JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 3, July 2018, Pages 661–697
http://dx.doi.org/10.1090/jams/893
Article electronically published on November 30, 2017

A PROOF OF THE SHUFFLE CONJECTURE

ERIK CARLSSON AND ANTON MELLIT

1. Introduction

The shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [HHL+05b] predicts a combinatorial formula for the Frobenius character $\mathcal{F}_{R_n}(X;q,t)$ of the diagonal coinvariant algebra R_n in n pairs of variables, which is a symmetric function in infinitely many variables with coefficients in $\mathbb{Z}_{\geqslant 0}[q,t]$. By a result of Haiman [Hai02], the Frobenius character is given explicitly by

$$\mathcal{F}_{R_n}(X;q,t) = (-1)^n \nabla e_n[X],$$

where, up to a sign convention, ∇ is the operator which is diagonal in the modified Macdonald basis defined in [BGHT99]. The original shuffle conjecture states

$$(1.1) \qquad (-1)^n \nabla e_n[X] = \sum_{\pi} \sum_{w \in \mathcal{WP}_\pi} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi,w)} x_w.$$

Here π is a Dyck path of length n, and w is some extra data called a "word parking function" depending on π. The functions (area, dinv) are statistics associated to a Dyck path and a parking function, and x_w is a monomial in the variables x. It was shown in [HHL+05b] that this sum, denoted $D_n(X;q,t)$, is symmetric in the x variables, and so it does at least define a symmetric function. It was also shown there that the shuffle conjecture included many previous conjectures and results about the q,t-Catalan numbers, and other special cases [GH96, GH02, Hag03, EHKK03, Hag04]. Remarkably, $D_n(X;q,t)$ had not even been proven to be symmetric in the q,t variables until now, even though the symmetry of $\mathcal{F}_{R_n}(X;q,t)$ is obvious. The name "shuffle conjecture" has to do with the fact that the coefficient of m_μ in equation (1.1) can be expressed in terms of parking functions that are "μ-shuffles". See Conjecture 6.1 of Haglund's book [Hag08] for a detailed explanation, and Chapter 6 in general for a thorough introduction to this topic.

In [HMZ12] Haglund, Morse, and Zabrocki conjectured a refinement of the original conjecture which partitions $D_n(X;q,t)$ by specifying the points where the Dyck path touches the diagonal called the "compositional shuffle conjecture". The refined conjecture states

$$(1.2) \qquad \nabla \left(C_\alpha[X;q]\right) = \sum_{\mathrm{touch}(\pi)=\alpha} \sum_{w \in \mathcal{WP}_\pi} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi,w)} x_w.$$

Received by the editors March 29, 2016, and, in revised form, August 29, 2017, and October 11, 2017.

2010 *Mathematics Subject Classification.* Primary 05E10; Secondary 05E05, 05A30, 33D52.

Here α is a composition, i.e., a finite list of positive integers specifying the gaps between the touch points of π. The function $C_\alpha[X;q]$ is defined below as a composition of creation operators for Hall–Littlewood polynomials in the variable $1/q$. They proved that

$$\sum_{|\alpha|=n} C_\alpha[X;q] = (-1)^n e_n[X],$$

implying that (1.2) does indeed generalize (1.1). The right-hand side of (1.2) will be denoted by $D_\alpha(X;q,t)$. A desirable approach to proving (1.2) would be to determine a recursive formula for $D_\alpha(X;q,t)$ and to interpret the result in terms of some commutation relations for ∇. Indeed, this approach has been applied in some important special cases; see [GH02, GXZ12, Hic12]. In [GXZ12], for instance, the authors devise a recursive formula (Proposition 3.12) to prove the Catalan case of the compositional conjecture, extending the results of [GH02]. Unfortunately, no such recursion is known in the general case, and so an even more refined function is needed.

In this paper, we will construct the desired refinement as an element of a larger vector space V_k of symmetric functions over $\mathbb{Q}(q,t)$ with k additional variables y_i adjoined, where k is the length of the composition α,

$$N_\alpha \in V_k = \mathrm{Sym}[X][y_1,\ldots,y_k].$$

In our first result (Theorem 4.11) we will explain how to recover $D_\alpha(X;q,t)$ from N_α, which is defined by an explicit recursion. In fact, while they live in different vector spaces, the recursions for N_α are similar to the recursions for the Catalan case in [GXZ12]. We make this connection precise in Proposition 4.14, which explains how the latter formulas follow as a special case.

We then define a pair of algebras $\mathbb{A}$ and $\mathbb{A}^*$ which are isomorphic by an antilinear isomorphism with respect to the conjugation $(q,t) \to (q^{-1},t^{-1})$, as well as an explicit action of each on the direct sum $V_* = \bigoplus_{k\geqslant 0} V_k$. We will then prove that there is an antilinear involution $\mathcal{N}$ on V_* which intertwines the two actions (Theorem 7.4) and represents an involutive automorphism on a larger algebra $\mathbb{A}, \mathbb{A}^* \subset \tilde{\mathbb{A}}$. This turns out to be the essential fact that relates the N_α to ∇.

The compositional shuffle conjecture (Theorem 7.5) then follows as a simple corollary from the following properties:

(i) There is a surjection coming from $\mathbb{A}, \mathbb{A}^*$

$$d_-^k : V_k \to V_0 = \mathrm{Sym}[X]$$

which maps a monomial y_α in the y variables to an element $B_\alpha[X;q]$ which is similar to $C_\alpha[X;q]$ and maps N_α to $D_\alpha(X;q,t)$, up to a sign.

(ii) The involution $\mathcal{N}$ commutes with d_- and maps y_α to N_α.

(iii) The restriction of $\mathcal{N}$ to $V_0 = \mathrm{Sym}[X]$ agrees with ∇ composed with a conjugation map which essentially exchanges the $B_\alpha[X;q]$ and $C_\alpha[X;q]$.

It then becomes clear that these properties imply (1.2).

While the compositional shuffle conjecture is clearly our main application, the shuffle conjecture has been further generalized in several remarkable directions such as the rational compositional shuffle conjecture, and relationships to knot invariants, double affine Hecke algebras, and the cohomology of the affine Springer fibers; see [BGLX14, GORS14, GN15, Neg13, Hik14, SV11, SV13]. We hope that future applications to these fascinating topics will be forthcoming.

2. The compositional shuffle conjecture

2.1. **Plethystic operators.** A λ-ring is a ring R with a family of ring endomorphisms $(p_i)_{i\in\mathbb{Z}_{>0}}$ satisfying

$$p_1[x] = x, \quad p_m[p_n[x]] = p_{mn}[x] \quad (x \in R, \quad m,n \in \mathbb{Z}_{>0}).$$

Unless stated otherwise, the endomorphisms are defined by $p_n(x) = x^n$ for each variable x such as q, t, u, v, z, x_i, y_i. The ring of symmetric functions over the λ-ring $\mathbb{Q}(q,t)$ is a free λ-ring with generator $X = x_1 + x_2 + \cdots$, and it will be denoted $\mathrm{Sym}[X]$. We will employ the standard notation used for plethystic substitution defined as follows: given an element $F \in \mathrm{Sym}[X]$ and A in some λ-ring R, the plethystic substitution $F[A]$ is the image of the homomorphism from $\mathrm{Sym}[X] \to R$ defined by replacing p_n by $p_n(A)$. For instance, we would have

$$p_1p_2[X/(1-q)] = p_1[X]p_2[X](1-q)^{-1}(1-q^2)^{-1}.$$

See [Hai01] for a reference.

The modified Macdonald polynomials [GHT99] will be denoted

$$\tilde{H}_\mu = t^{n(\mu)} J_\mu[X/(1-t^{-1}); q, t^{-1}] \in \mathrm{Sym}[X],$$

where J_μ is the integral form of the Macdonald polynomial [Mac95], and

$$n(\mu) = \sum_i (i-1)\mu_i.$$

The operator $\nabla : \mathrm{Sym}[X] \to \mathrm{Sym}[X]$ is defined by

$$\nabla \tilde{H}_\mu = \tilde{H}_\mu[-1]\tilde{H}_\mu = (-1)^{|\mu|} q^{n(\mu')} t^{n(\mu)} \tilde{H}_\mu. \tag{2.1}$$

Note that our definition differs from the usual one from [BGHT99] by the sign $(-1)^{|\mu|}$. We also have the sequences of operators $B_r, C_r : \mathrm{Sym}[X] \to \mathrm{Sym}[X]$ given by the formulas

$$(B_r F)[X] = F[X - (q-1)z^{-1}]\,\mathrm{Exp}[-zX]\big|_{z^r},$$
$$(C_r F)[X] = -q^{1-r} F[X + (q^{-1}-1)z^{-1}]\,\mathrm{Exp}[zX]\big|_{z^r},$$

where $\mathrm{Exp}[X] = \sum_{n=0}^{\infty} h_n[X]$ is the plethystic exponential and $|_{z^r}$ denotes the operation of taking the coefficient of z^r of a Laurent power series. Our definition again differs from the one in [HMZ12] by a factor $(-1)^r$. For any composition α, let C_α denote the composition $C_{\alpha_1} \cdots C_{\alpha_l}$, and similarly for B_α.

Finally, we denote by $x \mapsto \bar{x}$ the involutive automorphism of $\mathbb{Q}(q,t)$ obtained by sending q, t to q^{-1}, t^{-1}. We denote by ω the λ-ring automorphism of $\mathrm{Sym}[X]$ obtained by sending X to $-X$ and by $\bar{\omega}$ its composition with $\bar{*}$, i.e.,

$$(\omega F)[X] = F[-X], \quad (\bar{\omega} F)[X] = \bar{F}[-X].$$

2.2. **Parking functions.** We now recall the combinatorial background to state the shuffle conjecture, for which we refer to Haglund's book [Hag08]. We consider the infinite grid on the top right quadrant of the plane. Its intersection points are denoted as (i,j) with $i,j \in \mathbb{Z}$. For each cell of the grid, its coordinates (i,j) are the coordinates of the top right corner. Thus $i = 1, 2, \ldots$ indexes the columns and $j = 1, 2, \ldots$ indexes the rows. Let $\mathbb{D}$ be the set of Dyck paths of all lengths. A Dyck path of length n is a grid path, from $(0,0)$ to (n,n) consisting of North and East steps, that stays above the main diagonal $i = j$. For $\pi \in \mathbb{D}$ denote by $|\pi|$ its length n. For $\pi \in \mathbb{D}$, let

$$\operatorname{area}(\pi) := \#\operatorname{Area}(\pi), \quad \operatorname{Area}(\pi) := \{(i,j) : i < j,\ (i,j) \text{ under } \pi\}.$$

This is the set of cells between the path and the diagonal. Let a_j denote the number of cells $(i,j) \in \operatorname{Area}(\pi)$ in the row j. The *area sequence* is the sequence $a(\pi) = (a_1, a_2, \ldots, a_n)$, and we have $\operatorname{area}(\pi) = \sum_{j=1}^n a_n$.

Let $(x_1, 1), (x_2, 2), \ldots, (x_n, n)$ be the cells immediately to the right of the North steps. The sequence $x(\pi) = (x_1, x_2, \ldots, x_n)$ is called the *coarea sequence*, and we have $a_j + x_j = j$ for all j.

We have the dinv statistic and the Dinv set defined by

$$\begin{aligned}\operatorname{dinv}(\pi) &:= \#\operatorname{Dinv}(\pi),\\ \operatorname{Dinv}(\pi) &:= \operatorname{Dinv}^0(\pi) \cup \operatorname{Dinv}^1(\pi)\\ &= \left\{(j,j') : 1 \leqslant j < j' \leqslant n,\ a_j = a_{j'}\right\}\\ &\quad \cup \left\{(j,j') : 1 \leqslant j' < j \leqslant n,\ a_{j'} = a_j + 1\right\}.\end{aligned}$$

For $(j,j') \in \operatorname{Dinv}(\pi)$, we say that (x_j, j) *attacks* $(x_{j'}, j')$.

For any π, the set of *word parking functions* associated to π is defined by

$$\mathcal{WP}_\pi := \{w \in \mathbb{Z}_{>0}^n : w_j > w_{j+1} \text{ whenever } x_j = x_{j+1}\}.$$

In other words, the elements of $\mathcal{WP}_\pi$ are n-tuples w of positive integers which, when written from bottom to top to the right of each North step, are strictly decreasing on cells such that one is on top of the other. For any w, let

$$\operatorname{dinv}(\pi, w) := \#\operatorname{Dinv}(\pi, w), \quad \operatorname{Dinv}(\pi, w) := \left\{(j,j') \in \operatorname{Dinv}(\pi) : w_j > w_{j'}\right\}.$$

We note that both of these conditions differ from the usual notation in which parking functions are expected to increase rather than decrease, and in which the inequalities are reversed in the definition of dinv. This corresponds to choosing the opposite total ordering on $\mathbb{Z}_{>0}$ everywhere, which does not affect the final answer and is more convenient for the purposes of this paper.

Let us call $\alpha = (\alpha_1, \ldots, \alpha_k) = \operatorname{touch}(\pi)$ the *touch composition* of π if $\alpha_1, \ldots, \alpha_k$ are the lengths of the gaps between the points where π touches the main diagonal starting at the lower left. Equivalently, $\sum_{i=1}^k \alpha_i = n$ and the numbers 1, $1 + \alpha_1$, $1 + \alpha_1 + \alpha_2, \ldots, 1 + \alpha_1 + \cdots + \alpha_{k-1}$ are the positions of 0 in the area sequence $a(\pi)$.

Example 2.1. Let π be the following Dyck path of length 8 described in Figure 1. Then we have

$$\begin{aligned}\operatorname{Area}(\pi) &= \{(2,3),(2,4),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6),(7,8)\},\\ \operatorname{Dinv}(\pi) &= \{(1,2),(1,7),(2,7),(3,8),(4,5)\} \cup \{(7,3),(8,4),(8,5)\},\\ \operatorname{touch}(\pi) &= (1,5,2), \quad a(\pi) = (0,0,1,2,2,3,0,1),\\ x(\pi) &= (1,2,2,2,3,3,7,7)\end{aligned}$$

whence $\operatorname{area}(\pi) = 9$, $\operatorname{dinv}(\pi) = 5 + 3 = 8$. The labels shown above correspond to the vector $w = (9,5,2,1,5,2,3,2)$, which we can see is an element of $\mathcal{WP}_\pi$ because we have $5 > 2 > 1$, $5 > 2$, $3 > 2$. We then have

$$\operatorname{Dinv}(\pi, w) = \{(1,2),(1,7),(2,7)\} \cup \{(7,3),(8,4)\},$$

giving $\operatorname{dinv}(\pi, w) = 5$.

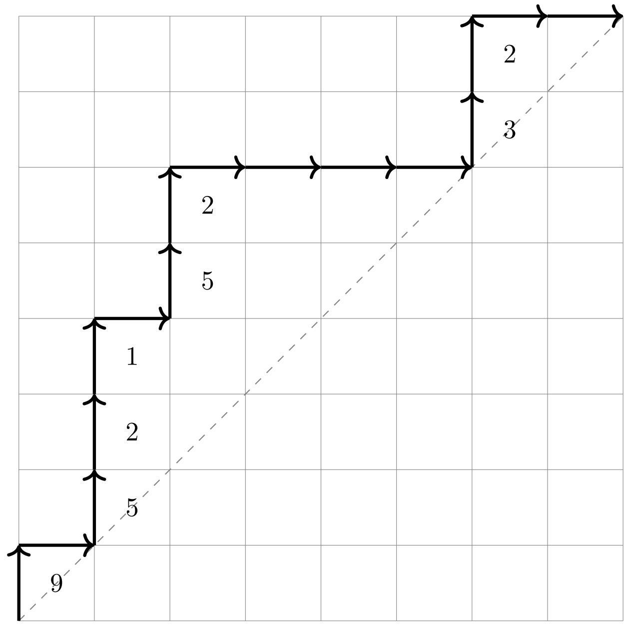

FIGURE 1. Example of a Dyck path of size 8.

2.3. **The shuffle conjectures.** For any infinite set of variables $X = \{x_1, x_2, \ldots\}$, let $x_w = x_{w_1} \cdots x_{w_n}$. In this notation, the original shuffle conjecture [HHL+05b] states

Conjecture ([HHL+05b])**.** We have

$$(-1)^n \nabla e_n = \sum_{|\pi|=n} t^{\mathrm{area}(\pi)} \sum_{w \in \mathcal{WP}_\pi} q^{\mathrm{dinv}(\pi,w)} x_w.$$

In particular, the right-hand side is symmetric in the x_i, and in q, t.

