrak{j}\subseteq\mathfrak{d}}} \mathfrak{p}_{\gamma_\mathfrak{j},\mathfrak{d}} = \mathrm{id}\,.$$

In particular, a point $z = y - \epsilon p^*(n)$ for small ϵ is contained in precisely those walls of $\mathfrak{D}_n$ containing $\mathfrak{j}$. But then $g_z(\mathfrak{D}_n) = \mathrm{id}$, contradicting minimality of C_n. Thus one finds that $C_n \subseteq \mathcal{H}_{k,-}$. Similarly, if $\langle e_k, p^*(n)\rangle \leq 0$, then $C_n \subseteq \mathcal{H}_{k,+}$. In particular, if $\langle e_k, p^*(n)\rangle = 0$, $C_n \subseteq e_k^\perp$, but there are no walls contained in $e_k^\perp$, so in this case $\mathfrak{D}_n = \emptyset$.

Now consider a joint $\mathfrak{j}$ of $\overline{\mathfrak{D}}'_{j+1}$ contained in $e_k^\perp$. There are three cases: either $\mathfrak{j}$ is perpendicular and positive, perpendicular and negative, or parallel. Consider the first case. Take a loop $\gamma_\mathfrak{j}$ around $\mathfrak{j}$ as in the positive case in Step IV. Because of positivity, if a wall $\mathfrak{d}$ of $\mathfrak{D}$ contains $\mathfrak{j}$, then with n chosen so that $\mathfrak{d} \in \mathfrak{D}_n$, we must have $\langle e_k, p^*(n)\rangle > 0$ and hence $\mathfrak{d}$ is contained in $\mathcal{H}_{k,-}$. Thus we have that to $\bar{d}$-order $j+1$,

$$\mathrm{id} = \mathfrak{p}_{\gamma_\mathfrak{j},\overline{\mathfrak{D}}'_{j+1}} = \mathfrak{p}_{\gamma_\mathfrak{j},\overline{\mathfrak{D}}_{j+1}\cup\mathfrak{D}} = \mathfrak{p}_{2,\mathfrak{D}} \circ \mathfrak{p}_{2,\overline{\mathfrak{D}}_{j+1}} \circ \mathfrak{p}_{\mathfrak{d}_k}^{-1} \circ \mathfrak{p}_{1,\overline{\mathfrak{D}}_{j+1}} \circ \mathfrak{p}_{\mathfrak{d}_k} = \mathfrak{p}_{2,\mathfrak{D}} = \mathfrak{p}_{\gamma_\mathfrak{j},\mathfrak{D}}$$

as in (C.11), where $\mathfrak{p}_{i,\mathfrak{D}}$ and $\mathfrak{p}_{i,\overline{\mathfrak{D}}_{j+1}}$ denote the contributions coming from the scattering diagrams $\mathfrak{D}$ and $\overline{\mathfrak{D}}_{j+1}$ and the pieces of $\gamma_\mathfrak{j}$ not crossing $e_k^\perp$. The same argument works for negative joints, while a parallel joint cannot contain any wall of $\mathfrak{D}$ (as we showed above that $\mathfrak{D}_n = \emptyset$ if $\langle e_k, p^*(n)\rangle = 0$) so that $\mathfrak{p}_{\gamma_\mathfrak{j},\mathfrak{D}} = \mathrm{id}$ trivially. We can now repeat the argument of the previous paragraph, taking for any n a general point $x \in C_n$ rather than $x \in C_n \setminus \mathcal{H}_{k,-}$. This allows us to conclude that $\mathfrak{D}_n = \emptyset$ for all n, proving uniqueness.

Step VII. Finishing the proof of Theorem 1.28. Having completed the induction step, we take $\overline{\mathfrak{D}}_\mathbf{s} = \bigcup_{j=0}^\infty \overline{\mathfrak{D}}_j$. We need to check it satisfies the stated conditions in Theorem 1.28. Certainly conditions (1) and (2) hold by construction.

