1. Introduction

A central problem in algebraic complexity theory is to prove that there is no efficient algorithm to evaluate the permanent

$$\text{per}_n := \sum_{\pi \in S_n} X_{1\pi(1)} \cdots X_{n\pi(n)}.$$  

The natural model of computation for studying this question is the one of straight-line programs (or arithmetic circuits), which perform arithmetic operations $+,-,\ast$ in the polynomial ring, starting with the variables $X_{ij}$ and complex constants. Efficient means that the number of arithmetic operations is bounded by a polynomial in $n$. The permanent arises in combinatorics and physics as a generating function. Its relevance for complexity theory derives from Valiant’s discovery [44,45] that the evaluation of the permanent is a complete problem for the complexity class VNP (and also for the class $\#P$ in the model of Turing machines); see [3,33] for more information.

The determinant

$$\text{det}_n := \sum_{\pi \in S_n} \text{sgn}(\pi) X_{1\pi(1)} \cdots X_{n\pi(n)}$$

is known to have an efficient algorithm. Its evaluation is complete for the complexity class $\text{VP}_{ws}$; see [43,44]. From the definition it is clear that $\text{VP}_{ws} \subseteq \text{VNP}$ and proving the separation $\text{VP}_{ws} \neq \text{VNP}$ is the flagship problem in algebraic complexity theory. It can be seen as an “easier” version of the famous $P \neq \text{NP}$ problem; see [4].

The conjecture $\text{VP}_{ws} \neq \text{VNP}$ can be restated without any reference to complexity classes by directly comparing permanents and determinants. The determinantal complexity $\text{dc}(f)$ of a polynomial $f \in \mathbb{C}[X_1, \ldots, X_N]$ is defined as the smallest nonnegative integer $n \in \mathbb{N}$ such that $f$ can be written as a determinant of an $n \times n$ matrix whose entries are affine linear forms in the variables $X_i$. Valiant [44] proved that the determinant is computationally universal in the sense that $\text{dc}(f) \leq n$ if $f$ can be written as an arithmetic expression involving fewer than $n$ operations $+,-,\ast$. Moreover, Valiant [44,46] and Toda [43] proved that $\text{VP}_{ws} \neq \text{VNP}$ is equivalent to the following conjecture.

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Conjecture 1.1 (Valiant 1979). The determinantal complexity \( dc(\text{per}_n) \) grows superpolynomially in \( n \).

It is known \(^{20}\) that \( dc(\text{per}_n) \leq 2^n - 1 \). Finding lower bounds on \( dc(\text{per}_n) \) is an active area of research \(^{21,13,24,29,35,48}\), but the best known lower bounds are only \( \Omega(n^2) \).

1(a). An attempt via algebraic geometry and representation theory. Toward answering Conjecture 1.1 Mulmuley and Sohoni \(^{36,38}\) proposed an approach based on algebraic geometry and representation theory, for which they coined the name geometric complexity theory (GCT).

We denote by \( \text{Sym}^n V^* \) the space of homogeneous polynomial functions of degree \( n \) on a finite-dimensional complex vector space \( V \). The group \( G := \text{GL}(V) \) acts on \( \text{Sym}^n V^* \) in a canonical way: \( (g \cdot f)(v) := f(g^{-1}v) \) for \( g \in G, f \in \text{Sym}^n V^* \), \( v \in V \). We denote by \( G \cdot f := \{ gf \mid g \in G \} \) the orbit of \( f \). We assume now \( V := \mathbb{C}^{n \times n} \), view the determinant \( \det_n \) as an element of \( \text{Sym}^n V^* \), and consider its orbit closure,

\[
\Omega_n := \text{GL}_{n^2} \cdot \det_n \subseteq \text{Sym}^n(\mathbb{C}^{n \times n})^*
\]

with respect to Euclidean topology. (By a general principle, this is the same as the closure with respect to Zariski topology; see \(^{39} \S 2.C.\).) It is easy to see that \( \Omega_2 = \text{Sym}^2(\mathbb{C}^{2 \times 2})^* \). For \( n = 3 \), the boundary of \( \Omega_n \) has been determined recently \(^{22}\), but for \( n = 4 \) it is already unknown.

For \( n > m \) we consider the padded permanent defined as

\[
X_{11}^{n-m}\text{per}_m \in \text{Sym}^n(\mathbb{C}^{m \times m})^*,
\]

where \( X_{11} \) denotes the linear form providing the \( (1, 1) \)-entry of a matrix in \( \mathbb{C}^{m \times m} \). Via the standard projection \( \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m} \), we can view \( X_{11}^{n-m}\text{per}_m \) as an element of the bigger space \( \text{Sym}^n(\mathbb{C}^{n \times n})^* \). (Sometimes the padding is achieved by using a linear form different from, e.g., \( X_{11} \), but this is irrelevant; see \(^{26} \) Appendix.)

The following conjecture was stated in \(^{37}\). We refer to \(^{12} \) Prop. 9.3.2 for an equivalent formulation in terms of complexity classes that goes back to \(^{5}\). (In particular see \(^{5} \) Problem 4.3.)