The stronger compositional shuffle conjecture [HMZ12] states

Conjecture ([HMZ12])**.** For any composition α, we have

$$(-1)^n \nabla C_\alpha(1) = \sum_{\mathrm{touch}(\pi)=\alpha} t^{\mathrm{area}(\pi)} \sum_{w \in \mathcal{WP}_\pi} q^{\mathrm{dinv}(\pi,w)} x_w. \tag{2.2}$$

2.4. **From** (area, dinv) **to** (bounce, area′)**.** In this paper, we will prove an equivalent version of this conjecture, as obtained in [LN14, Theorem 14], by applying the (area, dinv) to (bounce, area′) bijection from [HL05] and [Hag08]. We include our construction of this bijection because it seems to be different from the original one, and to demonstrate that it comes naturally from analysis of the attack relation. An important property of our construction is that it comes with a natural lift from Dyck paths to parking functions.

From any pair $\pi \in \mathbb{D}$, $w \in \mathcal{WP}_\pi$ we will obtain a pair $\pi' \in \mathbb{D}$, $w' \in \mathcal{WP}'_{\pi'}$ by a procedure described below. After the end of this section we will only work with π', w', so we will drop the apostrophe.

The Dyck path w' is obtained as follows: sort the cells (x_j, j) in the *reading order*, i.e., in increasing order by the corresponding labels a_j, using the row index

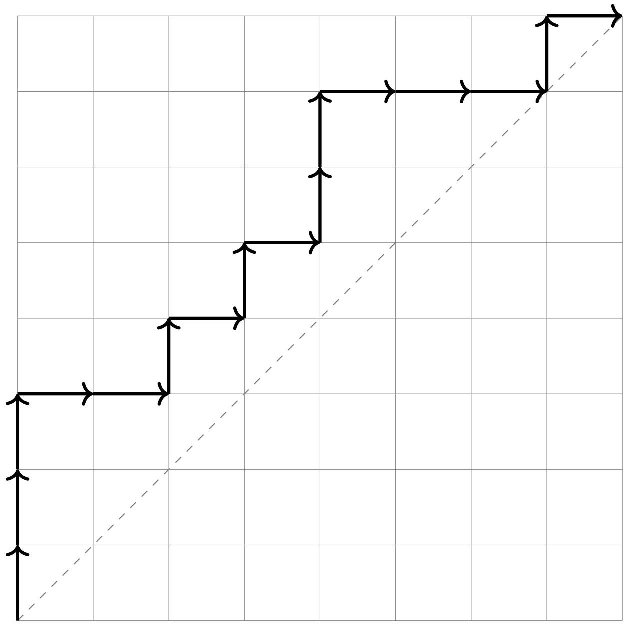

FIGURE 2. Image of the path from Figure 1 under the (area, dinv) to (bounce, area′) bijection.

j to break ties. Equivalently, we read the cells by diagonals from bottom to top, and from left to right in each diagonal. For instance, for the path π from Example 2.1, the list would be

$$\{(1,1),(2,2),(7,7),(2,3),(7,8),(2,4),(3,5),(3,6)\}. \tag{2.3}$$

Let σ_j be the position of the cell (x_j, j) in this list. This defines a permutation $\sigma \in S_n$. In the example case, we would get

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 6 & 7 & 8 & 3 & 5 \end{pmatrix}.$$

Now we observe that for each $j = 1, \ldots, n$, the cell (x_j, j) attacks all the subsequent cells in the reading order whose position is before the position where we would place $(x_j, j+1)$ if it were an element of the list.

More precisely, there is a unique Dyck path π' for which

$$\mathrm{Area}(\pi') = \sigma(\mathrm{Dinv}(\pi)) = \left\{(\sigma_j, \sigma_{j'}) : (j, j') \in \mathrm{Dinv}(\pi)\right\}.$$

The map $\pi \to \pi'$ is the desired bijection. To see the bijectivity one can either use [Hag08] or see Remark 2.3.

If π is the Dyck path from our example, then π' would be given by the path in Figure 2.

The above statistics can be translated into new statistics under this bijection. First, it is clear from the construction that $\mathrm{dinv}(\pi) = \mathrm{area}(\pi')$. We next explain how to calculate $\mathrm{area}(\pi)$ from π'. For any path, we obtain a new Dyck path called the "bounce path" as follows: Start at the origin $(0,0)$, and begin moving North until contact is made with the first East step of π. Then start moving East until contacting the diagonal. Then move North until contacting the path again, and so

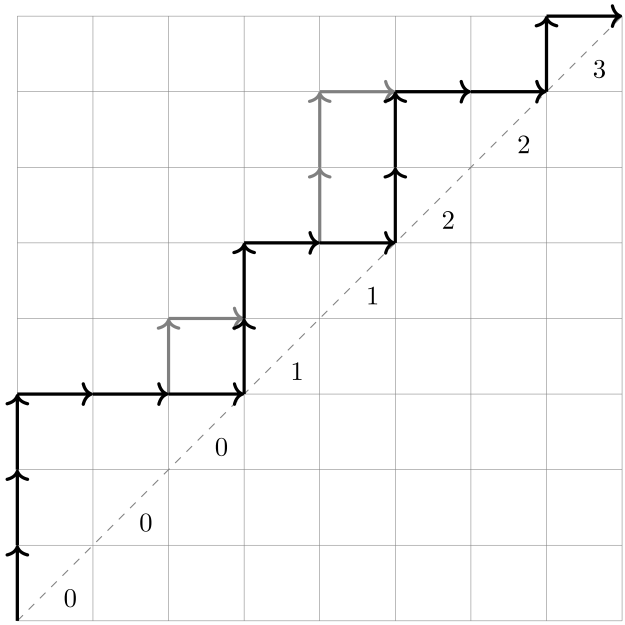

FIGURE 3. The bounce path of the path in Figure 2.

on. Note that contacting the path means running into the left endpoint of an East step, but passing by the rightmost endpoint does not count, as illustrated below. The bounce path splits the main diagonal into the *bounce blocks.* We number the bounce blocks starting from 0 and define the *bounce sequence* $b(\pi) = (b_1, b_2, \dots, b_n)$ in such a way that for any i the cell (i, i) belongs to the b_ith block. We then define

$$\mathrm{bounce}(\pi') := \sum_{i=1}^{n} b_i.$$

Another way to describe this construction is to say that $b_1 = 0$, $b_{i+1} \in \{b_i, b_i + 1\}$ and if i, i' are the smallest indices for which $b_i = c$ and $b_{i'} = c + 1$ for some c, then i' is the smallest index with $i' > i$ such that $(i, i') \notin \mathrm{Area}(\pi')$. This description and $\mathrm{Area}(\pi') = \sigma(\mathrm{Dinv}(\pi))$ implies $b_{\sigma_i} = a_i$, hence $\mathrm{bounce}(\pi') = \mathrm{area}(\pi)$; see [Hag08] for an alternative treatment.

For the path π' above, the bounce path is shown in Figure 3 with the original path in gray. The bounce sequence is given by the numbers written under the diagonal. We have

$$b(\pi') = (0, 0, 0, 1, 1, 2, 2, 3), \quad \mathrm{bounce}(\pi') = 9 = \mathrm{area}(\pi).$$

Next, we show how to reconstruct $\mathrm{touch}(\pi)$ from π'. For any path π' of length n, let l be the number of North steps from $(0, 0)$ until the first East step, which is the same as the length of the first bounce block. Let $\tilde{\pi}$ be the part of the path such that $\pi' = N^l E \tilde{\pi}$, the result of beginning with l North steps starting at the origin, followed by an East step, followed by the contents of $\tilde{\pi}$. Define numbers t_i by

$$t_i := \mathrm{bounce}\left(N^{i+1} E N^{l-i} E \tilde{\pi}\right), \quad 0 \leqslant i \leqslant l.$$

Note that the path $N^{i+1}EN^{l-i}E\tilde{\pi}$ has length $n+1$ for each i, and we have $t_0 = n + \mathrm{bounce}(\pi')$ and the t_i go down to $t_l = \mathrm{bounce}(\pi')$. Define

$$\mathrm{touch}'(\pi') := (t_0 - t_1, \ldots, t_{l-1} - t_l).$$

Proposition 2.2. *For every Dyck path* π

$$\mathrm{touch}'(\pi') = \mathrm{touch}(\pi).$$

For instance, in the example above we would have $l = 3$,

$$(t_0, t_1, t_2, t_3) = (17, 16, 11, 9), \quad \mathrm{touch}'(\pi') = \mathrm{touch}(\pi) = (1, 5, 2).$$

Proof. Consider the ith touch point (x, x) of π (we count the touch points starting from 0, i.e., $(0,0)$ is the 0th touch point.) It splits π into two parts: π_1 followed by π_2. Construct a new path $\hat{\pi}$ of length $n+1$ by taking a step North, then following a translated copy of π_2, then taking a step East, then following a translated copy of π_1. The new path has length $n+1$, and its area is bigger than the area of π by $n - x$. The (area,dinv) to (bounce,area′) map applied to $\hat{\pi}$ gives precisely the path $N^{i+1}EN^{l-i}E\tilde{\pi}$. Thus we have

$$\mathrm{area}(\hat{\pi}) = \mathrm{bounce}(N^{i+1}EN^{l-i}E\tilde{\pi}) = t_i,$$
$$\mathrm{area}(\hat{\pi}) = n - x + \mathrm{area}(\pi) = n - x + \mathrm{bounce}(\pi').$$

So the sizes of the gaps between the touch points of π are exactly the differences $t_{i-1} - t_i$. □

Remark 2.3. The construction we have used in the proof above can also be used to prove the bijectivity of the (area,dinv) to (bounce,area′) map. Here is an idea of a proof. First, every Dyck path arises as $\hat{\pi}$ above for unique π and i. On the other hand, every Dyck path can be uniquely written as $N^{i+1}EN^{l-i}E\tilde{\pi}$. Thus, iterating the construction, we obtain every Dyck path on each side of the (area,dinv) to (bounce,area′) map in a unique way.

Having analyzed the statistics associated to a Dyck path, we turn to the analysis of what happens to word parking functions. The dinv statistic is straightforward. For any $w' \in \mathbb{Z}^n_{>0}$, let

$$\mathrm{inv}(\pi', w') := \#\mathrm{Inv}(\pi', w'), \quad \mathrm{Inv}(\pi', w') := \left\{(i,j) \in \mathrm{Area}(\pi'),\ w'_i > w'_j\right\},$$

so that

$$\mathrm{Inv}(\pi', w') = \sigma\left(\mathrm{Dinv}(\pi, w)\right), \quad w'_{\sigma_i} = w_i.$$

For the value of w from Example 2.1, we would have

$$w' = (9, 5, 3, 2, 2, 1, 5, 2), \quad \mathrm{Inv}(\pi', w') = \{(1,2), (1,3), (2,3), (3,4), (5,6)\}.$$

In particular, $\mathrm{inv}(\pi', w') = \mathrm{dinv}(\pi, w) = 5$.

Finally, we reconstruct the word parking function condition. A cell (i,j) is called a *corner* of π' if it is above the path, but both its Southern and Eastern neighbors are below the path. Denote the set of corners by $c(\pi')$. One can check that the corners of π' correspond to pairs of cells with one on top of the other in π. For instance, from our example we have $c(\pi') = \{(2,4), (3,5), (4,6), (7,8)\}$. More precisely, we have

$$c(\pi') := \{(\sigma_j, \sigma_{j+1}) :\ 1 \leqslant j < n,\ x_j = x_{j+1}\}.$$

We therefore define

$$\mathcal{WP}'_{\pi'} := \left\{w' \in \mathbb{Z}_{>0}^n : w'_i > w'_j \text{ for } (i,j) \in c(\pi')\right\}, \tag{2.4}$$

so that the condition $w \in \mathcal{WP}_\pi$ is equivalent to $w' \in \mathcal{WP}'_{\pi'}$.

Putting this together, we have

Proposition 2.4. *For any composition α we have*

$$D_\alpha(q,t) = \sum_{\text{touch}'(\pi)=\alpha} t^{\text{bounce}(\pi)} \sum_{w \in \mathcal{WP}'_\pi} q^{\text{inv}(\pi,w)} x_w, \tag{2.5}$$

where $D_\alpha(q,t)$ is the right-hand side of (2.2).

3. Characteristic functions of Dyck paths

3.1. **Simple characteristic function.** We are going to study the summand in $D_\alpha(q,t)$ as a function of π. It is convenient to first introduce a simpler object where we drop the assumption $w \in \mathcal{WP}'_\pi$ and instead sum over all labelings. Given a Dyck path of length n, define $\chi(\pi) \in \text{Sym}[X]$ as follows:

Definition 3.1.

$$\chi(\pi) := \sum_{w \in \mathbb{Z}_{>0}^n} q^{\text{inv}(\pi,w)} x_w.$$

If $i < j$ and (i,j) is under π, i.e., $(i,j) \in \text{Area}(\pi)$, we say that i and j *attack* each other. It is not obvious from the definition that $\chi(\pi)$ defines a symmetric function. In fact, as we point out in Remark 3.6, $\chi(\pi)$ is actually an example of an LLT polynomial, but we present a proof in our setup here:

Proposition 3.2. *The expression for $\chi(\pi)$ above is symmetric in the variables $x_1, x_2, x_3, \ldots$, so that Definition* 3.1 *correctly defines an element of* $\text{Sym}[X]$.

Proof. We take the main idea from the proof of Lemma 10.2 from [HHL05a]. First note that for each n the correspondence $\pi \to \text{Area}(\pi)$ is a bijection between the set of Dyck paths of length n and the set of subsets $R \subset \{(j,j') : 1 \leqslant j < j' \leqslant n\}$ satisfying the property

(*) if $j < j' < j''$ and $(j,j'') \in R$, then both (j,j') and (j',j'') are in R.

In the proof we will work with R instead of π and write $\chi(R,n)$, $\text{inv}(R,w)$ instead of $\chi(\pi)$, $\chi(\pi,w)$. For each subset $S \subset \{1,2,\ldots,n\}$, $S = \{s_1 < s_2 < \cdots < s_{\#S}\}$, let $R_S = \{(j,j') : (s_j, s_{j'}) \in R\}$. Then R_S again satisfies (*).

It is enough to show that $\chi(R,n)$ is unaffected by interchange of x_i and x_{i+1} for any two neighboring indices $i, i+1$. For each subset $S \subset \{1,2,\ldots,n\}$ and a function $f : \{1,\ldots,n\} \to \mathbb{Z}_{>0}\backslash\{i,i+1\}$, let $\chi_{S,f}$ be the sub-sum of $\chi(R,n)$ corresponding to sequences w where the set of positions of i and $i+1$ in w is S, and the values of w outside of S are given by f. It is enough to show that $\chi_{S,f}$ is symmetric in x_i and x_{i+1}. We have

$$\text{inv}(R,w) = \#\{(j,j') \in R_S : w_{s_j} = i+1,\ w_{s_{j'}} = i\} + \text{inv}_{S,f},$$

where $\text{inv}_{S,f}$ depends only on S and f, but does not depend on the positions of i or $i+1$ in S. Thus we have

$$\chi_{S,f} = \chi(R_S, \#S)(x_i, x_{i+1})\, q^{\text{inv}_{S,f}} \prod_{j \notin S} x_{f(j)},$$

where

$$\chi(R,k)(x_1,x_2) = \sum_{w\in\{1,2\}^k} q^{\mathrm{inv}(R,w)} x_w.$$

So it is enough to show that $\chi(R,k)(x_1,x_2)$ is symmetric in x_1 and x_2 for any k, R satisfying (*). We proceed by induction on the size of R, the base case $\#R = 0$ being trivial. Fix k and $R \neq \varnothing$, and pick $(a,b) \in R$ maximal in the sense that $(a,j) \notin R$ for all $j > b$ and $(j,b) \notin R$ for all $j < a$. Then $R' = R\backslash\{(a,b)\}$ satisfies (*). Consider the difference

$$\chi(R,k) - \chi(R',k) = \sum_{w\in\{1,2\}^k} (q^{\mathrm{inv}(R,w)} - q^{\mathrm{inv}(R',w)}) x_w.$$

The coefficient $(q^{\mathrm{inv}(R,w)} - q^{\mathrm{inv}(R',w)})$ is nonzero only if $w_a = 2$ and $w_b = 1$. If this happens, then $\mathrm{inv}(R,w) = \mathrm{inv}(R',w) + 1$. Consider contributions of pairs of the form (a,j), (b,j), (j,a), (j,b) to $\mathrm{inv}(R',w)$. Since $w_a = 2$ pairs (j,a) do not contribute anything, and pairs (a,j) contribute 1 precisely when $a < j < b$ and $w_j = 1$. Since $w_b = 1$ pairs (b,j) do not contribute, and pairs (j,b) contribute 1 precisely when $a < j < b$ and $w_j = 2$. The net contribution is the number of j such that $a < j < b$, which is $b-a-1$. We see that $\mathrm{inv}(R',w) = b-a-1+\mathrm{inv}(R'_S, w_S)$, where $S = \{1,\dots,k\}\backslash\{a,b\}$, w_S denotes the sequence w with the entries w_a and w_b removed. Thus we obtain

$$\chi(R,k) - \chi(R',k) = (q-1)x_1x_2\chi(R'_S, k-2).$$

By the induction hypothesis $\chi(R',k)$ and $\chi(R'_S,k-2)$ are symmetric in x_1,x_2. Hence $\chi(R,k)$ is also symmetric. □

Another way to formulate this property is as follows: For a composition $c_1 + c_2 + \cdots + c_k = n$, consider the multiset $M_c = 1^{c_1}2^{c_2}\cdots k^{c_k}$. Consider the sum

$$\sum_{w \text{ a permutation of } M_c} q^{\mathrm{inv}(\pi,w)}.$$

Proposition 3.2 simply says that this sum does not depend on the order of the numbers $c_1, c_2, \dots, c_k$, or equivalently on the linear order on the set of labels. If λ is the partition with components $c_1, c_2, \dots, c_k$, then this sum computes the coefficient of the monomial symmetric function m_λ in $\chi(\pi)$, so we have (set $h_c = h_{c_1}\cdots h_{c_k}$)

$$(\chi(\pi), h_c) = \sum_{w \text{ a permutation of } M_c} q^{\mathrm{inv}(\pi,w)}. \tag{3.1}$$

We list here a few properties of χ so that the reader has a feeling of what kind of object it is.