For (3), first recall that because by construction $\overline{\mathfrak{D}}_\mathbf{s}$ can be viewed as a scattering diagram for $\tilde{G}$, it can also be viewed as a scattering diagram for G via the inclusion $\tilde{G} \subset G$, and in addition $\mathfrak{p}_{\mathfrak{d}_k} \in G$, so that $\overline{\mathfrak{D}}_\mathbf{s}$ is viewed as a scattering diagram for G in the sense of Definition 1.6, i.e., with no slab. Now as a scattering diagram for G, $\overline{\mathfrak{D}}_\mathbf{s}$ is equivalent to $\mathfrak{D}_\mathbf{s}$ by (C.12). By consistency of $\mathfrak{D}_\mathbf{s}$, $\mathfrak{p}_{\gamma,\overline{\mathfrak{D}}_\mathbf{s}}$ is independent of the endpoints of γ as an element of G. Now suppose g_1, g_2 are two automorphisms of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$, which induce automorphisms of $\widehat{\Bbbk[P]}$ (i.e., for $p \in P \subset \bar{P}$, $g_i(z^p) \in \widehat{\Bbbk[P]}$, giving a map $g_i : \widehat{\Bbbk[P]} \to \widehat{\Bbbk[P]}$ which is an automorphism), and agree as automorphisms of the latter ring. Then g_1, g_2 agree as automorphisms of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$. Thus in particular, $\mathfrak{p}_{\gamma,\overline{\mathfrak{D}}_{\mathbf{s}}}$ is independent of the endpoints of γ as an automorphism of $\widehat{\Bbbk[\bar{P}]}_{1+z^{v_k}}$. This gives condition (3).

The uniqueness of $\overline{\mathfrak{D}}_{\mathbf{s}}$ with these properties then follows from the induction hypothesis (2). Indeed, if $\overline{\mathfrak{D}}'_{\mathbf{s}}$ satisfies conditions (1)–(3) of Theorem 1.28, then working by induction on the order j, the induction hypothesis (1) holds for $\overline{\mathfrak{D}}'_{\mathbf{s}}$ (the existence of γ_{j} with $\mathfrak{p}_{\gamma_{\mathrm{j}},\overline{\mathfrak{D}}'_{\mathbf{s}}} \in \tilde{G}$ only being an issue for joints contained in $e_k^{\perp}$, and Step IV explains how to choose the loop γ_{j}). Thus by induction hypothesis (2), $\overline{\mathfrak{D}}_{\mathbf{s}}$ and $\overline{\mathfrak{D}}'_{s}$ are equivalent to order j.

This completes the proof of Theorem 1.28. $\square$

C.3. **The proof of Theorem 1.13.** The key point of the proof is just the positivity of the simplest scattering diagram as described in Example 1.14, which we use to analyze general two-dimensional scattering diagrams. We will consider a somewhat more general setup, but only in two dimensions, than that considered in the rest of this paper. In particular, we will follow the notation of [G11, §6.3.1], taking $M = \mathbb{Z}^2$, $N = \mathrm{Hom}(M, \mathbb{Z})$, and assume given a monoid P with a map $r : P \to M$, $\mathfrak{m} = P \setminus P^{\times}$. We will consider scattering diagrams $\mathfrak{D}$ for this data as in [G11, Def. 6.37], consisting of rays and lines which do not necessarily pass through the origin. Given any scattering diagram $\mathfrak{D}_{\mathrm{in}}$, the argument of Kontsevich and Soibelman from [KS06] (see [G11, Theorem 6.38] for an exposition of this particular case) adds rays to $\mathfrak{D}_{\mathrm{in}}$ to obtain a scattering diagram $\mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})$ such that $\mathfrak{p}_{\gamma,\mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})}$ is the identity for every loop γ. This diagram is unique up to equivalence.

The fundamental observation involves a kind of universal scattering diagram:

Proposition C.13. *In the above setup, suppose given $p_i \in \mathfrak{m} \subseteq P$, $1 \leq i \leq s$, with $r(p_i) \neq 0$, and positive integers $d_1, \ldots, d_s$. Consider the scattering diagram*

$$\mathfrak{D}_{\mathrm{in}} := \{(\mathbb{R}r(p_i), (1 + z^{p_i})^{d_i}) \,|\, 1 \leq i \leq s\},$$

$\mathfrak{D} := \mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})$. *We can choose $\mathfrak{D}$ within its equivalence class so that for any given ray $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \setminus \mathfrak{D}_{\mathrm{in}}$, we have*

$$f_{\mathfrak{d}} = (1 + z^{\sum_{i=1}^{s} n_i p_i})^c$$

for c a positive integer and the n_i nonnegative integers with at least two of them nonzero.