Conjecture 1.2 (Mulmuley and Sohoni 2001). For all \( c \in \mathbb{N}_{>1} \) we have \( X_{11}^{n-m}\text{per}_m \not\in \Omega_m \) for infinitely many \( m \).

Conjecture 1.2 implies Conjecture 1.1. Indeed, using that \( \text{GL}_{n^2} \) is dense in \( \mathbb{C}^{n^2 \times n^2} \), one shows (e.g., see \(^{6}\)) that \( dc(\text{per}_m) \leq n \) implies \( X_{11}^{n-m}\text{per}_m \in \Omega_n \). (The latter must be a point in the boundary of \( \Omega_n \) if \( n > m \).)

The following strategy toward Conjecture 1.2 was proposed in \(^{37}\). The action of the group \( G = \text{GL}(V) \) on \( \text{Sym}^n V^* \) induces an action on its (graded) coordinate ring \( \mathbb{C}[\text{Sym}^n V^*] = \bigoplus_{d \in \mathbb{N}} \text{Sym}^d \text{Sym}^n V^* \). The homogeneous parts \( \text{Sym}^d \text{Sym}^n V^* \) are called plethysms; in fact, we obtain the natural \( G \)-action on these spaces. The coordinate ring \( \mathbb{C}[\Omega_n] \) of the orbit closure \( \Omega_n \) is obtained as the homomorphic image of \( \mathbb{C}[\text{Sym}^n V^*] \) via the restriction of regular functions, and the \( G \)-action descends on this. In particular, we obtain the degree \( d \) part \( \mathbb{C}[\Omega_n]_d \) of \( \mathbb{C}[\Omega_n] \) as the homomorphic \( G \)-equivariant image of \( \text{Sym}^d \text{Sym}^n V^* \).

It is well known \(^{18}\) that the irreducible polynomial representations of \( G \) can be labeled by partitions \( \lambda \) into at most \( n^2 \) parts. The coordinate ring \( \mathbb{C}[\Omega_n] \) is a
direct sum of its irreducible submodules since $G$ is reductive. We say that $\lambda$ occurs in $\mathbb{C}[\Omega_n]$ if it contains an irreducible $G$-module of type $\lambda$.

Let $Z_{n,m}$ denote the orbit closure of the padded permanent $(n > m)$,

$$Z_{n,m} := \text{GL}_{n^2} \cdot X_{11}^{n-m} \text{per}_m \subseteq \text{Sym}^n(\mathbb{C}^{n \times n})^*.$$  \hspace{1cm} (1.2)

If $X_{11}^{n-m} \text{per}_m$ is contained in $\Omega_n$, then $Z_{n,m} \subseteq \Omega_n$, and the restriction defines a surjective $G$-equivariant homomorphism $\mathbb{C}[\Omega_n] \to \mathbb{C}[Z_{n,m}]$ of the coordinate rings. Schur’s lemma implies that if $\lambda$ occurs in $\mathbb{C}[Z_{n,m}]$, then it must also occur in $\mathbb{C}[\Omega_n]$. A partition $\lambda$ violating this condition is called an occurrence obstruction. Its existence thus proves that $Z_{n,m} \not\subseteq \Omega_n$ and, hence, $\text{dc} (\text{per}_m) > n$. It is known that occurrence obstructions $\lambda$ must satisfy $|\lambda| = nd$ and $\ell (\lambda) \leq m^2$; see \[12\][37][38]. Here $|\lambda| := \sum \lambda_i$ denotes the size of $\lambda$ and $\ell (\lambda)$ denotes the length of $\lambda$, which is defined as the number of nonzero parts of $\lambda$. We write $\lambda \vdash |\lambda|$, so in our case $\lambda \vdash nd$.

In \[37][38] it was suggested to prove Conjecture \[\ref{12}] by exhibiting occurrence obstructions. More specifically, the following conjecture was put forth.

**Conjecture 1.3** (Mulmuley and Sohoni 2001). For all $c \in \mathbb{N} \geq 1$, for infinitely many $m$, there exists a partition $\lambda$ occurring in $\mathbb{C}[Z_{mc,m}]$ but not in $\mathbb{C}[\Omega_{mc}]$.

This conjecture implies Conjecture \[\ref{12}] by the above reasoning.

Conjecture \[\ref{1.3}] on the existence of occurrence obstructions has stimulated much research and has been the main focus of researchers in geometric complexity theory in the past years; see Section \[\ref{1(b)}\]. Unfortunately, this conjecture is false. This is the main result of this work. More specifically, we show the following.

**Theorem 1.4** (Main Theorem). Let $n,d,m$ be positive integers with $n \geq m^2$ and $\lambda \vdash nd$. If $\lambda$ occurs in $\mathbb{C}[Z_{n,m}]$, then $\lambda$ also occurs in $\mathbb{C}[\Omega_n]$. In particular, Conjecture \[\ref{1.3}] is false.

One can likely improve the bound on $n$ by more careful analysis.