For a Dyck path π denote by π^{op} the reversed Dyck path, i.e., the path obtained by replacing each North step by East step and each East step by North step and reversing the order of steps. Reversing also the order of the components of c in (3.1), we see

Proposition 3.3.

$$\chi(\pi) = \chi(\pi^{op}).$$

Proofs of the following two statements are essentially taken from [HHL05a].

Proposition 3.4.

$$\bar{\omega}\chi(\pi) = (-1)^{|\pi|} q^{-\operatorname{area}(\pi)} \chi(\pi).$$

Proposition 3.5.

$$\chi(\pi)[(q-1)X] = (q-1)^{|\pi|} \sum_{w \in \mathbb{Z}_{>0}^{|\pi|} \text{ no attack}} q^{\operatorname{inv}(\pi,w)} x_w,$$

where "no attack" means that the summation is only over vectors w such that $w_i \neq w_j$ for $(i,j) \in \operatorname{Area}(\pi)$.

Proofs. We follow Chapter 4 of [HHL05a]. For an integer n and a subset $D \subset \{1, \ldots, n-1\}$, Gessel's *quasi-symmetric function* $Q_{n,D}$ in $x = (x_1, x_2, \ldots)$ is given by

$$Q_{n,D}(x) = \sum_{\substack{w_1 \leqslant \cdots \leqslant w_n \\ w_i = w_{i+1} \Rightarrow i \notin D}} x_w.$$

For each sequence $w \in \mathbb{Z}_{>0}^n$, its *standardization* is the unique permutation $\operatorname{Std}(w) \in S_n$ such that

$$w_i < w_j \text{ or } (w_i = w_j \text{ and } i < j) \Leftrightarrow \operatorname{Std}(w)_i < \operatorname{Std}(w)_j.$$

In other words, $\operatorname{Std}(w)$ sorts pairs (w_i, i) in lexicographic order. We notice the following properties:
(3.2)

$$\operatorname{Inv}(\pi, w) = \operatorname{Inv}(\pi, \operatorname{Std}(w)), \qquad \sum_{w : \operatorname{Std}(w) = \sigma} x_w = Q_{n,\operatorname{Des}(\sigma^{-1})}(x) \qquad (\sigma \in S_n),$$

where $\operatorname{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ is the descent set of σ. Thus the sum $\chi(\pi)$ splits as follows:

$$\chi(\pi) = \sum_{\sigma \in S_n} q^{\operatorname{inv}(\pi,\sigma)} Q_{n,\operatorname{Des}(\sigma^{-1})}.$$

Since $\chi(\pi)$ is symmetric by Proposition 3.2, we can apply Proposition 4.2 in [HHL05a]. Let $\mathcal{A}$ be the "super" alphabet

$$\mathcal{A} = \mathbb{Z}_+ \cup \mathbb{Z}_- = \{1, 2, 3, \ldots, \bar{1}, \bar{2}, \bar{3}, \ldots\}$$

consisting of *positive letters* $i \in Z_+$ and *negative letters* $\bar{i}$. Let

$$z_i = x_i \quad (i \in \mathbb{Z}_+), \qquad z_{\bar{i}} = -y_i \quad (\bar{i} \in \mathbb{Z}_-).$$

Then we have the following expression for $X = \sum_i x_i$, $Y = \sum_i y_i$:

$$\chi(\pi)[X - Y] = \sum_{\sigma \in S_n} q^{\operatorname{inv}(\pi,\sigma)} \tilde{Q}_{n,\operatorname{Des}(\sigma^{-1})}(x, y),$$

where

$$\tilde{Q}_{n,D} = \sum_{\substack{w_1 \leqslant \cdots \leqslant w_n \\ w_i = w_{i+1},\, w_i \in \mathbb{Z}_+ \Rightarrow i \notin D \\ w_i = w_{i+1},\, w_i \in \mathbb{Z}_- \Rightarrow i \in D}} z_w,$$

and the summation is over the sequences of elements of $\mathcal{A}$. The statement holds for an arbitrary choice of total ordering on $\mathcal{A}$. We work with the following ordering:

$$1 < \bar{1} < 2 < \bar{2} < \cdots.$$

We extend the definitions of Std, Inv, and inv to sequences of elements of $\mathcal{A}$,

$$w_i < w_j \text{ or } (w_i = w_j \in \mathbb{Z}_+ \text{ and } i < j) \text{ or } (w_i = w_j \in \mathbb{Z}_- \text{ and } i > j)$$

$$\Leftrightarrow \operatorname{Std}(w)_i < \operatorname{Std}(w)_j,$$

$\operatorname{inv}(\pi, w) := \# \operatorname{Inv}(\pi, w),\ \operatorname{Inv}(\pi, w) := \{(i,j) \in \operatorname{Area}(\pi) :\ w_i > w_j \text{ or } w_i = w_j \in \mathbb{Z}_-\}$,

so that the properties (3.2) are satisfied. Therefore, we have

$$\chi(\pi)[X - Y] = \sum_{w \in \mathcal{A}^n} q^{\operatorname{inv}(\pi,w)} z_w. \tag{3.3}$$

Setting $X = 0, Y = -X$, we obtain

$$\chi(\pi)[-X] = (-1)^n \sum_{w \in \mathbb{Z}_{>0}^n} q^{\operatorname{inv}'(\pi,w)} x_w,$$

where $\operatorname{inv}'(\pi, w)$ is the number of nonstrict inversions of w under the path,

$$\operatorname{inv}'(\pi, w) := \# \{(i,j) \in \operatorname{Area}(\pi),\ w_i \geqslant w_j\}.$$

Reversing the order of labels, we have

$$\chi(\pi)[-X] = (-1)^{|\pi|} \sum_{w \in \mathbb{Z}_{>0}^n} q^{\operatorname{area}(\pi) - \operatorname{inv}(\pi,w)} x_w,$$

which implies Proposition 3.4.

To prove Proposition 3.5, we set $X = qX$, $Y = X$ in (3.3). Applying the involution from the proof of Lemma 5.1 in [HHL05a] (flipping the sign of the last label that attacks a label with the same absolute value), we see that the terms for $w \in \mathcal{A}^n$, such that $|w_i| = |w_j|$ for some $(i,j) \in \operatorname{Area}(\pi)$, cancel out. In the remaining terms we have $|w_i| \neq |w_j|$ whenever $(i,j) \in \operatorname{Area}(\pi)$. Therefore the comparison between w_i and w_j depends only on $|w_i|$ and $|w_j|$. So we can first sum over sequences in $\mathbb{Z}_{>0}$ and then over the choices of signs. The latter summation produces an overall factor of $(q-1)^n$, and we obtain Proposition 3.5. ☐

3.2. **Weighted characteristic function.** To study the summand of $D_\alpha(q,t)$ in (2.5) as a function of π, we introduce a more general characteristic function. Given a function $\operatorname{wt} : c(\pi) \to R$ on the set of corners of some Dyck path π of size n, let

$$\chi(\pi, \operatorname{wt}) := \sum_{w \in \mathbb{Z}_{>0}^n} q^{\operatorname{inv}(\pi,w)} \left(\prod_{(i,j) \in c(\pi),\ w_i \leqslant w_j} \operatorname{wt}(i,j) \right) x_w, \tag{3.4}$$

so in particular (2.5) becomes

$$D_\alpha(q,t) = \sum_{\operatorname{touch}'(\pi) = \alpha} t^{\operatorname{bounce}(\pi)} \chi(\pi, 0).$$

For a constant function $\operatorname{wt} = 1$, we recover the simpler characteristic function

$$\chi(\pi, 1) = \chi(\pi). \tag{3.5}$$

It turns out that we can express the weighted characteristic function $\chi(\pi, \operatorname{wt})$ in terms the unweighted one evaluated at different paths. In particular this implies that $\chi(\pi, \operatorname{wt})$ is symmetric too.

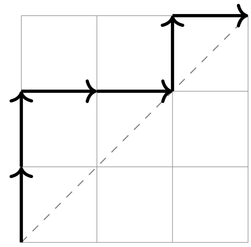

FIGURE 4

Remark 3.6. If π' is the image of π under the bijection from section 2.4, then we have that $\chi(\pi', 0) = F_\pi(X; q)$, where $F_\pi(X; q)$ are the *path symmetric functions* from Haglund's book [Hag08, page 95]. As Haglund explains, these functions are examples of LLT polynomials of vertical strips, using the description of Bylund and Haiman. In fact, $\chi(\pi', 1)$ is also an example of an LLT polynomial, but for a disjoint union of single boxes:

$$\chi(\pi', 1) = \mathrm{LLT}_{[a_n+1]/[a_n],\ldots,[a_1]/[a_1]}(X; q),$$

where $(a_1, \ldots, a_n) = a(\pi)$ is the area sequence.

Proposition 3.7. *We have that* $\chi(\pi, \mathrm{wt})$ *is symmetric in the* x_i *variables, and so it defines an element of* $\mathrm{Sym}[X]$.

Proof. Let π be a Dyck path, and let $(i, j) \in c(\pi)$ be one of its corners. We denote by wt_1 the weight on π which is obtained from wt by setting the weight of (i, j) to 1. Let π' be the Dyck path obtained from π by turning the corner inside out, in other words the Dyck path of smallest area which is both above π and above (i, j). Let wt_2 be the weight on π' which coincides with wt on all corners of π' which are also corners of π and is 1 on other corners. We claim that

$$\chi(\pi, \mathrm{wt}) = \frac{q\mathrm{wt}(i, j) - 1}{q - 1}\chi(\pi, \mathrm{wt}_1) + \frac{1 - \mathrm{wt}(i, j)}{q - 1}\chi(\pi', \mathrm{wt}_2). \tag{3.6}$$

To see this, notice that if we group the terms on the right-hand side, then both sides may be written as a sum over vectors $w \in \mathbb{Z}_{>0}$. Split both sums according to terms in which $w_i > w_j$, resulting in an additional factor of q, or $w_i \leqslant w_j$, resulting in an additional weight factor. It is easy to check that both sums agree on both the left and right sides.

The result now follows because we may recursively express any $\chi(\pi, \mathrm{wt})$ in terms of $\chi(\pi)$, which we have already remarked is symmetric. □

Example 3.8. In particular, we can use this to extract $\chi(\pi, 0)$ from $\chi(\pi', 1)$ for all π'. If $S \subset c(\pi)$ is any subset of the set of corners, let $\pi_S \in \mathbb{D}$ denote the path obtained by flipping the corners that are in S. Then equation (3.6) implies that

$$\chi(\pi, 0) = (1 - q)^{-|c(\pi)|} \sum_{S \subset c(\pi)} (-1)^{|S|}\chi(\pi_S, 1). \tag{3.7}$$

For instance, let π be the Dyck path in Figure 4. Then setting $x_i = 0$ for $i > 3$ reduces formula (3.4) to a finite sum over 27 terms, from which we can deduce that

$$\chi(\pi) = m_3 + (2 + q)m_{21} + (3 + 3q)m_{111} = s_3 + (1 + q)s_{21} + qs_{111}.$$

Similarly, if $\pi' = \pi_{\{(1,2)\}}$, we have

$$\chi(\pi') = s_3 + 2qs_{21} + q^2 s_{111}.$$

By formula (3.7) we obtain

$$\chi(\pi, 0) = (1-q)^{-1}\left(\chi(\pi) - \chi(\pi')\right) = s_{21} + qs_{111}.$$

Example 3.9. We can check that the Dyck path from Example 3.8 is the unique one satisfying $\mathrm{touch}'(\pi) = (1,2)$ and that $\mathrm{bounce}(\pi) = 1$. Therefore, using the calculation that followed, we have that

$$D_{(2,1)}(q,t) = t\chi(\pi, 0) = ts_{21} + qts_{111},$$

which can be seen to agree with $\nabla C_1 C_2(1)$.

Example 3.10. Though we will not need it, this weighted characteristic function can be used to describe an interesting reformulation of the formula for the modified Macdonald polynomial given in [HHL05a]. Let $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_l)$ be a partition of size n. Let us list the cells of μ in the reading order,

$$(l,1), (l,2), \ldots, (l,\mu_l), (l-1,1), \ldots, (l-1,\mu_{l-1}), \ldots, (1,1), \ldots, (1,\mu_1).$$

Denote the mth cell in this list by (i_m, j_m).

We say that a cell (i,j) attacks all cells which are after (i,j) and before $(i-1,j)$. Thus (i,j) attacks precisely $\mu_i - 1$ following cells if $i > 1$ and all following cells if $i = 1$. Next construct a Dyck path π_μ of length n in such a way that (m_1, m_2) with $m_1 < m_2$ is under the path if and only if (i_{m_1}, j_{m_1}) attacks (i_{m_2}, j_{m_2}). More specifically, the path begins with μ_l North steps, then it has μ_l pairs, of steps East-North, then $\mu_{l-1} - \mu_l$ North steps followed by μ_{l-1} East-North pairs and so on until we reach the point $(n-\mu_1, n)$. We complete the path by performing μ_1 East steps.

Note that the corners of π_μ precisely correspond to the pairs of cells (i,j), $(i-1,j)$. We set the weight of such a corner to $q^{\mathrm{arm}(i,j)} t^{-1-\mathrm{leg}(i,j)}$ and denote the weight function thus obtained by wt_μ. Note that in our convention for $\chi(\pi, \mathrm{wt})$ we should count noninversions in the corners, while in [HHL05a] they count "descents", which translates to counting inversions in the corners. Taking this into account, we obtain a translation of their Theorem 2.2:

$$\tilde{H}_\mu = q^{-n(\mu') + \binom{\mu_1}{2}} t^{n(\mu)} \chi(\pi_\mu, \mathrm{wt}_\mu).$$

4. Raising and lowering operators

Now let $\mathbb{D}_{k,n}$ be the set of Dyck paths from $(0,k)$ to (n,n), which we will call partial Dyck paths, and let $\mathbb{D}_k$ be their union over all n. For $\pi \in \mathbb{D}_{k,n}$, let $|\pi| = n-k$ denote the number of North steps. Unlike $\mathbb{D}$, the union of the sets $\mathbb{D}_k$ over all k is closed under the operation of adding a North or East step to the beginning of the path, and any Dyck path may be created in such a way starting with the empty path in $\mathbb{D}_0$. This is the set of paths for which we will develop a recursion. More precisely, we will define an extension of the function χ to a map from $\mathbb{D}_k$ to a new vector space V_k, and prove that certain operators on these vector spaces commute with adding North and East steps.