Proof.

Step I. The change of monoid trick. Note that if the $r(p_i)$ generate a rank 1 sublattice of M, then all the wall-crossing automorphisms of $\mathfrak{D}_{\mathrm{in}}$ commute and $\mathfrak{D} = \mathfrak{D}_{\mathrm{in}}$, so we are done. So assume from now on that the $r(p_i)$ generate a rank 2 sublattice of M.

Let $P' = \mathbb{N}^s$, generated by $e_1, \ldots, e_s$, define a map $u : P' \to P$ by $u(e_i) = p_i$, and a map $r' : P' \to M$ by $r'(e_i) = r(p_i)$. We extend u to a map $u : \widehat{\Bbbk[P']} \to \widehat{\Bbbk[P]}$, and define, for a scattering diagram $\mathfrak{D}$ for the monoid P', $u(\mathfrak{D}) := \{(\mathfrak{d}, u(f_{\mathfrak{d}})) \,|\, (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}\}$. Clearly, if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathrm{id}$, then $\mathfrak{p}_{\gamma,u(\mathfrak{D})} = \mathrm{id}$. Thus if $\mathfrak{D}' = \mathrm{Scatter}(\{(\mathbb{R}r(p_i), (1 + z^{e_i})^{d_i}) \,|\, 1 \leq i \leq s\})$, then $u(\mathfrak{D}')$ is equivalent to $\mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})$, by uniqueness of Scatter up to equivalence. So it is sufficient to show the result with $P = \mathbb{N}^s$, $p_i = e_i$.

Step II. Everything but the positivity of the exponents. We can construct $\mathfrak{D}$ specifically using the original method of [KS06], already explained here in Steps III and IV of the proof of Theorem 1.28: we construct $\mathfrak{D}$ order by order, constructing $\mathfrak{D}_d$ so that $\mathfrak{p}_{\gamma,\mathfrak{D}_d}$ is the identity modulo $\mathfrak{m}^d$ for γ a loop around the origin. Given a description

$$\mathfrak{p}_{\gamma,\mathfrak{D}_d} = \exp\Big(\sum c_i z^{m_i}\partial_{n_i}\Big) \mod \mathfrak{m}^{d+1}, \tag{C.14}$$

with the n_i primitive and the m_i all distinct, we add a collection of rays

$$\{(-\mathbb{R}_{\geq 0} r(m_i), (1+z^{m_i})^{\pm c_i})\}$$

for some $c_i \in \Bbbk$. However, inductively, we can show the c_i can be taken to be integers. Indeed, if all rays in $\mathfrak{D}_d$ have this property, then $\mathfrak{p}_{\gamma,\mathfrak{D}_d}$ is in fact an automorphism of $\widehat{\mathbb{Z}[P]}$, and thus the c_i appearing in (C.14) of $\mathfrak{p}_{\gamma,\mathfrak{D}_d}$ are also integers.

Next let us show that any exponent m_i is of the form $\sum n_j e_j \in P$ with at least two of the n_j nonzero. The pronilpotent group $\mathbb{V}$ in which all automorphisms live is given by the Lie algebra

$$\mathfrak{v} = \bigoplus_{\substack{m\in\mathfrak{m}\\ r(m)\neq 0}} z^m \Bbbk \otimes r(m)^\perp \subseteq \Theta(\Bbbk[P]),$$