1 (b). **Related work: Kronecker coefficients.** Kronecker coefficients are fundamental quantities that have been the object of study in algebraic combinatorics for a long time \[10\]. A difficulty in their study is that there is no known counting interpretation of them \[12\]. The **Kronecker coefficient** $k(\lambda, \mu, \nu)$ of three partitions $\lambda, \mu, \nu$ of the same size $d$ is defined as the dimension of the space of $S_d$-invariants of $[\lambda] \otimes [\mu] \otimes [\nu]$, where $[\lambda]$ denotes the irreducible $S_d$-module of type $\lambda$ and $S_d$ is the symmetric group on $d$ symbols; see \[32\] I§7, internal product). We write $n \times d$ for the rectangular partition $(d, \ldots, d)$ of size $nd$ and call $k_n(\lambda) := k(\lambda, n \times d, n \times d)$ the **rectangular Kronecker coefficient** of $\lambda \vdash nd$. In \[25\] it was shown that deciding positivity of Kronecker coefficients in general is NP-hard, but this proof fails for rectangular formats.

Let $K_n(\lambda)$ denote the multiplicity by which the irreducible $\text{GL}_{n^2}$-module of type $\lambda \vdash nd$ occurs in $\mathbb{C}[\Omega_n]$. We call the numbers $K_n(\lambda)$ the **GCT-coefficients**. In \[37\] it was realized that GCT-coefficients can be upper bounded by rectangular Kronecker coefficients: we have $K_n(\lambda) \leq k_n(\lambda)$ for $\lambda \vdash nd$. In fact, the multiplicity of $\lambda$ in the coordinate ring of the orbit $\text{GL}_{n^2} \cdot \det_n$ equals the so-called symmetric rectangular Kronecker coefficient \[12\], which is upper bounded by $k_n(\lambda)$.

Note that an occurrence obstruction for $Z_{n,m} \not\subseteq \Omega_n$ is a partition $\lambda$ for which $K_n(\lambda) = 0$ and such that $\lambda$ occurs in $\mathbb{C}[Z_{n,m}]$. Since hardly anything was known
about the GCT-coefficients, it was proposed in [37] to find \( \lambda \) for which the Kronecker coefficient \( k_n(\lambda) \) vanishes and such that \( \lambda \) occurs in \( \mathbb{C}[Z_{n,m}] \). This potential method of proving Conjecture 1.3 has stimulated research in algebraic combinatorics on these quantities. However, the recent article of Ikenmeyer and Panova [26] showed that proving Conjecture 1.3 in this way is not possible. Our work is greatly inspired by [26], even though it differs vastly in the details and techniques from the present paper (see §2).

Before [26] there were several papers showing that asymptotic considerations cannot resolve Conjecture 1.3. A first attempt toward finding occurrence obstructions was by asymptotic considerations (moment polytopes): for Kronecker coefficients this was ruled out in [8], where it was proven that for all \( \lambda \vdash nd \) there exists a stretching factor \( \ell \geq 1 \) such that \( k_n(\ell \lambda) > 0 \). Kumar [28] ruled out asymptotic considerations for the GCT-coefficients: assuming the Alon–Tarsi conjecture [1], he showed that \( K_n(n\lambda) > 0 \) for all \( \lambda \vdash nd \) and even \( n \). A similar conclusion, unconditional, although with less information on the stretching factor, was obtained in [9].

1(c). Future directions. While our main result, Theorem 1.4, rules out the possibility of proving Conjecture 1.2 via occurrence obstructions, there still remains the possibility that one may succeed by comparing multiplicities. If the orbit closure \( Z_{n,m} \) of the padded permanent \( \per_m \) is contained in \( \Omega_n \), then the restriction defines a surjective \( G \)-equivariant homomorphism \( \mathbb{C}[\Omega_n] \to \mathbb{C}[Z_{n,m}] \) of the coordinate rings, and hence the multiplicity of the type \( \lambda \) in \( \mathbb{C}[Z_{n,m}] \) is bounded from above by the GCT-coefficient \( K_n(\lambda) \). Thus, proving that \( K_n(\lambda) \) is strictly smaller than the latter multiplicity implies that \( Z_{n,m} \not\subseteq \Omega_n \). We note that the paper [16] rules out one natural asymptotic method for achieving this. Mulmuley pointed out to us a paper by Larsen and Pink [31] that is of potential interest in this connection. A preliminary version of this paper appeared as [11].

2. Overview of proof

The proof of our main result, Theorem 1.4, is intricate and combines various new techniques. We present below the main ingredients and outline the structure of the proof. Moreover, we shall discuss two aspects of our method, that we believe are of independent interest: fundamental invariants of forms and the lifting of highest weight vectors in plethysms.

2(a). Outline and ingredients. The only information we exploit about the orbit closure \( Z_{n,m} \) of the padded permanent (recall (1.2)) is an insight due to Kadish and Landsberg [27], which was also crucially used in Ikenmeyer and Panova [26]. Before stating it, we introduce the notion of the body \( \body \) of a partition \( \lambda \), which is obtained by deleting the first component of \( \lambda \) or, pictorially, by removing the first row in its diagram.

**Theorem 2.1** ([27]). If \( \lambda \vdash nd \) occurs in \( \mathbb{C}[Z_{n,m}]_d \), then \( \ell(\lambda) \leq m^2 \) and \( |\body| \leq md \).