Given a polynomial P depending on variables u, v, define

$$(\Delta_{uv}P)(u,v) = \frac{(q-1)vP(u,v) + (v-qu)P(v,u)}{v-u},$$
$$(\Delta^*_{uv}P)(u,v) = \frac{(q-1)uP(u,v) + (v-qu)P(v,u)}{v-u}.$$

We can easily check that $\Delta^*_{uv} = q\Delta^{-1}_{uv}$. We can recognize these operators as a simple modification of Demazure–Lusztig operators. The following can be checked by direct computation:

Proposition 4.1. *We have the following relations:*

$$(\Delta_{uv} - q)(\Delta_{uv} + 1) = 0, \qquad (\Delta^*_{uv} - 1)(\Delta^*_{uv} + q) = 0,$$
$$\Delta_{uv}\Delta_{vw}\Delta_{uv} = \Delta_{vw}\Delta_{uv}\Delta_{vw}, \qquad \Delta^*_{uv}\Delta^*_{vw}\Delta^*_{uv} = \Delta^*_{vw}\Delta^*_{uv}\Delta^*_{vw}.$$

Definition 4.2. Let $V_k = \mathrm{Sym}[X] \otimes \mathbb{Q}[y_1, y_2, \ldots, y_k]$, and let

$$T_i = \Delta^*_{y_i y_{i+1}} : V_k \to V_k, \quad i = 1, \ldots, k-1.$$

Define operators

$$d_+ : V_k \to V_{k+1}, \quad d_- : V_k \to V_{k-1}$$

by

$$(d_+F)[X] = T_1T_2 \cdots T_k \left(F[X + (q-1)y_{k+1}]\right), \tag{4.1}$$

and

$$(d_-F)[X] = -F[X - (q-1)y_k] \,\mathrm{Exp}[-y_k^{-1}X]\big|_{y_k^{-1}} \tag{4.2}$$

for $F \in V_k$.

Remark 4.3. Note that the operator d_- is related to the B_i operators,

$$d_-(y_k^i F) = -B_{i+1}F$$

for $F \in V_k$ which do not depend on y_k.

We now claim the following theorem:

Theorem 4.4. *For any Dyck path π of size n, let $\epsilon_1 \cdots \epsilon_{2n}$ denote the corresponding sequence of plus and minus symbols where a plus denotes an East step, and a minus denotes a North step reading π from bottom left to top right. Then*

$$\chi(\pi) = d_{\epsilon_1} \cdots d_{\epsilon_{2n}}(1)$$

as an element of $V_0 = \mathrm{Sym}[X]$.

Example 4.5. Let π be the Dyck path from Example 3.8. We have that

$$\begin{aligned} d_-d_-d_+d_+d_-d_+(1) &= d_-d_-d_+d_+d_-(1) = d_-d_-d_+d_+(s_1) \\ &= d_-d_-d_+\left(s_1 + (q-1)y_1\right) = d_-d_-\left(s_1 + (q-1)(y_1+y_2)\right) \\ &= d_-\left(s_2 + s_{11} + (q-1)s_1y_1\right) = s_3 + (1+q)s_{21} + qs_{111}, \end{aligned}$$

which agrees with the value calculated for $\chi(\pi)$.

Combining this result with equation (3.7) implies the following:

Corollary 4.6. *The following procedure computes $\chi(\pi, 0)$: start with $1 \in \mathrm{Sym}[X] = V_0$, follow the path from right to left applying $\frac{1}{q-1}[d_-, d_+]$ for each corner of w, and d_- (d_+) for each North (resp. East) step that is not a side of a corner.*

4.1. **Rank experiment.** The proof of Theorem 4.4 will be divided into several parts. However, before we proceed to the proof of Theorem 4.4, we would like to explain why we expected such a result to hold, and how we obtained it. In fact, the definition of χ_k from equation (4.5) in the proof below actually came first, and was discovered using computer experimentation, as we now explain.

First note that the number of Dyck paths of length n is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ which grows exponentially with n. The dimension of the degree n part of $\mathrm{Sym}[X]$ is the number of partitions of size n, which grows sub-exponentially. For instance for $n = 3$, we have five Dyck paths but only three partitions. Thus there must be linear dependences between different $\chi(\pi)$.

Now fix a partial Dyck path $\pi_1 \in \mathbb{D}_{k,n}$. For each partial Dyck path $\pi_2 \in \mathbb{D}_{k,n'}$, we can reflect π_2 and concatenate it with π_1 to obtain a full Dyck path $\pi_2^{op}\pi_1$ of length $n + n' - k$. We may then consider its character $\chi(\pi_2^{op}\pi_1)$. We keep n, π_1 fixed and vary n', π_2, thus obtaining a map $\varphi_{\pi_1} : \mathbb{D}_k \to \mathrm{Sym}[X]$. The map $\pi_1 \to \varphi_{\pi_1}$ is a map from $\mathbb{D}_k$ to the vector space of maps from $\mathbb{D}_k$ to $\mathrm{Sym}[X]$, which is very high dimensional, because both the set $\mathbb{D}_k$ is infinite and $\mathrm{Sym}[X]$ is infinite dimensional. A priori, it could be the case that the images of the elements of $\mathbb{D}_{k,n}$ in $\mathrm{Maps}(\mathbb{D}_k, \mathrm{Sym}[X])$ are linearly independent. However, computer experiments convinced us that it is not the case, and that there should be a vector space $V_{k,n}$ whose dimension is generally smaller than the size of $\mathbb{D}_{k,n}$. In fact, by restricting n' to be bounded by some arbitrary but large enough cut-off value, we were able to predict that the dimension of this space stabilizes to a very simple formula, which is the dimension of $V_{k,n}$, the degree $n - k$ component of V_k as it is defined above.

We therefore predicted the existence of a commutative diagram

$$\begin{array}{ccc} \mathbb{D}_{k,n} & \xrightarrow{\chi_{k,n}} & V_{k,n} \\ \big\downarrow{\scriptstyle\varphi} & \swarrow & \\ \mathrm{Maps}(\mathbb{D}_k, \mathrm{Sym}[X]) & & \end{array}$$

for some map $\chi_{k,n}$, whose image spans all of $V_{k,n}$. This ultimately led to the guess of the formula for χ_k in (4.5) as the correct extension of $\chi(\pi, 1)$. It is not at all trivial to deduce this formula from the dimension of $V_{k,n}$, and indeed, some substantial guesswork was required. However, the validity of any particular guess $\chi_{k,n}$ can be determined experimentally, by testing if its kernel in the $\mathbb{C}(q)$-span of $\mathbb{D}_{k,n}$ agrees with the kernel of φ. Clearly the existence of a testable criterion such as this makes the problem of determining $\chi_{k,n}$ experimentally much more reasonable.

Once the definition of $\chi_{k,n}$ was conjectured, finding formulas for $d_\pm$ that satisfy (4.6) turned out to be relatively straightforward.

4.2. **Characteristic functions of partial Dyck paths.** The following definition is motivated by Proposition 3.5. Let $\pi \in \mathbb{D}_{k,n}$. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in \mathbb{Z}_{>0}$ be a tuple of distinct numbers. The elements of $\mathrm{Im}(\sigma) \subset \mathbb{Z}_{>0}$ will be called *special*. Let

$$U_{\pi,\sigma} = \{w \in \mathbb{Z}_{>0}^n : w_i = \sigma_i \text{ for } i \leqslant k,\ w_i \neq w_j \text{ for } (i,j) \in \mathrm{Area}(\pi)\}.$$

The second condition on w is the "no attack" condition as before. The first condition says that we put the special labels in the positions $1, 2, \dots, k$ as prescribed by

σ. Let

$$\chi'_\sigma(\pi) = \sum_{w \in U_{\pi,\sigma}} q^{\mathrm{inv}(\pi,w)} z_w. \tag{4.3}$$

Here we use variables $z_1, z_2, \ldots$.

Suppose σ is a permutation, i.e., $\sigma_i \leqslant k$ for all i. Set $z_i = y_i$ for $i \leqslant k$ and $z_i = x_{i-k}$ for $i > k$. We denote

$$\chi'_k(\pi) = \chi'_{(1,2,\ldots,k)}(\pi).$$

Let us group the summands in (4.3) according to the positions of special labels. More precisely, let $S \subset \{1, \ldots, n\}$ such that $\{1, \ldots, k\} \subset S$ and $w^S : S \to \{1, \ldots, k\}$ such that $w_i^S = \sigma_i$ for $i = 1, 2, \ldots, k$ and $w_i \neq w_j$ for $i, j \in S$, $(i, j) \in \mathrm{Area}(\pi)$. Set

$$U_{\pi,\sigma}^{S,w^S} := \left\{ w \in U_{\pi,\sigma} : w_i = w_i^S \text{ for } i \in S,\ w_i > k \text{ for } i \notin S \right\},$$
$$\Sigma_{\pi,\sigma}^{S,w^S} := \sum_{w \in U_{\pi,\sigma}^{S,w^S}} q^{\mathrm{inv}(\pi,w)} x_w, \qquad \chi'_\sigma(\pi) = \sum_{S,w^S} \Sigma_{\pi,\sigma}^{S,w^S}.$$

Let $m_1 < m_2 < \cdots < m_r$ be all the positions not in S. Let π_S be the unique Dyck path of length r such that $(i, j) \in \mathrm{Area}(\pi_S)$ if and only if $m_i, m_j \in \mathrm{Area}(\pi)$. We have

$$\Sigma_{\pi,\sigma}^{S,w^S} = q^A \prod_{i \in S} y_{w_i} \sum_{w \in \mathbb{Z}_{>0}^r \text{no attack}} q^{\mathrm{inv}(\pi_S,w)} x_w,$$

where

$$A = \#\{(i, j) \in \mathrm{Area}(\pi) : (i \in S, j \in S, w_i^S > w_j^S) \text{ or } (i \notin S, j \in S)\}.$$

By Proposition 3.5 we have

$$\Sigma_{\pi,\sigma}^{S,w^S} = q^A (q-1)^{|S|-n} \chi(\pi^S) \left[(q-1)X\right] \prod_{i \in S} y_{w_i}. \tag{4.4}$$

In particular $\chi_\sigma(\pi)$ is a symmetric function in $x_1, x_2, \ldots$, and it makes sense to define

$$\chi_\sigma(\pi)[X] := \frac{1}{y_1 y_2 \cdots y_k} (q-1)^{|\pi|} \chi'_\sigma(\pi) \left[\frac{X}{q-1}\right] \in V_k, \quad \chi_k(\pi) := \chi_{\mathrm{Id}_k}(\pi). \tag{4.5}$$

Remark 4.7. The identity (4.4) also implies that the coefficients of $\chi_\sigma(\pi)[X]$ are polynomials in q, and it gives a way to express χ_σ in terms of the characteristic functions $\chi(\pi_S)$ for all S.

For $k = 0$, we recover $\chi(\pi)$:

$$\chi_0(\pi) = \chi(\pi) \qquad (\pi \in \mathbb{D}_0 = \mathbb{D}).$$

Thus, it suffices to prove that

$$\chi_{k+1}(E\pi) = d_+ \chi_k(\pi), \quad \chi_{k-1}(N\pi) = d_- \chi_k(\pi) \quad (\pi \in \mathbb{D}_k). \tag{4.6}$$

4.3. **Raising operator.** We begin with the first case. Let $\pi \in \mathbb{D}_{k,n}$ so that $E\pi \in \mathbb{D}_{k+1,n+1}$, and we need to express $\chi_{k+1}(E\pi)$ in terms of $\chi_k(\pi)$. Let σ be the following sequence:

$$\sigma = (k+1, 1, 2, \ldots, k).$$

Then we have a bijection $f : U_{\pi,\mathrm{Id}_k} \to U_{E\pi,\sigma}$ obtained by sending

$$w = (1, 2, \ldots, k, w_{k+1}, \ldots, w_n)$$

to

$$f(w) := (k+1, 1, 2, \ldots, k, w_{k+1}, \ldots, w_n).$$

This is possible because 1 does not attack $k+1$ in $E\pi$. We clearly have $\mathrm{inv}(E\pi, f(w)) = \mathrm{inv}(\pi, w) + k$, which implies

$$\chi'_\sigma(E\pi) = z_{k+1} q^k \chi'_k(\pi),$$

where both sides are written in terms of the variables z_i. When we pass to the variables x_i, y_i on the left, we have

$$(z_1, z_2, \ldots) = (y_1, y_2, \ldots, y_{k+1}, x_1, x_2, \ldots),$$

but on the right we have

$$(z_1, z_2, \ldots) = (y_1, y_2, \ldots, y_k, x_1, x_2, \ldots).$$

Thus we need to perform the substitution $X = y_{k+1} + X$,

$$\chi'_\sigma(E\pi)[X] = y_{k+1} q^k \chi'_k(\pi)[X + y_{k+1}].$$

Performing the transformation (4.5), we obtain

$$\chi_\sigma(E\pi) = q^k \chi_k(\pi) \left[X + (q-1) y_{k+1}\right]. \tag{4.7}$$

To finish the computation, we need to relate $\chi_{k+1} = \chi_{\mathrm{Id}_{k+1}}$ and χ_σ. We first note that σ can be obtained from Id_{k+1} by successively swapping neighboring labels. Let $\sigma^{(1)} = \mathrm{Id}_{k+1}$ and

$$\sigma^{(i)} = (i, 1, 2, \ldots, i-1, i+1, \ldots, k+1) \qquad (i = 2, 3, \ldots, k+1)$$

so that $\sigma = \sigma^{(k+1)}$. It is clear that $\sigma^{(i+1)}$ can be obtained from $\sigma^{(i)}$ by interchanging the labels i and $i+1$.

We show below (Proposition 4.8) that this kind of interchange is controlled by the operator $\Delta_{y_i, y_{i+1}}$:

$$\chi_{\sigma^{(i+1)}}(E\pi) = \Delta_{y_i, y_{i+1}} \chi_{\sigma^{(i)}}(E\pi). \tag{4.8}$$

This implies

$$\chi_\sigma(E\pi) = \Delta_{y_{k-1}, y_k} \cdots \Delta_{y_1, y_2} \chi_{k+1}(E\pi).$$

When we insert this equation into (4.7), we arrive at

$$\chi_{k+1}(E\pi) = T_1 \cdots T_k \left(\chi_k(\pi) \left[X + (q-1) y_{k+1}\right]\right) = d_+ \chi_k(\pi).$$

4.4. **Swapping operators.**

Proposition 4.8. *For any $w \in \mathbb{D}_k$, σ as above and m special, suppose that $m+1$ is not special or $\sigma^{-1}(m) < \sigma^{-1}(m+1)$. Then we have*

$$\chi'_{\tau_m \sigma}(w) = \Delta_{z_m, z_{m+1}} \chi'_\sigma,$$

where τ_m is the transposition $m \leftrightarrow m+1$, $(\tau_m \sigma)_i = \tau_m(\sigma_i)$ for $i = 1, \ldots, k$.

Proof. We decompose both sides as follows. For any $w \in U_{\pi,\sigma}$, let $S(w)$ be the set of indices j where $w_j \in \{m, m+1\}$. For $w, w' \in U_{\pi,\sigma}$ write $w \sim w'$ if $S(w) = S(w')$ and $w_i = w'_i$ for all $i \notin S(w)$. This defines an equivalence relation on $U_{\pi,\sigma}$. The sum (4.3) is then decomposed as follows:

$$\chi'_\sigma(\pi) = \sum_{[w] \in U_{\pi,\sigma}/\sim} q^{\mathrm{inv}_1(\pi,w)} \prod_{i \notin S} z_{w_i} \sum_{w' \sim w} a(w'), \tag{4.9}$$

where

$$\mathrm{inv}_1(\pi, w) = \#\{(i,j) \in \mathrm{Area}(\pi) : w_i > w_j, i \notin S(w) \text{ or } j \notin S(w)\},$$

which does not depend on the choice of a representative w in the equivalence class $[w]$, and

$$a(w) = q^{\mathrm{inv}_2(\pi,w)} \prod_{i \in S} z_{w_i},$$

$$\mathrm{inv}_2(\pi, w) = \#\{(i,j) \in \mathrm{Area}(\pi) : w_i > w_j, i, j \in S(w)\}.$$

Let $f : U_{\pi,\sigma} \to U_{\pi,\tau_m\sigma}$ be the bijection defined by $f(w)_i = \tau_m(w_i)$. This bijection respects the equivalence relation $\sim$, and we have $S(f(w)) = S(w)$. Moreover, we have $\mathrm{inv}_1(\pi, w) = \mathrm{inv}_1(\pi, f(w))$. We now make the stronger claim that for any $w \in U_{\pi,\sigma}$

$$\sum_{w' \sim f(w)} a(w') = \Delta_{z_m, z_{m+1}} \sum_{w' \sim w} a(w'), \tag{4.10}$$

which would imply the statement by summing over all equivalence classes.