following the notation of [G11, pp. 290–291]. This contains a subalgebra $\mathfrak{v}'$ where the sum is taken over all $m \in \mathfrak{m}$ not proportional to one of the e_i. Then clearly $[\mathfrak{v},\mathfrak{v}'] \subseteq \mathfrak{v}'$, so the corresponding pronilpotent group $\mathbb{V}'$ is normal in $\mathbb{V}$. Furthermore, $\mathfrak{v}/\mathfrak{v}'$ is abelian, hence so is $\mathbb{V}/\mathbb{V}'$. For any loop γ, the image of $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathrm{in}}}$ is thus the identity in $\mathbb{V}/\mathbb{V}'$, as every wall in $\mathfrak{D}_{\mathrm{in}}$ contributes twice to $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathrm{in}}}$, but with inverse automorphisms. Assume inductively that $\mathfrak{D}_d \setminus \mathfrak{D}_{\mathrm{in}}$ only contains rays whose attached functions $(1+z^{m_i})^{c_i}$ have m_i not proportional to any e_j. Then the wall-crossing automorphisms associated to these rays lie in $\mathbb{V}'$, so $\mathfrak{p}_{\gamma,\mathfrak{D}_d}$ is the identity in $\mathbb{V}/\mathbb{V}'$, i.e., it lies in $\mathbb{V}'$. Thus the expression $\sum c_i z^{m_i}\partial_{n_i}$ of (C.14) lies in $\mathfrak{v}'$, hence the inductive step follows.

It remains to show that each wall added is of the form $(\mathfrak{d},(1+z^m)^c)$ with c positive.

Step III. The perturbation trick. We will now show the result for all monoids $P=\mathbb{N}^\alpha$ for all α, all choices of $r: P\to M$, all choices of $p_i \in P\setminus\{0\}$ with $r(p_i)\neq 0$, and all positive choices of d_i. (Note by Step I this is a bit more than we need, as we do not take the p_i to necessarily be generators of P.) All cases are dealt with simultaneously by induction.

We define for $p\in P$ the *order* $\mathrm{ord}(p)$, which is the unique $n\in\mathbb{Z}_{\geq 0}$ such that $p \in \mathfrak{m}^n\setminus\mathfrak{m}^{n+1}$. For a ray $(\mathfrak{d},(1+z^p)^c)$, we write $\mathrm{ord}(\mathfrak{d}) := \mathrm{ord}(p)$, and say $\mathfrak{d}$ is a ray of order $\mathrm{ord}(\mathfrak{d})$. We will go by induction on the order, showing that a ray $(\mathfrak{d},(1+z^p)^c)$ in $\mathfrak{D}$ of order $\leq k$ for any choice of data has c positive. This is obviously the case for $k=1$, as all elements of $\mathfrak{D}\setminus\mathfrak{D}_{\mathrm{in}}$ have order at least 2. So assume the induction hypothesis is true for all orders $< k$, and we need to show rays added of order k have positive exponent.

We will use the *perturbation trick* repeatedly. Given a scattering diagram $\mathfrak{D}_{\mathrm{in}}$ for which we would like to compute $\mathfrak{D}=\mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})$, choose general $v_{\mathfrak{d}}\in M_{\mathbb{R}}$ for each $\mathfrak{d}\in\mathfrak{D}_{\mathrm{in}}$. Define $\mathfrak{D}'_{\mathrm{in}} := \{(\mathfrak{d}+v_{\mathfrak{d}}, f_{\mathfrak{d}}) \,|\, \mathfrak{d}\in\mathfrak{D}_{\mathrm{in}}\}$; this is the perturbed diagram. We can then run the Kontsevich–Soibelman algorithm for $\mathfrak{D}'_{\mathrm{in}}$, for example as described in [G11, Theorem 6.38]. This gives a scattering diagram $\mathfrak{D}'=\mathrm{Scatter}(\mathfrak{D}'_{\mathrm{in}})$ with the property that $\mathfrak{p}_{\gamma,\mathfrak{D}'}$ is the identity for every loop γ. This is the case in particular for γ a very large loop around the origin which contains all singular points of $\mathfrak{D}'$. We can assume as usual that $\mathfrak{D}'$ has been constructed only by adding rays of the form $(1+z^m)^c$.