This means that if \( \lambda \vdash nd \) occurs in \( \mathbb{C}[Z_{n,m}]_d \), then \( \lambda \) must have a very long first row if \( n \) is substantially larger than \( m \). (Note that \( |\body| \leq md \) is equivalent to \( \lambda_1 \geq (n - m)d \).)

It is remarkable that Proposition 2.1 still holds if the permanent \( \per_m \) is replaced by any homogeneous polynomial of degree \( m \) in \( m^2 \) variables: this can be checked...
by tracing the proof that we provide in Section 4(b). So no specific property of the permanent is used: only the padding (multiplication with a power of a linear form) is relevant. Let us point out the recent paper [19] which indicates that in other models of computation also expressing VP \(\neq\) VNP, but avoiding the padding (trace of matrix powers), no lower bounds can be obtained via occurrence obstructions.

For the proof of Theorem 1.4 we need to show that many partitions \(\lambda\) occur in \(\mathbb{C}[\Omega_n]\). For this we shall establish the occurrence of certain basic shapes in \(\mathbb{C}[\Omega_n]\). Then we get more shapes by the following semigroup property.

**Lemma 2.2.** If \(\lambda\) occurs in \(\mathbb{C}[\Omega_n]\) and \(\mu\) occurs in \(\mathbb{C}[\Omega_n]\), then \(\lambda + \mu\) occurs in \(\mathbb{C}[\Omega_n]\).

**Proof.** \(\lambda\) occurs in \(\mathbb{C}[\Omega_n]\) iff \(\mathbb{C}[\Omega_n]\) contains a highest weight vector of weight \(\lambda\) (see Section 3(b) for basics on highest weight vectors). Moreover, it is immediate that the product of a highest weight vector of weight \(\lambda\) with a highest weight vector of weight \(\mu\) is a highest weight vector of weight \(\lambda + \mu\). Nonzeroness of this product follows from the irreducibility of the variety \(\Omega_n\). \(\square\)

For the basis shapes it turns out to be sufficient to take rectangular diagrams to which there are appended a first column and a (long) first row (see Theorem 6.2). See Proposition 6.1 for details. Fortunately, this case can be dealt with by essentially the same techniques as for Proposition 2.4.

**Proposition 2.3.** Let \(n \geq k\ell\), and let \(\ell\) be even. Then \((k \times \ell)^{\sharp n k}\) occurs in \(\mathbb{C}[\Omega_n]_k\).

The strategy of the proof of the Main Theorem 1.4 is now as follows: Suppose we are given an even \(\lambda \vdash nd\) such that \(n \geq m^{25}\) and \(\lambda\) occurs in \(\mathbb{C}[Z_{n,m}]\). By Proposition 2.1 we have \(\ell(\lambda) \leq m^2\) and \(|\lambda| \leq md\).

We distinguish two cases. If the degree \(d\) is large (say \(d \geq 24m^6\)), we proceed as in [20]. We decompose the body \(\bar{\lambda}\) of \(\lambda\) into a sum of even rectangles \(k \times \ell\) (i.e., \(\ell\) is even). Since \(n\) and \(d\) are sufficiently large in comparison with \(m\), it turns out that we can write \(\lambda\) as a sum of row extended rectangles \((k \times \ell)^{\sharp nk}\), where \(n \geq k\ell\). By Proposition 2.3 all \((k \times \ell)^{\sharp nk}\) occur in \(\mathbb{C}[\Omega_n]_k\). The semigroup property then implies that \(\lambda\) occurs in \(\mathbb{C}[\Omega_n]_d\). (See Proposition 6.3 for more details.)

If the degree \(d\) is small, we rely on the following result. We again assume that \(V = \mathbb{C}^{n \times n}\).

**Proposition 2.4.** Let \(\lambda \vdash nd\) be such that there exists a positive integer \(m\) satisfying \(|\lambda| \leq md\) and \(md^2 \leq n\). Then every highest weight vector of weight \(\lambda\) in \(\text{Sym}^d\text{Sym}^m V\), viewed as a degree \(d\) polynomial function on \(\text{Sym}^n V^*\), does not vanish on \(\Omega_n\). In particular, if \(\lambda\) occurs in \(\text{Sym}^d\text{Sym}^m V\), then \(\lambda\) occurs in \(\mathbb{C}[\Omega_n]_d\).

In fact, in order to treat the general case of noneven partitions \(\lambda\), we need to make a further case distinction to treat separately the case \(|\lambda| < m^{10}\) of a very small body (or extremely long first row). See Proposition 6.4 for details. Fortunately, this case can be dealt with by essentially the same techniques as for Proposition 2.4.
In order to prove the above two propositions, we exploit very little information on the orbit closure $\Omega_n$ of the determinant. The only property we use is that $\Omega_n$ contains many padded power sums. Here is the formal statement.

**Theorem 2.5.** Let $X, \varphi_1, \ldots, \varphi_k$ be linear forms on $\mathbb{C}^{n \times n}$ and assume $n \geq sk$. Then the power sum $X^{n-s}(\varphi_1^s + \cdots + \varphi_k^s)$ of $k$ terms of degree $s$, padded to degree $n$, is contained in $\Omega_n$.