For each $w \in U_{\pi,\sigma}$, the set $S(w)$ is decomposed into a disjoint union of *runs*, i.e., subsets

$$R = \{j_1, \ldots, j_l\} \subset \{1, \ldots, n\}, \quad j_1 < \cdots < j_l$$

such that in each run j_a attacks j_{a+1} for all a and elements of different runs do not attack each other. Because of the nonattacking condition, the labels w_{j_a} must alternate between $m, m+1$, and j_a does not attack j_{a+2}. Thus to fix w in each equivalence class, it is enough to fix w_{j_1} for each run. Suppose the runs of $S(w)$ have lengths $l_1, l_2, \ldots, l_r$ and the first values of w in each run are $c_1, c_2, \ldots, c_r$, respectively.

With this information $a(w)$ can be computed as follows:

$$a(w) = \prod_{i=1}^{r} a(l_i, c_i),$$

where

$$a(l,c) := \begin{cases} q^{l'-1} z_m^{l'} z_{m+1}^{l'} & l = 2l', c = m, \\ q^{l'} z_m^{l'+1} z_{m+1}^{l'} & l = 2l'+1, c = m, \\ q^{l'} z_m^{l'} z_{m+1}^{l'} & l = 2l', c = m+1, \\ q^{l'} z_m^{l'} z_{m+1}^{l'+1} & l = 2l'+1, c = m+1. \end{cases}$$

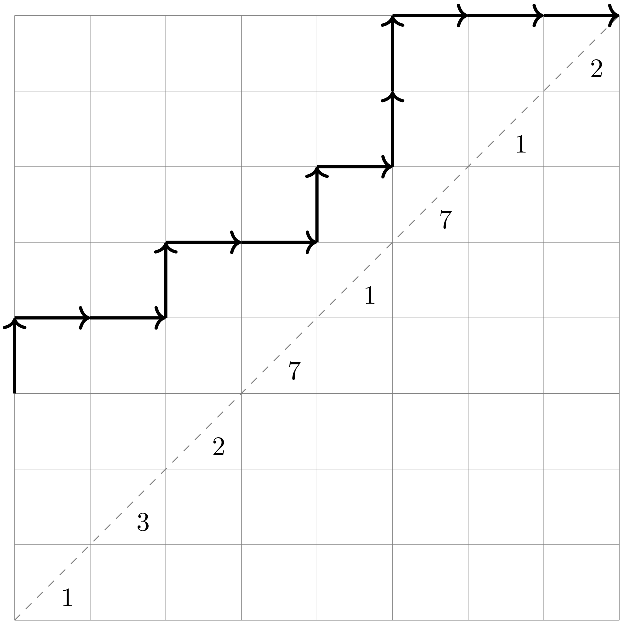

Figure 5

For instance, let $k = 3$, and let π be the Dyck path in Figure 5, and let

$$w = (1, 3, 2, 7, 1, 7, 1, 2) \in U_{\pi,(132)}.$$

Let $m = 1$. Then we have $S(w) = \{1, 3, 5, 7, 8\}$, which decomposes into two runs $\{1, 3, 5\}$ and $\{7, 8\}$. So we have $r = 2$, $(l_1, l_2) = (3, 2)$, $(c_1, c_2) = (1, 1)$ and we obtain

$$a(w) = a(3, 1)a(2, 1) = qz_1^2 z_2 z_1 z_2 = qz_1^3 z_2^2.$$

Note that by the assumption on σ, we have $c_1 = m$, while c_i can take arbitrary values $\{m, m + 1\}$ for $i > 1$. This implies

$$\sum_{w' \sim w} a(w') = a(l_1, m) \prod_{i=2}^{r} (a(l_i, m) + a(l_i, m + 1)).$$

On the other hand we have

$$\sum_{w' \sim f(w)} a(w') = \sum_{w' \sim w} a(f(w')) = a(l_1, m + 1) \prod_{i=2}^{r} (a(l_i, m) + a(l_i, m + 1)).$$

Now notice that for all l the sum $a(l, m) + a(l, m + 1)$ is symmetric in z_m, z_{m+1}. The operator $\Delta_{z_m, z_{m+1}}$ commutes with multiplication by symmetric functions and satisfies

$$\Delta_{z_m, z_{m+1}}(a(l, m)) = a(l, m + 1).$$

This establishes (4.10) and the proof is complete. □

Remark 4.9. The arguments used in the proof can be used to show that in the case when $m, m + 1$ are both not special, the function $\chi'_\sigma(\pi)$ is symmetric in z_m, z_{m+1}. In particular, we can obtain a direct proof of the fact that χ'_σ is symmetric in the variables $z_m, z_{m+1}, z_{m+2}, \ldots$ for $i = \max(\sigma) + 1$, without use of Proposition 3.5.

4.5. **Lowering operator.** We now turn to the remaining identity $\chi_{k-1}(N\pi) = d_-\chi_k(\pi)$. Assume $\pi \in \mathbb{D}_{k,n}$, so that $N\pi \in \mathbb{D}_{k-1,n}$. We observe that

$$\chi'_{k-1}(N\pi)[X + y_k] = \sum_{r\geqslant 0} \chi'_{k,r}(\pi)[X],$$

where

$$\chi'_{k,r}(\pi) = \chi'_\sigma(\pi), \quad \sigma = (1, 2, \ldots, k-1, k+r),$$

and to get to the second equality, we have summed over all possible values of $r = w_k - k$ that do not result in an attack. It is convenient to set $x_0 = y_k$. Using Proposition 4.8, we can characterize $\chi'_{k,r}(\pi)$ by

$$\chi'_{k,0}(\pi) = \chi'_k(\pi), \quad \chi'_{k,r+1}(\pi) = \Delta_{x_r,x_{r+1}}\chi'_{k,r}(\pi) \quad (r \geqslant 0). \tag{4.11}$$

Now notice that there is a unique expansion

$$\chi'_k(\pi)[X] = \sum_{j\geqslant 1} y_k^j g_j(\pi)[X + y_k], \quad g_j(\pi) \in V_{k-1}.$$

The advantage over the more obvious expansion in powers of y_k is that each coefficient $g_j[X + y_k]$ is symmetric in the variables $y_k, x_1, \ldots$. As a result, we have that

$$\chi'_{k,r}(\pi)[X] = \Delta_{x_{r-1},x_r} \cdots \Delta_{x_2,x_1}\Delta_{y_k,x_1} \sum_{i\geqslant 1} y_k^i g_i(\pi)[X + y_k] = \sum_{i\geqslant 1} f_{i,r} g_i(\pi)[X + y_k],$$

where

$$f_{i,r} = \Delta_{x_{r-1},x_r} \cdots \Delta_{x_1,x_2}\Delta_{y_k,x_1}(y_k^i) \quad (i \geqslant 1, r \geqslant 0).$$

The extra symmetry in the y_k variable is used to pass Δ_{y_k,x_1} by multiplication by $g_i(\pi)[X + y_k]$.

Now we need an explicit formula for $f_{i,r}$:

Proposition 4.10. *Denote* $X_r = y_k + x_1 + \cdots + x_r$, $X_{-1} = 0$. *We have*

$$f_{i,r} = \frac{h_i[(1-q)X_r] - h_i[(1-q)X_{r-1}]}{1-q}.$$

Proof. Denote the right-hand side by $f'_{i,r}$. The proof goes by induction on r. For $r = 0$ both sides are equal to y_k^i. Thus it is enough to show that

$$\Delta_{x_r,x_{r+1}}(f'_{i,r}) = f'_{i,r+1}. \tag{4.12}$$

Use $X_r = X_{r-1} + x_r$ to write $f'_{i,r}$ as

$$f'_{i,r} = \sum_{j=1}^{i} x_r^j h_{i-j}[(1-q)X_{r-1}] = x_r h_{i-1}[(1-q)X_{r-1} + x_r]. \tag{4.13}$$

Now X_{r-1} does not contain the variables x_r, x_{r+1}, so we have

$$\Delta_{x_r,x_{r+1}}(f'_{i,r}) = \sum_{j=1}^{i} h_{i-j}[(1-q)X_{r-1}]\Delta_{x_r,x_{r+1}} x_r^j.$$

Using the formula

$$\Delta_{x_r,x_{r+1}} x_r^j = x_{r+1} h_{j-1}[(1-q)x_r + x_{r+1}],$$

which is straightforward to check, we can evaluate

$$\begin{aligned}\Delta_{x_r x_{r+1}} f'_{i,r} &= x_{r+1} \sum_{j=1}^{i} h_{j-1}[(1-q)x_r + x_{r+1}] h_{i-j}[(1-q)X_{r-1}] \\ &= x_{r+1} h_{i-1}[(1-q)X_r + x_{r+1}],\end{aligned}$$

which matches $f'_{i,r+1}$ by (4.13). □

Now, if we sum over all r, we obtain

$$\sum_{r\geqslant 0} f_{i,r} = (1-q)^{-1} h_i \left[(1-q)(X+y_k)\right]. \tag{4.14}$$

Thus

$$\chi'_{k-1}(N\pi)[X+y_k] = (1-q)^{-1} \sum_{i\geqslant 1} h_i[(1-q)(X+y_k)] g_i(\pi)[X+y_k].$$

This implies

$$\chi_{k-1}(N\pi)[X] = -\frac{(q-1)^{n-k}}{y_1\cdots y_{k-1}} \sum_{i\geqslant 1} h_i[-X] g_i(\pi) \left[\frac{X}{q-1}\right]. \tag{4.15}$$

On the other hand $g_i(\pi)$ were defined in such a way that

$$\chi_k(\pi)[(q-1)X] = \frac{(q-1)^{n-k}}{y_1\cdots y_k} \sum_{i\geqslant 1} y_k^i g_i(\pi)[X+y_k].$$

Substituting $\frac{1}{q-1}X - y_k$ for X gives

$$\chi_k(\pi)[X-(q-1)y_k] = \frac{(q-1)^{n-k}}{y_1\cdots y_k} \sum_{i\geqslant 1} y_k^i g_i(\pi) \left[\frac{X}{q-1}\right]. \tag{4.16}$$

Comparing (4.15) and (4.16), we obtain

$$\chi_{k-1}(N\pi)[X] = \sum_{i\geqslant 0} -h_{i+1}[-X] \left(\chi_k(\pi)[X-(q-1)y_k]|_{y_k^i}\right).$$

This can be seen to coincide with $d_-\chi_k(\pi)$, establishing the second case of (4.6). Thus the proof of Theorem 4.4 is complete.

4.6. **Main recursion.** We now show how to express all of $D_\alpha(q,t)$ using our operators:

Theorem 4.11. *If α is a composition of length l, we have*

$$D_\alpha(q,t) = d_-^l(N_\alpha),$$

where $N_\alpha \in V_l$ is defined by the recursion relations

$$N_\varnothing = 1, \quad N_{[1,\alpha]} = d_+ N_\alpha, \quad N_{a\alpha} = \frac{t^{a-1}}{q-1}[d_-,d_+] \sum_{\beta\models a-1} d_-^{l(\beta)-1} N_{\alpha\beta}. \tag{4.17}$$

Proof. For any $k>0$, let $\mathbb{D}_k^0 \subset \mathbb{D}_k$ denote the subset of partial Dyck paths that begin with an East step. For $k=0$, let $\mathbb{D}_0^0 = \{\varnothing\}$. Define functions $\chi^0 : \mathbb{D}_k^0 \to V_k$ by

$$\chi^0(\varnothing) = 1, \quad \chi^0(EN^i\pi) = \frac{1}{q-1}[d_-,d_+] d_-^{i-1} \chi^0(\pi),$$
$$\chi^0(E\pi) = d_+ \chi^0(\pi).$$

Given a composition α of length l, let

$$\mathbb{D}_\alpha = \left\{\pi \in \mathbb{D} : \text{touch}'(\pi) = \alpha\right\}.$$

By the definition of touch$'$, every element of $\mathbb{D}_\alpha$ is of the form $\pi = N^l\tilde{\pi}$ for a unique element $\tilde{\pi} \in \mathbb{D}_l^0$ so that by Corollary 4.6 we have

$$\chi(\pi, 0) = d_-^l \chi_l^0(\tilde{\pi}).$$

Let

$$N'_\alpha = \sum_{\pi \in \mathbb{D}_\alpha} t^{\text{bounce}(\pi)} \chi_l^0(\tilde{\pi}) \in V_l,$$

so that $D_\alpha(q,t) = d_-^l(N'_\alpha)$. It suffices to show that N'_α satisfies the relations (4.17), and so agrees with N_α.

For a composition α of length l and $0 \leqslant r \leqslant l$, we have a map $\gamma_{\alpha,r} : \mathbb{D}_\alpha \to \mathbb{D}$ as follows: $\gamma_{\alpha,r}(\pi) = N^{r+1}EN^{l-r}\tilde{\pi}$. Clearly $|\gamma_{\alpha,r}(\pi)| = |\pi| + 1$. From the definition of touch$'$, we see the following relation:

$$\text{bounce}(\gamma_{\alpha,r}(\pi)) = \text{bounce}(\pi) + \sum_{i>r} \alpha_i.$$

Next we compute touch$'(\gamma_{\alpha,r}(\pi))$. For $0 \leqslant i \leqslant r$, we have

$$\text{bounce}(N^{i+2}EN^{r-i}EN^{l-r}\tilde{\pi}) = \text{bounce}(N^{i+1}EN^{l-i}\tilde{\pi}).$$

This implies

$$\text{touch}'(\gamma_{\alpha,r}(\pi)) = \left(1 + \sum_{i>r} \alpha_i,\ \alpha_1,\ \alpha_2, \dots,\ \alpha_r\right),$$

so in particular touch$'(\gamma_{\alpha,r}(\pi))$ depends only on α and r.

Since every nonempty Dyck path can be obtained as $\gamma_{\alpha,r}(\pi)$ in a unique way, we obtain for every composition α of length r,

$$\mathbb{D}_{a\alpha} = \bigsqcup_{\beta \models a-1} \gamma_{\alpha\beta,r}(\mathbb{D}_{\alpha\beta}).$$

It is not hard to see that this identity precisely translates to the relations (4.17) for N'_α. ☐

Example 4.12. Using Theorem 4.11, we find that

$$\begin{aligned} N_{31} &= \frac{t^3}{(q-1)^2}\left(d_{-++-++} - d_{-+++-+} - d_{+-+-++} + d_{+-++-+}\right) \\ &\quad + [\frac{t^2}{q-1}\left(d_{-+-+++} - d_{+--+++}\right) = qt^3y_1^2 - qt^2y_1e_1 \in V_2, \end{aligned}$$

where $d_{\epsilon_1\cdots\epsilon_n} = d_{\epsilon_1}\cdots d_{\epsilon_n}(1)$. We may then check that

$$d_-^2 N_{31} = qt^3B_3B_1(1) + qt^2B_2B_1B_1(1) = \nabla C_3C_1(1).$$

Example 4.13. Let $\alpha = (a_1, \dots, a_k)$ be a composition of norm n and length k, and consider the polynomials in q,t given by

$$D_\alpha^{(-)}(q,t) = \langle D_\alpha(q,t), e_n\rangle = (-1)^n D_\alpha(q,t)[-1],$$

which encode the Catalan case of the compositional shuffle conjecture. This case of the conjecture was proved in [GXZ12], using recursions (Proposition 3.12) that are similar to those defining N_α (4.17), and it is natural to ask if they follow as a special case. Indeed, this can be achieved with help of the following proposition:

Proposition 4.14. *Let π be Dyck path of size N ending with k East steps, and let $(\epsilon_1, \ldots, \epsilon_{2N-k}, +^k)$ denote the corresponding sequence of plus and minus symbols, as in Theorem 4.4. Then for $f \in V_k$, we have*

$$\left(d_{\epsilon_1} \cdots d_{\epsilon_{2N-k}}(f)\right)[-1] = (-1)^N q^{\operatorname{area}(\pi)} f[-q^k]\big|_{y_i=q^{i-1}}. \tag{4.18}$$

Proof. We prove this by induction on $2N-k$, the number of $d_\pm$ symbols.