Then, up to equivalence, $\mathfrak{D}$ can be obtained from $\mathfrak{D}'$ by taking the *asymptotic scattering diagram of* $\mathfrak{D}'$; i.e., just translate each line of $\mathfrak{D}'$ so it passes through the origin and each ray of $\mathfrak{D}'$ so its endpoint is the origin. See §1.4 of [GPS] for more detail. If after performing this translation, we obtain a number of rays with the same support of the form $(\mathfrak{d},(1+z^m)^{c_i})$, i in some index set, we can replace all these rays with a single ray $(\mathfrak{d},(1+z^m)^{\sum c_i})$ without affecting the equivalence class. Thus if we want to show positivity of the exponents for $\mathfrak{D}$, it is enough to show the desired positivity for $\mathfrak{D}'$.

We will typically use an induction hypothesis to show positivity for $\mathfrak{D}'$. Indeed, for each order, we will run the Kontsevich–Soibelman algorithm at each singular point, and the behavior at each singular point is equivalent to a scattering diagram of the general type being considered. Indeed, if p is a singular point of some $\mathfrak{D}'_d$ constructed to order d, we obtain a local version $\mathfrak{D}^{\mathrm{loc}}_p$ of the scattering diagram at p by replacing each $\mathfrak{d}$ with $p\in\mathfrak{d}$ with $\mathfrak{d}-p$, and replacing such translated rays with the line spanned by the ray if the translated ray does not have the origin as its endpoint. As long as all attached functions of rays and lines passing through p are of the form $(1+z^m)^c$ with c a positive integer, we are back in the original situation of the proposition. We shall write $\mathfrak{D}^{\mathrm{loc}}_{p,\mathrm{in}}$ for the set of lines in $\mathfrak{D}^{\mathrm{loc}}_p$.

We first observe that using the perturbation trick it is enough to show the induction hypothesis for order k when at most two of the p_i have $\mathrm{ord}(p_i)=1$. Indeed, after perturbing, the lines of $\mathfrak{D}'_{\mathrm{in}}$ only intersect pairwise, but as more rays are added as the Kontsevich–Soibelman algorithm is run, one might have more complicated behavior at singular points. However, any ray added has order >1. Thus we only have to analyze initial scattering diagrams $\mathfrak{D}^{\mathrm{loc}}_{p,\mathrm{in}}$ with at most two lines of order 1.

Next we observe the induction hypothesis allows us to show the result only for $s=2$, with both lines having order 1. Indeed, write $\mathfrak{D}^{\mathrm{loc}}_{p,\mathrm{in}}$ as $(\mathfrak{d}_i,(1+z^{p_i})^{c_i})$ and order the p_i so that $\mathrm{ord}(p_1)\le\mathrm{ord}(p_2)\le\cdots$. Apply Step I, getting a map $u:P'\to P$ with $u(e_i)=p_i$. We are trying to prove that rays with P-order k have positive exponent. But consider a ray $(\mathfrak{d},(1+z^{\sum n_ip_i})^c)$, which is the image under u of a ray $(\mathfrak{d},(1+z^{\sum n_ie_i})^c)$ appearing in $\mathrm{Scatter}(\{(1+z^{e_i})^{c_i}\,|\,1\le i\le s\})$ with $\mathrm{ord}_P(\sum n_ip_i)=k$ and at least one of $n_j, j\ge 3$ nonzero. Then $\mathrm{ord}_{P'}\sum n_ie_i<k$, so by the induction hypothesis we can assume c is positive. On the other hand, rays of the form $(\mathfrak{d},(1+z^{\sum n_ie_i})^c)$ with $n_j=0$ for $j\ge 3$ appearing in $\mathrm{Scatter}(\{(1+z^{e_i})^{c_i}\,|\,1\le i\le s\})$ already appear in $\mathrm{Scatter}(\{(1+z^{e_i})^{c_i}\,|\,1\le i\le 2\})$, as follows easily by working modulo the ideal in P' generated by the $e_j, j\ge 3$. Thus we are only concerned about rays which arise from scattering the two order 1 lines. Thus it is sufficient to show the result when $s=2$.