**Proof.** The case $n = 1$ is trivial, so we assume $k < n^2$ without loss of generality. Let $X_1, \ldots, X_n$ denote the standard basis of $(\mathbb{C}^{n \times n})^*$. Writing the power sum $X_1^s + \cdots + X_n^s$ as a formula requires at most $(s-1)k + k - 1 = sk - 1$ many additions and multiplications. Valiant’s construction [14] implies that $X_1^s + \cdots + X_n^s$ has the determinantal complexity at most $sk \leq n$, i.e., it can be written as the determinant of an $n \times n$ matrix with affine linear entries in $X_1, \ldots, X_n$. By substituting $X_i$ with $X_i/X_{k+1}$ and multiplying with $X_{k+1}^n$, we see that $X_{k+1}^n(X_1^s + \cdots + X_n^s)$ equals the determinant of an $n \times n$ matrix with homogeneous linear entries in $X_1, \ldots, X_{k+1}$. It follows that $X_{n-s}(X_1^s + \cdots + X_n^s)$ lies in $\Omega_n$. Let now $X, \varphi_1, \ldots, \varphi_k$ be linear forms in $(\mathbb{C}^{n \times n})^*$. The assertion follows by applying a linear map sending $X_{k+1}$ to $X$ and $X_i$ to $\varphi_i$ for $1 \leq i \leq k$. \hfill \Box

We next discuss the ideas underlying the proof of Proposition 2.3 and Proposition 2.4.

2 (b). **Generic fundamental invariants of tensors.** A theorem by Howe [21] tells us about the smallest degree $d$ such that the plethysm $\text{Sym}^d\text{Sym}^n C^N$ has nonzero $\text{SL}_N$-invariants:

$$\dim (\text{Sym}^d\text{Sym}^n C^N)^{\text{SL}_N} = \begin{cases} 0 & \text{if } d < N, \\ 1 & \text{if } d = N \text{ and } n \text{ is even}, \\ 0 & \text{if } d = N \text{ and } n \text{ is odd}. \end{cases}$$

So if $n$ is even, there is (up to scaling) a unique nonzero invariant $F_{n,N} \in \text{Sym}^N\text{Sym}^n C^N$. This invariant was already known to Cayley [15], and it is sometimes called a hyperdeterminant. We prefer to call it the fundamental invariant of $\text{Sym}^n C^N$; see [10] eq. (3.8)] for an explicit formula and generalizations.

Let $V = \mathbb{C}^N$, and denote by $e_1, \ldots, e_N$ the standard basis of $V$ and by $X_1, \ldots, X_N$ be the basis of $V^*$ dual to the standard basis of $V$. We interpret $F_{n,N}$ as a polynomial function on the space $\text{Sym}^n V^*$. The basic observation is as follows: if some $p \in \text{Sym}^n V^*$ satisfies $F_{n,N}(p) \neq 0$, then the rectangular partition $d \times n$ occurs in the orbit closure of $\text{GL}_N \cdot p$. Indeed, the restriction of coordinate rings $\mathbb{C}[\text{Sym}^n V^*] \to \mathbb{C}[\text{GL}_N \cdot p]$ maps $F_{n,N}$ to a nonzero function and, hence, the $d \times n$-isotypical component of $\mathbb{C}[\text{GL}_N \cdot p]$ is nonzero.

Implementing this idea is not as easy as it may look. For instance, consider $p = X_1 \cdots X_n$ for even $n$. Then $F_{n,N}(p) \neq 0$ turns out to be equivalent to the Alon–Tarsi conjecture [1], which states that the number of column-even Latin squares of size $n$ is different from the number of column-odd Latin squares of size $n$ (see [10] [28]). This conjecture is still open.

However, for the power sum $p = c_1 X_1^n + \cdots + c_N X_N^n$ with nonzero coefficients $c_i$, one can show that $F_{n,N}(p) \neq 0$; see [10] Thm. 3.19. This is based on a technique introduced in [7]. An extension of this method leads to the proof of Proposition 2.3 which is provided in Section 6 (b).
2 (c). **Lifting of highest weight vectors.** The proof of Proposition 2.4 has led us to insights that are of independent interest and that we are going to discuss now. Consider the plethysm $\text{Sym}^d \text{Sym}^m V$, where again $V = \mathbb{C}^N$. The multiplicity of the irreducible $\text{GL}_N$-module of type $\mu \vdash dm$ in $\text{Sym}^d \text{Sym}^m V$ equals the dimension of its space $\text{HWV}_\mu(\text{Sym}^d \text{Sym}^m V)$ of highest weight vectors of weight $\mu \vdash dm$; see Section 3(b) for these notions. The known stability property of plethysms ([47, Cor. 1.8], [14]) can be expressed as

$$\dim \text{HWV}_\mu(\text{Sym}^d \text{Sym}^m V) = \dim \text{HWV}_{\mu^{\leq dn}}(\text{Sym}^d \text{Sym}^n V)$$

for $\mu \vdash md$, $n \geq m$, provided $\mu_2 \leq m$. Recall that $\mu^{\leq dn}$ denotes the shape obtained from $\mu$ by adding $d(n - m)$ boxes to its first row.