First, suppose the final step ϵ_{2N-k} is a plus. Let ρ_k denote the homomorphism $f \mapsto f[-q^k]|_{y_i=q^{i-1}}$ on the right-hand side of (4.18). Using the definition (4.1) of d_+, and the observation that $\rho_k T_i = \rho_k$, we have that

$$\rho_{k+1}(d_+ f) = \rho_{k+1}(f[X + (q-1)y_{k+1}]) = \rho_k(f).$$

The case now follows from the above equation, the induction step, and the fact that the path π remains the same.

The second case follows easily from the formula

$$\rho_k(d_-(f y_{k+1}^a)) = -q^{d+1-(k-1)a} \rho_k(f),$$

for $f \in V_k$ homogeneous of degree d in the symmetric function part. The base case is obvious. $\square$

The recursions of [GXZ12] now follow fairly easily from (4.17) as a special case, if Theorem 4.11 is taken as the definition of $D_\alpha(q,t)$. The relations for N_α were not discovered this way, and the authors only noticed this connection after following up on a useful suggestion from one of the referees, for which the authors are grateful.

5. Operator relations

We have operators

$$e_k, d_\pm, T_i \circlearrowright V_* = V_0 \oplus V_1 \oplus \cdots, \tag{5.1}$$

where e_k is the projection onto V_k, and the others are defined as above. It is natural to ask for a complete set of relations between them. They are formalized in the following algebra:

Definition 5.1. The Dyck path algebra $\mathbb{A} = \mathbb{A}_q$ (over R) is the path algebra of the quiver with vertex set $\mathbb{Z}_{\geqslant 0}$, arrows d_+ from i to $i+1$, arrows d_- from $i+1$ to i for $i \in \mathbb{Z}_{\geqslant 0}$, and loops $T_1, T_2, \ldots, T_{k-1}$ from k to k subject to the following relations:

$$(T_i - 1)(T_i + q) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i-j| > 1),$$

$$T_i d_- = d_- T_i, \quad d_+ T_i = T_{i+1} d_+, \quad T_1 d_+^2 = d_+^2, \quad d_-^2 T_{k-1} = d_-^2,$$

$$d_-(d_+ d_- - d_- d_+) T_{k-1} = q(d_+ d_- - d_- d_+) d_- \quad (k \geqslant 2),$$

$$T_1(d_+ d_- - d_- d_+) d_+ = q d_+ (d_+ d_- - d_- d_+),$$

where in each identity k denotes the index of the vertex where the respective paths begin. We have used the same letters $T_i, d_\pm$ to label the ith loop at every node k to match with the previous notation. To distinguish between different nodes, we will use $T_i e_k$, where e_k is the idempotent associated with node k.

We will prove:

Theorem 5.2. *The operators* (5.1) *define a representation of* $\mathbb{A}$ *on* V_**. Furthermore, we have an isomorphism of representations*

$$\varphi : \mathbb{A}e_0 \xrightarrow{\sim} V_*,$$

which sends e_0 *to* $1 \in V_0$*, and maps* $e_k\mathbb{A}e_0$ *isomorphically onto* V_k*.*

The proof will occupy the rest of this section. We begin by establishing that we have a defined a representation of the algebra.

Lemma 5.3. *The operators* T_i *and* $d_\pm$ *satisfy the relations of Definition* 5.1.

Proof. The first line is just Proposition 4.1, and most follow from definition. The first one that does not is the commutation relation of d_+ with T_i. We have

$$\begin{aligned} d_+T_i(F) &= T_1 \cdots T_k\left((T_iF)[X + (q-1)y_{k+1}]\right) \\ &= T_1 \cdots T_iT_{i+1}T_i \cdots T_k(F)[X + (q-1)y_{k+1}] \\ &= T_1 \cdots T_{i+1}T_iT_{i+1} \cdots T_k(F)[X + (q-1)y_{k+1}] = T_{i+1}d_+(F), \end{aligned}$$

using the braiding relations.

For the next, we have

$$\begin{aligned} d_+^2 F &= T_1T_2 \cdots T_{k+1}T_1T_2 \cdots T_k(F[X + (q-1)y_{k+1} + (q-1)y_{k+2}]) \\ &= T_2T_3 \cdots T_{k+1}T_1T_2 \cdots T_{k+1}(F[X + (q-1)y_{k+1} + (q-1)y_{k+2}]). \end{aligned}$$

The last T_{k+1} can be removed because its argument is symmetric in y_{k+1} and y_{k+2}, and we obtain $T_1^{-1}d_+^2F$.

The next identity is more technical. The operator image of $T_{k-1} - 1$ consists of elements of the form $(qy_{k-1} - y_k)F$, where F is symmetric in y_{k-1} and y_k. Thus we need to check that d_-^2 vanishes on such elements. Let us evaluate d_-^2 on

$$(qy_{k-1} - y_k)y_{k-1}^a y_k^b F,$$

where F does not contain the variables y_{k-1} and y_k, and $a, b \in \mathbb{Z}_{\geqslant 0}$. We obtain

$$(qB_{a+2}B_{b+1} - B_{a+1}B_{b+2})F.$$

This expression is antisymmetric in a, b by Corollary 3.4 of [HMZ12], which implies our identity.

Next using the previous relations and Lemma 5.4 below write

$$\begin{aligned} d_-(d_+d_- - d_-d_+)T_{k-1} &= (q-1)d_-T_1T_2 \cdots T_{k-1}y_kT_{k-1} \\ &= q(q-1)d_-T_1T_2 \cdots T_{k-2}y_{k-1} \\ &= q(q-1)T_1T_2 \cdots T_{k-2}y_{k-1}d_- = q(d_+d_- - d_-d_+)d_-. \end{aligned} \tag{5.2}$$

Similarly,

$$\begin{aligned} T_1(d_+d_- - d_-d_+)d_+ &= (q-1)T_1T_1T_2 \cdots T_ky_{k+1}d_+ \\ &= (q-1)q^kT_1y_1T_1^{-1}T_2^{-1} \cdots T_k^{-1}d_+ = (q-1)q^kT_1y_1T_1^{-1}d_+T_1^{-1} \cdots T_{k-1}^{-1} \\ &= (q-1)q^kd_+y_1T_1^{-1} \cdots T_{k-1}^{-1} = qd_+(d_+d_- - d_-d_+). \end{aligned} \tag{5.3}$$

□

To establish the isomorphism, we first show that we can produce the operators of multiplication by y_i from $\mathbb{A}$.

Lemma 5.4. *For $F \in V_k$, we have*

$$(d_-d_+ - d_+d_-)F = (q-1)T_1T_2\cdots T_{k-1}(-y_kF), \quad y_i = \frac{1}{q}T_iy_{i+1}T_i. \tag{5.4}$$

Proof. First, we endow V_k with the following twisted action of $\mathrm{Sym}[X]$:

$$(F*G)[X] = F\left[X + (q-1)\sum_{i=1}^k y_i\right]G,$$

for $F \in \mathrm{Sym}[X]$, and $G \in V_k$. It can be checked that the operators d_+, d_- intertwine this action:

$$d_+(F*G) = F*d_+G, \quad d_-(F*G) = F*d_-G. \tag{5.5}$$

For the second one, for instance, it suffices to assume that $k=1$. Then by the definition of d_- given in (4.2), we have

$$d_-(F[X+(q-1)y_1]G) = -F[X]G[X-(q-1)y_1]\,\mathrm{Exp}[-y_1^{-1}X]\big|_{y_1^{-1}} = F*d_-G.$$

We will not need this, but in fact, if $\pi_1 \in \mathbb{D}_k$, $\pi_2 \in \mathbb{D}$, and $\pi_1\cdot\pi_2 \in \mathbb{D}_k$ is their concatenation, then we must also have that

$$\chi_k(w_1\cdot w_2) = \chi(w_2)*\chi_k(w_1).$$

Since the operators on both sides commute with the twisted action of $\mathrm{Sym}[X]$ introduced above, we may assume without loss of generality that F is a polynomial of $y_1, y_2, \ldots, y_k$.

Write the left-hand side of the first identity as

$$d_-T_1\cdots T_{k-1}T_kF - T_1\cdots T_{k-1}\left((d_-F)[X+(q-1)y_k]\right).$$

The operator d_- in the first summand involves only the variable y_{k+1}. Thus we can write the left-hand side as

$$T_1\cdots T_{k-1}(d_-T_kF - (d_-F)[X+(q-1)y_k]).$$

Hence it is enough to prove

$$d_-T_kF - (d_-F)[X+(q-1)y_k] = (1-q)y_kF.$$

It is clear that none of the operations involve the variables $y_1, y_2, \ldots, y_{k-1}$. Thus we can assume $F = y_k^i$ for $i \in \mathbb{Z}_{\geqslant 0}$ without loss of generality. Direct computation gives

$$T_k(y_k^i) = y_{k+1}^i + (1-q)\sum_{j=1}^i y_k^j y_{k+1}^{i-j}.$$

Thus the left-hand side equals

$$-h_{i+1}[-X] - (1-q)\sum_{j=1}^i y_k^j h_{i-j+1}[-X] + h_{i+1}[-X-(q-1)y_k]$$
$$= -(1-q)\sum_{j=1}^i y_k^j h_{i-j+1}[-X] + (1-q)\sum_{j=1}^{i+1} y_k^j h_{i-j+1}[-X] = (1-q)y_k^{i+1}.$$

The second relation is easy. ∎

The operators of multiplication by y_i are characterized by these relations and therefore come from elements of $\mathbb{A}$. We next establish the relations these operators satisfy within $\mathbb{A}$:

Lemma 5.5. *For $k \in \mathbb{Z}_{>0}$ define elements $y_1, \ldots, y_k \in e_k \mathbb{A} e_k$ by solving for $y_i F$ in the identities* (5.4). *Then the following identities hold in $\mathbb{A}$:*

$$\begin{aligned} y_i T_j &= T_j y_i \qquad \text{for } i \notin \{j, j+1\}, \\ y_i d_- &= d_- y_i, \quad d_+ y_i = T_1 T_2 \cdots T_i y_i (T_1 T_2 \cdots T_i)^{-1} d_+, \\ y_i y_j &= y_j y_i \quad \text{for any } i, j. \end{aligned}$$

Proof. Note that y_1 can be written as

$$y_1 = \frac{1}{q^{k-1}(q-1)} (d_+ d_- - d_- d_+) T_{k-1} \cdots T_1.$$

Our task becomes easier if we notice that it is enough to check the first identity for $i = 1$ and $i = k$, the second one for $i = k$, the third one for $i = 1$, and the last one for $i = 1$, $j = k$. The other cases can be deduced from these by applying the T-operators.

For $j > 1$, we have

$$\begin{aligned} y_1 T_j &= \frac{1}{q^{k-1}(q-1)} (d_+ d_- - d_- d_+) T_{k-1} \cdots T_1 T_j \\ &= \frac{1}{q^{k-1}(q-1)} (d_+ d_- - d_- d_+) T_{j-1} T_{k-1} \cdots T_1 = T_j y_1. \end{aligned}$$

Similarly, we verify that y_k commutes with T_j^{-1} hence with T_j for $j < k-1$.

Reversing the arguments in (5.2) and (5.3), we verify the second and the third identities.

Thus it is left to check that $y_k y_1 = y_1 y_k$. We assume $k \geqslant 2$. Write the left-hand side as

$$\begin{aligned} y_k y_1 &= \frac{1}{q-1} T_{k-1}^{-1} \cdots T_1^{-1} (d_+ d_- - d_- d_+) y_1 \\ &= \frac{1}{q-1} T_{k-1}^{-1} \cdots T_1^{-1} (T_1 y_1 T_1^{-1})(d_+ d_- - d_- d_+), \end{aligned}$$

using the already established commutation relations and that $k \geqslant 2$ to swap $T_1 y_1 T_1^{-1}$ and d_-. Performing the cancellation, we obtain $y_1 y_k$. ☐

The following lemma completes the proof of the theorem:

Lemma 5.6. *The elements of the form*

$$d_-^m y_1^{a_1} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} e_0 \tag{5.6}$$

with $a_{k+1} \geqslant a_{k+2} \geqslant \cdots \geqslant a_{k+m}$ form a basis of $\mathbb{A} e_0$. Furthermore, the representation φ maps these elements to a basis of V_*.

Proof. We first show that elements of the form (5.6), with no condition on the a_i span $\mathbb{A}$. It suffices to check that the span of these elements is invariant under d_-, T_i, and d_+. This can be done by applying the following reduction rules that follow from the definition of $\mathbb{A}$ and Lemma 5.5:

$$T_i d_- \to d_- T_i, \quad T_j y_i \to y_i T_j \quad (i \notin \{j, j+1\}),$$
$$T_i y_i \to y_{i+1} T_i + (1-q) y_i, \quad T_i y_{i+1} \to y_i T_i + (q-1) y_i,$$
$$T_i d_+^{k+m} e_0 \to d_+^{k+m} e_0,$$
$$d_+ d_- \to d_- d_+ + (q-1) T_1 T_2 \cdots T_{k-1} y_k, \quad y_i d_- \to d_- y_i.$$

The next step is to reduce the spanning set. We can use the following identity, which follows from $d_-^2 T_{k-1} = d_-^2$:

$$d_-^m (1 - T_j) y_1^{a_1} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} e_0 = 0 \qquad (k < j < k+m).$$

Note that T_j commutes with $y_j y_{j+1}$. Suppose $a_j < a_{j+1}$. Then we can rewrite the above identity as

$$0 = d_-^m y_1^{a_1} \cdots y_j^{a_j} y_{j+1}^{a_j} (1 - T_j) y_{j+1}^{a_{j+1} - a_j} y_{j+2}^{a_{j+2}} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} e_0.$$

Using $T_j y_{j+1} = y_j (T_j + (q-1))$, $T_j y_r = y_r T_j$ for $r > j+1$, and $T_j d_+^{k+m} e_0 = d_+^{k+m} e_0$, we can rewrite the identity as vanishing of a linear combination of terms of the form (5.6), and the lexicographically smallest term is precisely

$$d_-^m y_1^{a_1} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} e_0.$$

Thus we can always reduce terms of the form (5.6) which violate the condition $a_{k+1} \geqslant a_{k+2} \geqslant \cdots \geqslant a_{k+m}$ to a linear combination of lexicographically greater terms, showing that the subspace in the lemma at least spans $\mathbb{A} e_0$.

We now show that they map to a basis of V_*, which also establishes that they are independent, completing the proof. Consider the image of the elements of our spanning set

$$d_-^m y_1^{a_1} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} (1) = d_-^m (y_1^{a_1} \cdots y_{k+m}^{a_{k+m}}),$$

$$(-1)^m y_1^{a_1} y_2^{a_2} \cdots y_k^{a_k} B_{a_{k+1}+1} B_{a_{k+2}+1} \cdots B_{a_{k+m}+1} (1). \tag{5.7}$$

Notice that $\lambda := (a_{k+1}+1, a_{k+2}+1, \ldots, a_{k+m}+1)$ is a partition, so

$$B_{a_{k+1}+1} B_{a_{k+2}+1} \cdots B_{a_{k+m}+1} (1)$$

is a multiple of the Hall–Littlewood polynomial $\tilde{H}_{\lambda'}[-X; 1/q, 0]$. These polynomials form a basis of the space of symmetric functions, thus the elements (5.7) form a basis of $\bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} V_k$. □

6. Conjugate structure

It is natural to ask if there is a way to extend ∇ to the spaces V_k recovering the original operator at $k = 0$. What we have found is that it is simpler to extend the composition

$$\mathcal{N}(F) = \nabla \bar{\omega} F = \nabla \omega \overline{F}, \tag{6.1}$$

where the conjugation simply makes the substitution $(q, t) = (q^{-1}, t^{-1})$, $\omega(F) = F[-X]$ is the Weyl involution up to a sign, and $\bar{\omega}$ denotes the composition. This is a very interesting operator, which in fact is an antilinear involution on $\mathrm{Sym}[X]$ corresponding to dualizing vector bundles and tensoring with $\mathcal{O}(1)$ in the Haiman–Bridgeland–King–Reid picture, which identifies $\mathrm{Sym}[X]$ with the equivariant K-theory of the Hilbert scheme of points in the complex plane [BKR01]. The key to our proof is to extend this operator to an antilinear involution on every V_k, suggesting that V_k should have a geometric interpretation as well. Since this paper was completed, E. Gorsky and the authors have discovered this connection in terms of a smooth subscheme of the flag Hilbert scheme. This will be explained in an upcoming paper, which also explains a family of ways to extend the operator ∇ itself in addition to the involution $\mathcal{N}$, also in terms of line bundles.