Step IV. The change of lattice trick. To deal with the case where $\mathfrak{D}_{\mathrm{in}}$ consists of two lines, we use the *change of lattice* trick to reduce to a simpler expression for the scattering diagram. By Step I, we can take $P=\mathbb{N}^2$, $p_i=e_i$. Let $M^\circ\subseteq M$ be the sublattice generated by $v_1=r(e_1), v_2=r(e_2)$. Note as in Step I we can assume that this is a rank 2 sublattice, as otherwise the automorphisms associated to the two lines commute. Then $N^\circ:=\mathrm{Hom}(M^\circ,\mathbb{Z})$ is a superlattice of N, with dual basis v_1^*, v_2^*. In what follows, we will talk about scattering diagrams defined using both the lattice M and M°. Bear in mind that a wall $(\mathfrak{d}, f_\mathfrak{d})$ could be interpreted using either lattice, and the automorphism induced by crossing such a wall depends on which lattice we are using, as primitive vectors in N differ from primitive vectors in N°.

To see the relationship between these automorphisms, for $w \in N^\circ \setminus \{0\}$, let

$$e(w) = \min\{e > 0 \,|\, ew \in N\}.$$

Then a wall $(\mathfrak{d}, f_\mathfrak{d})$ for M induces a wall-crossing automorphism of $\widehat{\Bbbk[P]}$ which is the same as the automorphism induced by the wall $(\mathfrak{d}, f_\mathfrak{d}^{e(n_\mathfrak{d})})$ for M°, where $n_\mathfrak{d} \in N^\circ$ is primitive and annihilates $\mathfrak{d}$.

Consider

$$\mathfrak{D}^\circ_{\mathrm{in}} := \{(\mathbb{R}v_1, (1+z^{e_1})^{d_1 e(v_2^*)}), (\mathbb{R}v_2, (1+z^{e_2})^{d_2 e(v_1^*)})\}$$

as a scattering diagram for the lattice M°. Let $\mathfrak{D}^\circ = \mathrm{Scatter}(\mathfrak{D}^\circ_{\mathrm{in}})$. Let $\mathfrak{D}'$ be the scattering diagram for M obtained by replacing every wall $(\mathfrak{d}, (1+z^p)^c) \in \mathfrak{D}^\circ$ with $(\mathfrak{d}, (1+z^p)^{c/e(n_\mathfrak{d})})$. Thus the wall-crossing automorphism for each wall in $\mathfrak{D}'$ as a scattering diagram for the lattice M is the same automorphism for the corresponding wall in $\mathfrak{D}^\circ$. Then $\mathfrak{p}_{\gamma,\mathfrak{D}'}$ is the identity. Thus by uniqueness of the scattering process up to equivalence, $\mathfrak{D}'$ is equivalent to $\mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})$. (Note this implies that $c/e(n_\mathfrak{d}) \in \mathbb{Z}$ also, as $\mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})$ only involves integer exponents.)

Thus it is enough to prove the desired positivity for the scattering diagram $\mathfrak{D}^\circ$. To do so, we use a variant of the perturbation trick, factoring the two lines in $\mathfrak{D}^\circ_{\mathrm{in}}$. We choose general $v^1_{j_1}, v^2_{j_2} \in M_\mathbb{R}$, with $1 \le j_1 \le d_1 e(v_2^*)$, $1 \le j_2 \le d_2 e(v_1^*)$. Define

$$\tilde{\mathfrak{D}}^\circ_{\mathrm{in}} := \{(v^1_j + \mathbb{R}v_1, 1+z^{e_1}) \,|\, 1 \le j \le d_1 e(v_2^*)\} \cup \{(v^2_j + \mathbb{R}v_2, 1+z^{e_2}) \,|\, 1 \le j \le d_2 e(v_1^*)\}.$$

Again, we initially only have pairwise intersections. The first stage of this algorithm will then only involve points where two lines of the form $(v^1_j + \mathbb{R}v_1, 1+z^{e_1})$ and $(v^2_{j'} + \mathbb{R}v_2, 1+z^{e_2})$ intersect. The algorithm only adds one ray in the direction $-v_1 - v_2$ with endpoint the intersection point and attached function $1 + z^{e_1+e_2}$, as follows from Example 1.14. This now accounts for all new rays of order 2. We continue to higher degree, but now we can use the induction hypothesis at every singular point p as we did in Step III, because every line in $\mathfrak{D}^{\mathrm{loc}}_{p,\mathrm{in}}$ has order ≥ 2 except for possibly one or two of the given lines of order 1, and we have already accounted for all rays produced by collisions of two lines of order 1. $\square$