We deepen our understanding of this numerical identity by constructing an explicit injective linear lifting map between the corresponding spaces

$$\kappa^d_{m,n} : \text{Sym}^d \text{Sym}^m V \rightarrow \text{Sym}^d \text{Sym}^n V,$$

which maps highest weight vectors of weight $\mu \vdash md$ to highest weight vectors of weight $\mu^{\leq dn}$. The map $\kappa^d_{m,n}$ arises as the $d$-fold symmetric power of the linear map $M : \text{Sym}^m V \rightarrow \text{Sym}^n V$, $p \mapsto pe_1^{n-m}$, which is the multiplication with $e_1^{n-m}$.

We show how to label a system of generators $v_T$ of $\text{HWV}_\mu(\text{Sym}^d \text{Sym}^m V)$ by tableaux $T$ of shape $\mu$ with content $d \times m$ (see Section 4(a)). Similar to [7, 23], we work out a combinatorial rule for contracting $v_T$ with a tensor of rank 1, which is a technical tool used throughout our developments (see Theorem 4.7).

The lifting of generators turns out to have a beautifully simple combinatorial description in terms of the tableaux; see Theorem 5.5. This also leads to a proof of the stability property of plethysms that is different from the proofs in [14, 47]. As a consequence we obtain the following insight, which is crucial for our purposes. If $\lambda \vdash nd$ satisfies $\lambda_2 + |\lambda| \leq md$, then every highest weight vector of weight $\lambda$ in the right-hand side of (2.1) is obtained by lifting a highest weight vector in the left-hand side.

In order to understand the effect of liftings with regard to polynomial evaluation, we use duality to show that $\langle \kappa^d_{m,n}(f), q^d \rangle = \langle f, M^*(q)^d \rangle$ for $f \in \text{Sym}^d \text{Sym}^m V$, and $q \in \text{Sym}^n V^*$; see Theorem 5.3. Here $M^* : \text{Sym}^m V^* \rightarrow \text{Sym}^n V^*$ denotes the dual map of $M$. It turns out that $M^*(q)$ essentially equals the $(n-m)$-fold partial derivative of $q$ in the direction of $e_1$. The proof of Proposition 2.4 which is given in Section 6(a) combines all the statements mentioned so far, plus Proposition 3.2.

2 (d). **No occurrence obstructions for Waring rank.** We shed further light on the method of occurrence obstructions by investigating what happens when replacing $\det_n$ by the sum of $n$th powers of linear forms, thus inquiring about the Waring rank of polynomials. The Waring rank is a considerably weaker model of computation, which can be seen from the fact that for this complexity measure, exponential lower bounds for the permanent can be proven. More specifically, the Waring rank $R(p)$ of a polynomial $p \in \text{Sym}^n V^*$ is defined as the minimum $r$ such that there exists a representation $p = \varphi_1^n + \cdots + \varphi_r^n$ as a sum of $r$ powers of linear forms $\varphi_i \in V^*$. One can show that $R(\det_n)$ and $R(\text{per}_n)$ are both below by $\left(\begin{array}{c}n \\ [n/2]\end{array}\right)^2$; see [30].
Before proceeding, we note that \( R(X^aY^b) \leq b + 1 \) if \( a < b \), which follows from the identity
\[
\sum_{j=0}^{b-1} (X + \zeta^j Y)^{a+b} = bX^{a+b} + b\binom{a+b}{a} X^a Y^b,
\]
where \( \zeta \) is a primitive \( b \)th root of unity. (We can even do better: if \( \omega \) is a primitive \( a \)th root of unity, the identity
\[
\sum_{j=0}^{a-1} (\varepsilon \omega^j X + Y)^{a+b} = Y^{a+b} + a\binom{a+b}{a} X^a Y^b \varepsilon^a + O(\varepsilon^{a+1})
\]
shows that \( X^a Y^b \) is a limit of polynomials with Waring rank at most \( a + 1 \).

One may think of proving lower bounds on the Waring rank by studying the orbit closure
\[
\text{PS}_n := \text{GL}_{n^2} \cdot (X_1^n + \cdots + X_n^n) \subseteq \text{Sym}^n(\mathbb{C}^n)^2
\]
of the power sum with \( n^2 \) terms. Indeed, suppose that \( p \in \text{Sym}^m V^* \) satisfies \( R(p) \leq n \) for some \( n > 2m \). Consider the padded power sum \( X^{n-m} p \), where \( X \in V^* \). It satisfies \( R(X^{n-m} p) \leq n^2 \), since we showed \( R(X^{n-m} X_i^m) \leq n - m + 1 \leq n \) above. Therefore, \( X^{n-m} p \in \text{PS}_n \). Thus a possible strategy for showing lower bounds on \( R(p) \) could be to disprove that the padded polynomial \( X^{n-m} p \) is contained in \( \text{PS}_n \). By analogy, replacing \( \Omega_n \) with the simpler orbit closure \( \text{PS}_n \), we may ask whether the separation can be achieved using occurrence obstructions. Unfortunately, the answer turns out to be no!