We will define the operator, which was discovered experimentally to have nice properties, by explicitly constructing the action of $\mathbb{A}$ conjugated by the conjectural involution $\mathcal{N}$. Let $\mathbb{A}^* = \mathbb{A}_{q^{-1}}$, and label the corresponding generators by $d_\pm^*, T_i^*, e_i^*$. Denote by z_i the image of y_i under the isomorphism from $\mathbb{A}$ to $\mathbb{A}^*$ which sends generators to generators, and which is antilinear with respect to $q \mapsto q^{-1}$.

Theorem 6.1. *There is an action of* $\mathbb{A}^*$ *on* V_* *given by the assignment*

$$T_i^* = T_i^{-1}, \quad d_-^* = d_-, \quad e_i^* = e_i, \quad (d_+^* F)[X] = \gamma F[X + (q-1) y_{k+1}], \tag{6.2}$$

where $F \in V_k$ *and* γ *is the operator which sends* y_i *to* y_{i+1} *for* $i = 1, \dots, k$ *and* y_{k+1} *to* $t y_1$. *Furthermore, it satisfies the additional relations*

$$z_{i+1} d_+ = d_+ z_i, \quad y_{i+1} d_+^* = d_+^* y_i, \quad z_1 d_+ = -y_1 d_+^* t q^{k+1}, \quad d_+^* d_+^m(1) = d_+^{m+1}(1) \tag{6.3}$$

for any $m \geqslant 0$.

The statement is equivalent to validity of a certain set of relations satisfied by the operators. These will be verified in the following propositions.

First we list the obvious relations:

Proposition 6.2.

$$d_+^* T_i^{-1} = T_{i+1}^{-1} d_+^*, \qquad T_1^{-1} d_+^{*2} = d_+^{*2}, \qquad d_+^* y_i = y_{i+1} d_+^*.$$

Proof. Easy from the definition. □

To verify the rest of the relations, we use an approach similar to the one used in the proof of Lemma 5.4. Now we need not just one, but a family of twisted multiplications: For $F \in \mathrm{Sym}[X]$, $G \in V_k$, $m = 0, 1, 2, \dots, k$, put

$$(F *_m G)[X] = F\left[X + (q-1)\left(\sum_{i=1}^{m} t y_i + \sum_{i=m+1}^{k} y_i\right)\right] G.$$

It is not hard to show that they satisfy

$$d_+^*(F *_m G) = F *_{m+1} d_+^* G, \quad d_-(F *_m G) = F *_m d_- G, \tag{6.4}$$

where $F \in \mathrm{Sym}[X]$, and $G \in V_k$, the first identity holds for $0 \leqslant m \leqslant k$, and the second one for $0 \leqslant m < k$.

Let us first verify

Proposition 6.3.

$$d_-(d_+^* d_- - d_- d_+^*) T_{k-1}^{-1} = q^{-1}(d_+^* d_- - d_- d_+^*) d_- \quad (k \geqslant 2). \tag{6.5}$$

Proof. Rewrite it as

$$d_+^* d_-^2 - d_- d_+^* d_- (T_{k-1} + q) + q d_-^2 d_+^* = 0.$$

Multiplying both sides by $q - 1 = T_{k-1} - 1 + q - T_{k-1}$ produces an equivalent relation, which can be reduced to

$$(d_+^* d_-^2 - (q+1) d_- d_+^* d_- + q d_-^2 d_+^*)(T_{k-1} + q) = 0.$$

Note that the image of $T_{k-1}+q$ consists of elements which are symmetric in y_{k-1}, y_k. Let

$$A = d_+^* d_-^2 - (q+1) d_- d_+^* d_- + q d_-^2 d_+^*.$$

It is enough to show that A vanishes on elements of V_k that are symmetric in y_{k-1}, y_k. We have (recall that $k \leqslant 2$)

$$A(F * G) = F *_1 A(G), \quad A y_i = y_{i+1} A \quad (F \in \mathrm{Sym}[X],\, G \in V_k,\, i < k-1).$$

Thus it is enough to verify vanishing of A on symmetric polynomials of y_{k-1}, y_k. We evaluate A on $y_{k-1}^a y_k^b$:

$$A(y_{k-1}^a y_k^b) = \big(\Gamma_+(t(q-1)y_1) B_{a+1} B_{b+1} - (q+1) B_{a+1} \Gamma_+(t(q-1)y_1) B_{b+1} \\ + q B_{a+1} B_{b+1} \Gamma_+(t(q-1)y_1)\big) 1,$$

where $\Gamma_+(Z)$ is the operator $F[X] \to F[X+Z]$. For any monomial u and integer i, we have operator identities

$$\Gamma_+(u) B_i = (B_i - u B_{i-1}) \Gamma_+(u), \quad B_i \Gamma_+(-u) = \Gamma_+(-u)(B_i - u B_{i-1}),$$

thus we have

$$\begin{aligned} \Gamma_+(t(q-1)y_1) B_{a+1} B_{b+1} &= \Gamma_+(-t y_1)(B_{a+1} - q t y_1 B_a)(B_{a+1} - q t y_1 B_a) \Gamma_+(q t y_1), \\ B_{a+1} \Gamma_+(t(q-1)y_1) B_{b+1} &= \Gamma_+(-t y_1)(B_{a+1} - t y_1 B_a)(B_{a+1} - q t y_1 B_a) \Gamma_+(q t y_1), \\ B_{a+1} B_{b+1} \Gamma_+(t(q-1)y_1) &= \Gamma_+(-t y_1)(B_{a+1} - t y_1 B_a)(B_{a+1} - t y_1 B_a) \Gamma_+(q t y_1). \end{aligned}$$

Performing the cancellations, we arrive at

$$A(y_{k-1}^a y_k^b) = \Gamma_+(-t y_1)(t y_1 (1-q)(B_a B_{b+1} - q B_{a+1} B_b)) 1.$$

This expression is antisymmetric in a, b by [HMZ12, Corollary 3.4]. Thus (6.5) is true. □

Next we have to check:

Proposition 6.4.

$$T_1^{-1}(d_+^* d_- - d_- d_+^*) d_+^* = q^{-1} d_+^* (d_+^* d_- - d_- d_+^*).$$

Proof. Multiplying both sides by qT_1 and using the easier relations, rewrite it as

$$d_+^{*2} d_- - (T_1 + q) d_+^* d_- d_+^* + q d_- d_+^{*2} = 0.$$

Again, because of the commutation relations with the twisted multiplication by symmetric functions and y_i, it is enough to evaluate the left-hand side on y_k^a for all $a \in \mathbb{Z}_{\geqslant 0}$. We obtain

$$-h_{a+1}[-X - t(q-1)(y_1+y_2)] + (T_1+q) h_{a+1}[-X - t(q-1) y_1] - q h_{a+1}[-X].$$

We use the identity $h_n[X+Y] = \sum_{i+j=n} h_i[X] h_j[Y]$ to write the left-hand side as a linear combination of terms $h_{a+1-b}[-X]$ with $b > 0$. The coefficient in front of each term with $b > 0$ is

$$-h_b[t(1-q)(y_1+y_2)] + (T_1+q) h_b[t(1-q) y_1].$$

By a direct computation,

$$(T_1+q)h_b[t(1-q)y_1] = (T_1+q)(1-q)t^b y_1^b$$
$$= (1-q)t^b\Big(y_1^b + (1-q)\sum_{i=1}^{b-1} y_1^i y_2^{b-i} + y_2^b\Big) = h_b[t(1-q)(y_1+y_2)],$$

and we are done. □

At this point, we have established the fact that the operators given by (6.2) define an action of $\mathbb{A}^*$ on V_*. Also we have established the second relation in (6.3). The last relation is obvious. The first and the third are verified below:

Proposition 6.5.

$$z_1 d_+ = -y_1 d_+^* t q^{k+1}, \qquad z_{i+1} d_+ = d_+ z_i.$$

Proof. Using (6.4) we see that the operator $y_1 d_+^*$ satisfies the two properties,

$$y_1 d_+^* y_i = y_{i+1} y_1 d_+^*, \qquad y_1 d_+^*(F * G) = F *_1 y_1 d_+^*(G)$$

for $F \in \mathrm{Sym}[X]$, $G \in V_k$, $i = 1, \ldots, k$. By definition (on V_k)

$$z_1 = \frac{q^{k-1}}{q^{-1}-1}(d_+^* d_- - d_- d_+^*) T_{k-1}^{-1} \cdots T_1^{-1},$$

thus (again, on V_k)

$$z_1 d_+ = \frac{q^k}{q^{-1}-1}(d_+^* d_- - d_- d_+^*) T_k^{-1} \cdots T_1^{-1} d_+.$$

From this expression we see, using (6.4) once again, that $z_1 d_+$ satisfies the same two properties as the operator $y_1 d_+^*$,

$$z_1 d_+ y_i = y_{i+1} z_1 d_+, \qquad z_1 d_+(F * G) = F *_1 z_1 d_+(G)$$

for $F \in \mathrm{Sym}[X]$, $G \in V_k$, $i = 1, \ldots, k$. Thus it is enough to verify the first identity on $1 \in V_k$. The right-hand side is $-tq^{k+1} y_1$. The left-hand side is

$$\frac{q^k}{q^{-1}-1}(d_+^* - 1) d_-(1) = \frac{q^k}{q^{-1}-1}(X + t(q-1)y_1 - X) = -tq^{k+1} y_1,$$

so the first identity holds.

It is enough to verify the second identity for $i = 1$ because the general case can be deduced from this one by applying the T-operators. For the identity $z_2 d_+ = d_+ z_1$, expressing z_1, z_2 in terms of d_-, d_+^* and the T-operators, we arrive at the equivalent identity

$$T_1^{-1} d_+ (d_+^* d_- - d_- d_+^*) = (d_+^* d_- - d_- d_+^*) d_+.$$

If we denote by A either of the two sides, we have

$$A(F * G) = F *_1 A(G), \qquad A y_i = T_2 T_3 \cdots T_{i+1} y_{i+1} (T_2 T_3 \cdots T_{i+1})^{-1} A$$

for $F \in \mathrm{Sym}[X]$, $G \in V_k$, $i = 1, \ldots, k-1$. Thus it is enough to verify the identity on $y_k^a \in V_k$ ($a \in \mathbb{Z}_{\geqslant 0}$). Applying $T_k^{-1} T_{k-1}^{-1} \cdots T_2^{-1}$ to both sides, the identity to be verified is

$$T_k^{-1} T_{k-1}^{-1} \cdots T_1^{-1} d_+ (d_+^* d_- - d_- d_+^*)(y_k^a) = (d_+^* d_- - d_- d_+^*) T_{k-1}^{-1} \cdots T_1^{-1} d_+ (y_k^a).$$

The left-hand side is evaluated to

$$-h_{a+1}[-X - t(q-1)y_1 - (q-1)y_{k+1}] + h_{a+1}[-X - (q-1)y_{k+1}].$$

The right-hand side is evaluated to

$$(d_+^* d_- - d_- d_+^*) T_k(y_{k+1}^a) = F[X + t(q-1)y_1] - F[X]$$

with

$$F[X] = -h_{a+1}[-X] - (1-q) \sum_{i=0}^{a-1} y_{k+1}^{a-i} h_{i+1}[-X]$$
$$= -h_{a+1}[-X + (1-q)y_{k+1}] + (1-q)y_{k+1}^{a+1},$$

and the identity follows. □

This completes our proof of Theorem 6.1.

We also have the following proposition, which we will use to connect the conjugate action with N_α.

Proposition 6.6. *For a composition α of length k, let*

$$y_\alpha = y_1^{\alpha_1 - 1} \cdots y_k^{\alpha_k - 1} \in V_k.$$

Then the following recursions hold:

$$y_{1\alpha} = d_+^* y_\alpha, \quad y_{a\alpha} = \frac{t^{1-a}}{q-1} (d_+^* d_- - d_- d_+^*) \sum_{\beta \models a-1} q^{1-l(\beta)} d_-^{l(\beta)-1}(y_{\alpha\beta}) \quad (a > 1).$$

Proof. The first identity easily follows from the explicit formula for d_+^*. For $i = 1, 2, \ldots, k-1$, we have

$$(d_- d_+^* - d_+^* d_-) y_i = y_{i+1}(d_- d_+^* - d_+^* d_-).$$

Therefore it is enough to verify the following identity for any $a \in \mathbb{Z}_{\geqslant 1}$:

$$(q-1)t^a y_1^a = (d_+^* d_- - d_- d_+^*) \sum_{\beta \models a} q^{1-l(\beta)} d_-^{l(\beta)-1}(y_k^{\beta_1 - 1} \cdots y_{k+l(\beta)-1}^{\beta_{l(\beta)}-1}) \in V_k. \tag{6.6}$$

We group the terms on the right-hand side by $b = \beta_1 - 1$, and the sum becomes

$$\sum_{b=0}^{a-1} y_k^b \sum_{\beta \models a-b-1} q^{-l(\beta)} d_-^{l(\beta)} \left(y_{k+1}^{\beta_1 - 1} \cdots y_{k+l(\beta)}^{\beta_{l(\beta)}-1} \right)$$
$$= \sum_{b=0}^{a-1} y_k^b \sum_{\beta \models a-b-1} q^{-l(\beta)} (-1)^{l(\beta)} B_{\beta_1} \cdots B_{\beta_{l(\beta)}}(1) = \sum_{b=0}^{a-1} y_k^b h_{a-b-1}[q^{-1}X].$$

We have used the identity

$$h_n[q^{-1}X] = \sum_{\alpha \models n} q^{-l(\alpha)} (-1)^{l(\alpha)} B_\alpha(1), \tag{6.7}$$

which can be obtained by applying $\bar{\omega}$ to Proposition 5.2 of [HMZ12]:

$$h_n[-X] = \sum_{\alpha \models n} C_\alpha(1).$$

Thus the right-hand side of (6.6) is evaluated to the expression

$$(d_+^* d_- - d_- d_+^*) \sum_{b=0}^{a-1} y_k^b q^{-(a-b-1)} h_{a-b-1}[X]$$
$$= -\sum_{b=0}^{a-1} \left(\Gamma_+(t(q-1)y_1) B_{b+1} - B_{b+1}\Gamma_+(t(q-1)y_1)\right) h_{a-b-1}[q^{-1}X].$$
$$= -\sum_{b=0}^{a-1} \Gamma_+(-ty_1)\left((B_{b+1} - qty_1 B_b) - (B_{b+1} - ty_1 B_b)\right)\left(h_{a-b-1}[q^{-1}X + ty_1]\right)$$
$$= (q-1)ty_1 \Gamma_+(-ty_1) \sum_{b=0}^{a-1} B_b(h_{a-b-1}[q^{-1}X + ty_1]).$$

Thus we need to prove

$$\sum_{b=0}^{a-1} B_b(h_{a-b-1}[q^{-1}X + ty_1]) = t^{a-1} y_1^{a-1}.$$

Then the left-hand side as a polynomial in y_1 indeed has the right coefficient of y_1^{a-1}. The coefficient of y_1^i for $i < a-1$ is

$$t^i \sum_{b=0}^{a-1-i} B_b(h_{a-b-1-i}[q^{-1}X]).$$

So it is enough to show

$$\sum_{b=0}^{m} B_b(h_{m-b}[q^{-1}X]) = 0 \quad (m \in \mathbb{Z}_{>0}).$$

Using (6.7) again, we see that the left-hand side equals

$$B_0(h_m[q^{-1}X]) - qh_m[q^{-1}X] = (B_0 - q)(-q^{-1}C_m(1)) = 0$$

because $B_0 C_m = qC_m B_0$ by [HMZ12, Proposition 3.5] and $B_0(1) = 1$. □

7. The involution

Definition 7.1. Consider $\mathbb{A}$ and $\mathbb{A}^*$ as algebras over $\mathbb{Q}(q,t)$, and let $\tilde{\mathbb{A}} = \tilde{\mathbb{A}}_{q,t}$ be the quotient of the free product of $\mathbb{A}$ and $\mathbb{A}^*$ by the relations

$$d_-^* = d_-, \quad T_i^* = T_i^{-1}, \quad e_i^* = e_i,$$
$$z_{i+1} d_+ = d_+ z_i, \quad y_{i+1} d_+^* = d_+^* y_i, \quad z_1 d_+ = -y_1 d_+^* t q^{k+1}.$$

Remark 7.2. For any $k \geqslant 0$, the *affine Hecke algebra* AHA_k is the algebra generated over $\mathbb{Q}(q)$ by $T_1, \ldots, T_{k-1}, y_1^{\pm 1}, \ldots, y_k^{\pm 1}$ modulo relations

$$(T_i - 1)(T_i + q) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i-j| > 1),$$
$$y_i T_j = T_j y_i \quad (i \notin \{j, j+1\}), \quad y_i y_j = y_j y_i, \quad T_i y_{i+1} T_i = q y_i.$$