Corollary C.15. *In the situation of Proposition* C.13*, suppose instead that*

$$\mathfrak{D}_{\mathrm{in}} := \{(\mathbb{R}r(p_i), (1+\alpha_i z^{p_i})^{d_i}) \,|\, 1 \le i \le s\},$$

where now $\alpha_i \in \Bbbk$, the ground field. Choosing $\mathfrak{D} = \mathrm{Scatter}(\mathfrak{D}_{\mathrm{in}})$ up to equivalence, we can assume that each ray $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D} \setminus \mathfrak{D}_{\mathrm{in}}$ satisfies

$$f_\mathfrak{d} = (1 + \prod_i (\alpha_i z^{p_i})^{a_i})^c$$

for some choice of nonnegative integers a_i and where c is a positive integer.

Proof. This follows easily from from Proposition C.13. First, using the change of monoid trick (Step I of the proof of Proposition C.13), we may assume $P = \mathbb{N}^s$ and $p_i = e_i$. Consider the automorphism $\nu : \widehat{\Bbbk[P]} \to \widehat{\Bbbk[P]}$ defined by $\nu(z^{e_i}) = \alpha_i z^{e_i}$. Applying ν to the function attached to each wall of $\mathrm{Scatter}(\{(\mathbb{R}r(e_i), (1+z^{e_i})^{d_i})\})$ gives a scattering diagram $\mathfrak{D}'$ whose incoming walls are precisely those of $\mathfrak{D}_{\mathrm{in}}$, and

$\mathfrak{p}_{\gamma,\mathfrak{D}'} = \mathrm{id}$ for γ a loop around the origin. Thus we can take $\mathfrak{D} = \mathfrak{D}'$ and the result follows from Proposition C.13. $\square$

Proof of Theorem 1.13. In fact one can use the $\mathfrak{D}_{\mathbf{s}}$ as constructed explicitly in the algorithm of the proof of Theorem 1.28. The only issue is that we need to know that the walls added at each joint have the desired positivity property. Note that the statement of Theorem 1.13 involves scattering diagrams without slabs, while the proof of Theorem 1.28 given involves a slab. So for the purpose of this discussion, we can ignore all issues concerning the slab in the proof of Theorem 1.28, and the only thing we need to do is look at the procedure for producing $\mathfrak{D}[\mathfrak{j}]$ in Step II of the proof of Theorem 1.28.

For a perpendicular joint $\mathfrak{j}$ of $\mathfrak{D}_d$, we can split $M = \Lambda_{\mathfrak{j}} \oplus M'$, where M' is a rank 2 lattice. For each wall $\mathfrak{d} \in \mathfrak{D}_d$ containing $\mathfrak{j}$, we can inductively assume that $f_{\mathfrak{d}} = (1+z^m)^c$ for some positive integer c, and split $z^m = z^{m_{\mathfrak{j}}} z^{m'}$, with $m_{\mathfrak{j}} \in \Lambda_{\mathfrak{j}}$ and $m' \in M'$. Because $\mathfrak{j}$ is perpendicular, we have $m' \neq 0$. We will apply Corollary C.15 to the case where the monoid P is the one being used in Theorem 1.21, and $r : P \to M'$ is the projection. We can then view the computation at the joint as a two-dimensional scattering situation in the lattice M' over the ground field $\Bbbk(\Lambda_{\mathfrak{j}})$, the quotient field of $\Bbbk[\Lambda_{\mathfrak{j}}]$. To obtain the relevant two-dimensional scattering diagram, we replace each wall $(\mathfrak{d}, f_{\mathfrak{d}})$ with $\mathfrak{j} \subseteq \mathfrak{d}$ with $\big((\mathfrak{d} + \Lambda_{\mathfrak{j}} \otimes \mathbb{R})/(\Lambda_{\mathfrak{j}} \otimes \mathbb{R}), f_{\mathfrak{d}}\big)$ in $M'_{\mathbb{R}} = M_{\mathbb{R}}/\Lambda_{\mathfrak{j}} \otimes \mathbb{R}$. We are then in the situation of Corollary C.15, and the result follows. $\square$

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