The only information used about the orbit closure of the determinant \( \Omega_n \) in the proof of our main result is that it contains certain padded power sums (Theorem 2.5). More specifically, for the proof of Theorem 1.1, we consider highest weights \( \lambda \) occurring in \( \mathbb{C}[\Omega_n]_d \) for different values of \( d \)—Proposition 2.4 and Proposition 6.4. In both cases we show that each highest weight vector \( f \in \text{Sym}^d \text{Sym}^n V \) of weight \( \lambda \) of interest appears in \( \mathbb{C}[\Omega_n]_d \) by showing that there is a corresponding power sum \( p = \varphi_1^k + \cdots + \varphi_k^k \) on which \( f \) does not vanish. In Proposition 2.4 we have \( k = d < n, s = md < n \), and in Proposition 6.4 we have \( k = m^2 s, m^2 s^2 \leq n \). For all values of \( d \) and the corresponding \( k \), we have that \( R(p) \leq k \) and \( R(X_1^{n-s} p) \leq nk \leq n^2 \), so \( X_1^{n-s} p \in \text{PS}_n \), and hence, \( \lambda \) occurs in \( \mathbb{C}[\text{PS}_n]_d \). Hence we can replace \( \Omega_n \) by \( \text{PS}_n \), and we obtain the following result.

**Corollary 2.6.** Let \( n, d, m \) be positive integers with \( n \geq m^{25} \) and \( \lambda \vdash nd \). If \( \lambda \) occurs in \( \mathbb{C}[Z_{n,m}] \), then \( \lambda \) also occurs in \( \mathbb{C}[\text{PS}_n] \).

Recall that in this statement, \( Z_{n,m} \) is the orbit closure of the padded permanent \( X^{n-m} \text{per}_m \); see (12). Tracing the proof reveals that the permanent can be replaced by any homogeneous polynomial \( p \) of degree \( m \) in \( m^2 \) variables.

So we obtain a dramatic result: the strategy of occurrence obstructions cannot even be used in the weaker model of \( \text{PS}_n \) against padded polynomials.

### 3. Preliminaries

3 (a). **Symmetric powers.** We refer to [11] for background on multilinear algebra. Let us begin with some notational conventions. Let \( V \) be a finite-dimensional complex vector space. We write \( v_i \) for vectors in \( V \) and \( \varphi_j \) for linear forms in \( V^* \), and we write \( \langle v_i, \varphi_j \rangle \) or \( \langle \varphi_j, v_i \rangle \) for the contraction \( \varphi_j(v_i) \) of \( v_i \) with \( \varphi_j \). If \( V = \mathbb{C}^N \), then \( e_1, \ldots, e_N \) denotes the standard basis of \( V \), and we denote by \( X_1, \ldots, X_N \) the
basis of $V^*$ dual to it, so that $\langle e_i, X_j \rangle = \delta_{ij}$. A basis of the $d$th tensor power $\bigotimes^d V$ is given by the tensors $e_I := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$, where $I$ runs over all tuples $I = (i_1, \ldots, i_d) \in [N]^d$, where we write $[N] := \{1, 2, \ldots, N\}$. Similarly, we define the dual basis $X_I$ of $\bigotimes^d V^*$ satisfying $\langle e_I, X_J \rangle = \delta_{I,J}$.

The symmetric group $\mathfrak{S}_d$ on $d$ symbols acts on the $d$th tensor power $\bigotimes^d V$ by permuting the factors. The $d$th symmetric power $\text{Sym}^d V$ is the subspace of $\bigotimes^d V$ consisting of the $\mathfrak{S}_d$-invariant tensors. It is obtained as the image of the symmetrizing projection

$$
\Pi_d := \frac{1}{d!} \sum_{\pi \in \mathfrak{S}_d} \pi
$$

which we can view as an element of the the group algebra $\mathbb{C}[\mathfrak{S}_d]$. The multiplication maps

$$
\text{Sym}^d V \times \text{Sym}^d V \to \text{Sym}^{d_1+d_2} V,
$$

turn $\text{Sym} V := \bigoplus_{d \in \mathbb{N}} \text{Sym}^d V$ into an associative graded $\mathbb{C}$-algebra on which $\text{GL}(V)$ acts by algebra automorphisms. We note that the construction of the symmetric power is functorial: a linear map $\Phi : V \to W$ of finite-dimensional $\mathbb{C}$-vector spaces defines a unique linear map $\text{Sym}^d \Phi : \text{Sym}^d V \to \text{Sym}^d W$ satisfying $(\text{Sym}^d \Phi)(v^d) = \Phi(v)^d$. Moreover, $\text{Sym}^d(\Psi \circ \Phi) = \text{Sym}^d \Psi \circ \text{Sym}^d \Phi$ for linear maps $\Psi, \Phi$ that can be composed. In particular, the symmetric product of $d$ vectors $v_1, \ldots, v_d \in V$ is defined as $v_1 \cdots v_d := \Pi_d(v_1 \otimes \cdots \otimes v_d)$. It is well known that $\text{Sym}^d V$ is spanned by the $d$th powers $v^d = v \otimes^d v$.