The *positive part* AHA_k^+ is defined as the subalgebra of AHA_k generated by T_i and y_i, or equivalently as the algebra generated over $\mathbb{Q}(q)$ by $T_1, \ldots, T_{k-1}, y_1, \ldots, y_k$ modulo the same relations. We have a natural homomorphism $\mathrm{AHA}_k^+ \to e_k \mathbb{A} e_k$ which can be shown to be injective using Lemma 5.6. It is tempting to guess that in a similar way the subalgebra of $e_k \tilde{\mathbb{A}} e_k$ generated by T_i, y_i, and z_i is isomorphic to the positive part DAHA_k^{++} of the *double affine Hecke algebra* DAHA_k. To fix a version of DAHA_k^{++} which is close to our notations, we start with relations [SV11, (2.1)–(2.7)] and perform substitutions $q = t^{-1}$, $v = q^{\frac{1}{2}}$, $T_i = q^{\frac{1}{2}} T_i^{-1}$, $X_i = y_i$, $Y_i = z_i$ followed by reversal of the order of generators in each monomial. So DAHA_k^{++} is defined over $\mathbb{Q}(q,t)$ by generators $T_1, \ldots, T_{k-1}, y_1, \ldots, y_k, z_1, \ldots, z_k$ and relations of two copies of AHA_k (the second copy is transformed by $T_i \to T_i^{-1}$, $q \to q^{-1}$)

$$(T_i - 1)(T_i + q) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i-j| > 1),$$

$$y_i T_j = T_j y_i \quad (i \notin \{j, j+1\}), \quad y_i y_j = y_j y_i, \quad T_i y_{i+1} T_i = q y_i,$$

$$z_i T_j = T_j z_i \quad (i \notin \{j, j+1\}), \quad z_i z_j = z_j z_i, \quad T_i z_i T_i = q z_{i+1},$$

and two extra relations. The first one is

$$z_2 y_1 = q y_1 T_1^{-2} z_2 \quad \Leftrightarrow \quad q y_2 z_1 = z_1 T_1^2 y_2,$$

which *can* be deduced in $\tilde{A}$ from (5.4) and $z_2 d_+ = d_+ z_1$. The second relation is

$$z_1 y_1 \cdots y_k = t y_1 \cdots y_k z_1.$$

The following identity can be deduced from the rest of the relations:

$$y_2 \cdots y_k z_1 = q^{1-k} z_1 T_1 \cdots T_{k-1} T_{k-1} \cdots T_1 y_2 \cdots y_k.$$

Thus we expect to have

$$z_1 y_1 = q^{1-k}\, t\, y_1 z_1 T_1 \cdots T_{k-1} T_{k-1} \cdots T_1.$$

However this *does not* hold in $\tilde{A}$. Instead we have

$$z_1 y_1 = q^{1-k}\, t\, y_1 z_1 T_1 \cdots T_{k-1} T_{k-1} \cdots T_1 \; + \; q t y_1 d_- d_+^* T_{k-1} \cdots T_1.$$

So we see that we *do not* obtain a natural homomorphism $\mathrm{DAHA}_k^{++} \to e_k \tilde{\mathbb{A}} e_k$. One way to repair the situation is to introduce the "partially symmetrized" $\mathrm{SDAHA}_{k,\infty}^{++}$ by starting with the DAHA in infinitely many generators T_i, z_i, y_i $(i = 1, 2, 3, \ldots)$, and then symmetrizing in generators with $i > k$. For instance for $k = 0$, we expect $e_0 \tilde{\mathbb{A}} e_0$ to coincide with the positive part of the elliptic Hall algebra, which is the stable limit of spherical DAHAs as shown in [SV11]. Details of this construction will be provided in a future publication.

We now prove

Theorem 7.3. *The operations T_i, d_-, d_+, d_+^*, e_i define an action of $\tilde{\mathbb{A}}$ on V_*. Furthermore, the kernel of the natural map $\tilde{\mathbb{A}} e_0 \to V_*$ that sends $f e_0$ to $f(1)$ is given by $I e_0$ where $I \subset \tilde{\mathbb{A}}$ is the ideal generated by*

$$I = \langle d_+^* d_+^m - d_+^{m+1} | \quad m \geqslant 0 \rangle. \tag{7.1}$$

In particular, we have an isomorphism $V_* \cong \tilde{\mathbb{A}} e_0 / I e_0$.

Proof. Theorem 6.1 shows that we have a map of modules $\tilde{\mathbb{A}} e_0 \to V_*$ that restricts to the isomorphism of Theorem 5.2 on the subspace $\mathbb{A} e_0$, so in particular is surjective. Furthermore, the last relation of (6.3) shows that it descends to a map

$\tilde{\mathbb{A}}e_0/Ie_0 \mapsto V_*$, which must still be surjective. We have the following commutative diagram.

$$\begin{array}{ccc} \tilde{\mathbb{A}}e_0/Ie_0 & \longrightarrow & V_* \\ \uparrow & \overset{\sim}{\nearrow} & \\ \mathbb{A}e_0 & & \end{array}$$

Thus we have an inclusion $\mathbb{A}e_0 \subset \tilde{\mathbb{A}}e_0/Ie_0$ and it remains to show that the image of $\mathbb{A}e_0$ in $\tilde{\mathbb{A}}e_0/Ie_0$ is the entire space. We do so by induction: notice that both $\mathbb{A}e_0$ and $\tilde{\mathbb{A}}e_0/Ie_0$ have a grading by the total degree in d_+, d_+^*, and d_-, as all the relations are homogeneous. For instance, y_i and z_i have degree 2, and T_i has degree 0 for all i. Denote the space of elements of degree m in $\mathbb{A}e_0$, $\tilde{\mathbb{A}}e_0/Ie_0$ by $V^{(m)}$, $W^{(m)}$, respectively. We need to prove $V^{(m)} = W^{(m)}$. The base cases $m = 0$, $m = 1$ are clear.

For the induction step, suppose $m > 0$, $V^{(i)} = W^{(i)}$ for $i \leqslant m$, and let $F \in V^{(m)}$. It is enough to show that $d_+^* F \in V^{(m+1)}$. By Lemma 5.6, we can assume that F is in the canonical form (5.6). We therefore must check three cases: $F = d_+^m(1)$ for $1 \in V_0$, $F = y_i G$ for $G \in V^{(m-2)}$, and $F = d_-(G)$ for $G \in V^{(m-1)}$. In the first case we have $d_+^* F = d_+^{m+1}1$. In the second case we have $d_+^*(F) = y_{i+1} d_+^*(G)$. In the third case we have

$$d_+^* F = d_+^* d_- G = d_- d_+^* G + (q^{-1} - 1) T_1^{-1} \cdots T_{k-1}^{-1} z_k G.$$

Now we use expansion of G in terms of the generators T_i, d_+, and d_-. Because of the commutation relations between T_i and z_j, it is enough to consider two cases, $G = d_+ G'$ and $G = d_- G'$ for $G' \in V^{(m-2)}$. In the first case we have $z_k G = d_+ z_{k-1} G'$ if $k > 1$ and $z_1 G = -y_1 d_+^* t G'$ if $k = 1$. In the second case we have $z_k G = d_- z_k G'$. In all cases the claim is reduced to the induction hypothesis. □

Now by looking at the defining relations of $\tilde{\mathbb{A}}$, we make the remarkable observation that there exists an involution ι of $\tilde{\mathbb{A}}$ that permutes $\mathbb{A}$ and $\mathbb{A}^*$ and is antilinear with respect to the conjugation $(q, t) \mapsto (q^{-1}, t^{-1})$ on the ground field $\mathbb{Q}(q, t)$! Furthermore, this involution preserves the ideal I, and therefore induces an involution on V_* via the isomorphism of Theorem 7.3.

Theorem 7.4. *There exists a unique antilinear degree-preserving automorphism* $\mathcal{N} : V_* \to V_*$ *satisfying*

$$\mathcal{N}(1) = 1, \quad \mathcal{N} T_i = T_i^{-1} \mathcal{N}, \quad \mathcal{N} d_- = d_- \mathcal{N}, \quad \mathcal{N} d_+ = d_+^* \mathcal{N}, \quad \mathcal{N} y_i = z_i \mathcal{N}.$$

Moreover, we have

(i) $\mathcal{N}$ *is an involution, i.e.,* $\mathcal{N}^2 = \mathrm{Id}$.

(ii) *For any composition* α *we have*

$$\mathcal{N}(y_\alpha) = q^{\sum(\alpha_i - 1)} N_\alpha.$$

(iii) *On* $V_0 = \mathrm{Sym}[X]$, *we have* $\mathcal{N} = \nabla \bar{\omega}$, *where* $\bar{\omega}$ *is the involution sending* q, t, X *to* q^{-1}, t^{-1}, $-X$, *respectively* (*see* (6.1)).

Proof. The automorphism is induced from the involution of $\tilde{\mathbb{A}}$, from which part (i) follows immediately. Part (ii) follows from applying $\mathcal{N}$ to the relations of Proposition 6.6.

Finally, let $D_1, D_1^* : V_0 \to V_0$ be the operators

$$(D_1 F)[X] = F[X + (1-q)(1-t)u^{-1}]\,\mathrm{Exp}[-uX]|_{u^1},$$
$$(D_1^* F)[X] = F[X - (1-q^{-1})(1-t^{-1})u^{-1}]\,\mathrm{Exp}[uX]|_{u^1},$$

and let $\underline{e}_1 : V_0 \to V_0$ be the operator of multiplication by $e_1[X] = X$. It is easy to verify that

$$D_1 = -d_- d_+^*, \quad \underline{e}_1 = d_- d_+, \quad \bar{\omega} D_1 = D_1^* \bar{\omega}.$$

Thus it follows that

$$\mathcal{N} D_1 = -\underline{e}_1 \mathcal{N}, \quad \mathcal{N} \underline{e}_1 = -D_1 \mathcal{N}.$$

Let $\nabla' = \mathcal{N}\bar{\omega}$ on V_0. Then

$$\nabla'(1) = 1, \quad \nabla' \underline{e}_1 = D_1 \nabla', \quad \nabla' D_1^* = -\underline{e}_1 \nabla'.$$

It was shown in [GHT99] that ∇ satisfies the same commutation relations, and that one can obtain all symmetric functions starting from 1 and successively applying $\underline{e}_1$ and D_1^*. Thus $\nabla = \nabla'$, proving part (iii). □

The compositional shuffle conjecture now follows easily:

Theorem 7.5. *For a composition α of length k, we have*

$$\nabla C_{\alpha_1} \cdots C_{\alpha_k}(1) = D_\alpha(X; q, t).$$

Proof. Using Theorems 4.11 and 7.4, we have

$$D_\alpha(q,t) = d_-^k(N_\alpha) = d_-^k(\mathcal{N}(q^{|\alpha|-k} y_\alpha)) = \mathcal{N}(q^{|\alpha|-k} d_-^k(y_\alpha))$$
$$= \mathcal{N}\left(q^{|\alpha|-k}(-1)^k B_\alpha(1)\right) = \mathcal{N}\bar{\omega} C_\alpha(1) = \nabla C_\alpha(1). \qquad \square$$

Acknowledgments

The authors would like to thank François Bergeron, Adriano Garsia, Mark Haiman, Jim Haglund, Fernando Rodriguez-Villegas, and Guoce Xin for many valuable discussions on this and related topics. The authors also acknowledge the International Center for Theoretical Physics, Trieste, Italy, where most of the research for this paper was performed. Erik Carlsson was also supported by the Center for Mathematical Sciences and Applications at Harvard University during some of this period, which he gratefully acknowledges.

References

[BGHT99] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler, *Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions*, Methods Appl. Anal. **6** (1999), no. 3, 363–420. Dedicated to Richard A. Askey on the occasion of his 65th birthday, Part III. MR1803316

[BGLX14] F. Bergeron, A. Garsia, E. Sergel Leven, and G. Xin, *Compositional (km, kn)-shuffle conjectures*, Int. Math. Res. Not. IMRN **14** (2016), 4229–4270. MR3556418

[BKR01] T. Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554. MR1824990

[EHKK03] E. S. Egge, J. Haglund, K. Killpatrick, and D. Kremer, *A Schröder generalization of Haglund's statistic on Catalan paths*, Electron. J. Combin. **10** (2003), Research Paper 16, 21. MR1975766

[GH96] A. M. Garsia and M. Haiman, *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combin. **5** (1996), no. 3, 191–244. MR1394305

[GH02] A. M. Garsia and J. Haglund, *A proof of the q,t-Catalan positivity conjecture*, Discrete Math. **256** (2002), no. 3, 677–717. LaCIM 2000 Conference on Combinatorics, Computer Science and Applications (Montreal, QC). MR1935784

[GHT99] A. M. Garsia, M. Haiman, and G. Tesler, *Explicit plethystic formulas for Macdonald q,t-Kostka coefficients*, Sém. Lothar. Combin. **42** (1999), Art. B42m, 45. The Andrews Festschrift (Maratea, 1998). MR1701592

[GN15] E. Gorsky and A. Neguţ, *Refined knot invariants and Hilbert schemes* (English, with English and French summaries), J. Math. Pures Appl. (9) **104** (2015), no. 3, 403–435. MR3383172

[GORS14] E. Gorsky, A. Oblomkov, J. Rasmussen, and V. Shende, *Torus knots and the rational DAHA*, Duke Math. J. **163** (2014), no. 14, 2709–2794. MR3273582

[GXZ12] A. M. Garsia, G. Xin, and M. Zabrocki, *Hall-Littlewood operators in the theory of parking functions and diagonal harmonics*, Int. Math. Res. Not. IMRN **6** (2012), 1264–1299. MR2899952

[Hag03] J. Haglund, *Conjectured statistics for the q,t-Catalan numbers*, Adv. Math. **175** (2003), no. 2, 319–334. MR1972636

[Hag04] J. Haglund, *A proof of the q,t-Schröder conjecture*, Internat. Math. Res. Notices **11** (2004), 525–560.

[Hag08] J. Haglund, *The q,t-Catalan numbers and the space of diagonal harmonics*, University Lecture Series, vol. 41, American Mathematical Society, Providence, RI, 2008. With an appendix on the combinatorics of Macdonald polynomials. MR2371044

[Hai01] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), no. 4, 941–1006. MR1839919

[Hai02] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), no. 2, 371–407. MR1918676

[HHL05a] J. Haglund, M. Haiman, and N. Loehr, *A combinatorial formula for Macdonald polynomials*, J. Amer. Math. Soc. **18** (2005), no. 3, 735–761. MR2138143

[HHL+05b] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov, *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126** (2005), no. 2, 195–232. MR2115257

[Hic12] A. S. Hicks, *Two parking function bijections: a sharpening of the q, t-Catalan and Shröder theorems*, Int. Math. Res. Not. IMRN **13** (2012), 3064–3088. MR2946232

[Hik14] T. Hikita, *Affine Springer fibers of type A and combinatorics of diagonal coinvariants*, Adv. Math. **263** (2014), 88–122. MR3239135

[HL05] J. Haglund and N. Loehr, *A conjectured combinatorial formula for the Hilbert series for diagonal harmonics*, Discrete Math. **298** (2005), no. 1-3, 189–204. MR2163448

[HMZ12] J. Haglund, J. Morse, and M. Zabrocki, *A compositional shuffle conjecture specifying touch points of the Dyck path*, Canad. J. Math. **64** (2012), no. 4, 822–844. MR2957232

[LN14] N. A. Loehr and E. Niese, *New combinatorial formulations of the shuffle conjecture*, Adv. in Appl. Math. **55** (2014), 22–47. MR3176715

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky; Oxford Science Publications. MR1354144

[Neg13] A. Negut, *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22** (2014), 6242–6275. MR3283004

[SV11] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, Compos. Math. **147** (2011), no. 1, 188–234. MR2771130

[SV13] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$*, Duke Math. J. **162** (2013), no. 2, 279–366. MR3018956

International Centre for Theoretical Physics, Str. Costiera, 11, 34151 Trieste, Italy

Current address: Department of Mathematics, University of California, Davis, 1 Shields Ave., Davis, California 95616

Email address: `ecarlsson@math.ucdavis.edu`

International Centre for Theoretical Physics, Str. Costiera, 11, 34151 Trieste, Italy

Current address: Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

Email address: `anton.mellit@univie.ac.at`