We can identify a symmetric tensor $p \in \text{Sym}^d V^*$ with the homogeneous polynomial of degree $d$ given by $V \to \mathbb{C}, v \mapsto \langle p, v^d \rangle$: one calls $p$ the polarization of the corresponding homogeneous polynomial. This way, we can view $\text{Sym} V^*$ as the $\mathbb{C}$-algebra of polynomials in $N$ variables. The interpretation of homogeneous degree $d$ polynomials in the variables $X_i$ as symmetric tensors in $\text{Sym}^d V^*$ is highly essential for our work.

For $\alpha \in \mathbb{N}^N$ with $|\alpha| = d$, we define the monomial

$$
e^\alpha := e_1^{\alpha_1} \cdots e_N^{\alpha_N} = \Pi_d(e_1^{\otimes \alpha_1} \otimes \cdots \otimes e_N^{\otimes \alpha_N}) = \frac{1}{d!} \sum_{\pi \in \mathfrak{S}_d} \pi e_1^{\otimes \alpha_1} \otimes \cdots \otimes e_N^{\otimes \alpha_N} = \frac{1}{d!} \sum_{I \in [N]^d} e_I,
$$

where the sum is over all tuples $I \in [N]^d$ with the frequency $\alpha$; i.e., $\alpha_k$ is the number of occurrences of $k$ in $I$. It is well known that the $e^\alpha$ form a basis of $\text{Sym}^d V$. Similarly, the $X^\alpha := X_1^{\alpha_1} \cdots X_N^{\alpha_N}$ form a basis of $\text{Sym}^d V^*$. A straightforward calculation shows that

$$
\langle e^\alpha, X^\beta \rangle = \frac{1}{d!} \delta_{\alpha, \beta},
$$

where $\binom{d}{\alpha}$ denotes the multinomial coefficient. Therefore, $(\binom{d}{\beta} X^\beta)$ is the dual basis of $(e^\alpha)$.

Let $p \in \text{Sym}^d V^*$ with $d \geq 1$ and $v \in V$. Viewing $p$ as a polynomial function on $V$, there is a well-defined directional derivative of $p$ at $u \in V$ in direction $v$,

$$
\partial_v p(u) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} (p(u + \epsilon v) - p(u)).
$$
This defines $\partial_v p \in \text{Sym}^{d-1} V^*$. In coordinates, we have $\partial_v p = \sum v_i \frac{\partial p}{\partial X_i}$, where $v = \sum v_i e_i$. So for fixed $v$, we obtain a linear map $\partial_v : \text{Sym}^d V^* \to \text{Sym}^{d-1} V^*$, $p \mapsto \partial_v p$, which we define to be 0 if $d = 0$. We denote by $\partial^k_v$ the $k$-fold composition of $\partial_v$.

After recalling these general facts, we present now a useful lemma on the evaluation of polynomials at points of low rank.

**Lemma 3.1.** Let $W$ be a finite-dimensional $\mathbb{C}$-vector space, and let $p \in \text{Sym}^d W^*$ such that $\langle p, (\sum_{j=1}^r w_j)^d \rangle \neq 0$ for some $w_1, \ldots, w_r \in W$. Then there exists $J \subseteq [r]$ with $|J| \leq d$ and $\langle p, (\sum_{j \in J} w_j)^d \rangle \neq 0$.

**Proof.** By multilinearity we have

$$0 \neq \langle p, (\sum_{j=1}^r w_j)^d \rangle = \sum_{j_1, \ldots, j_d} \langle p, w_{j_1} \otimes \cdots \otimes w_{j_d} \rangle,$$

hence there exist $j_1, \ldots, j_d \in [r]$ such that $\langle p, w_{j_1} \otimes \cdots \otimes w_{j_d} \rangle \neq 0$. The polarization formula [17, p. 5] gives

$$(3.2) \quad \langle p, w_{j_1} \otimes \cdots \otimes w_{j_d} \rangle = \frac{1}{d!} \sum_{I \subseteq [d]} (-1)^{|I|} \langle p, (\sum_{i \in I} w_{j_i})^d \rangle.$$

Hence there must be a nonzero contribution for some $I$, and the assertion follows. \hfill $\Box$

**Proposition 3.2.** Let $V$ be a finite-dimensional $\mathbb{C}$-vector space, and let $d, n \geq 1$. If $f \in \text{Sym}^d \text{Sym}^n V$ is nonzero, then $\langle f, (\varphi_1^n + \cdots + \varphi_d^n)^d \rangle \neq 0$ for Zariski almost all $(\varphi_1, \ldots, \varphi_d) \in (V^*)^d$. This means that $f$, viewed as a homogeneous polynomial function of degree $d$ on $\text{Sym}^n V^*$, does not vanish on almost all power sums $\varphi_1^n + \cdots + \varphi_d^n$ with $d$ terms.

**Proof.** Let $w \in \text{Sym}^n V^*$ be such that $\langle f, w^d \rangle \neq 0$. There exist $r \in \mathbb{N}$ and $\psi_1, \ldots, \psi_r \in V^*$ such that $w = \psi_1^n + \cdots + \psi_r^n$. By Lemma 3.1 there are $1 \leq j_1, \ldots, j_d \leq r$ such that $\langle f, (\psi_{j_1}^n + \cdots + \psi_{j_d}^n)^d \rangle \neq 0$. Therefore, the open set $\{ (\varphi_1, \ldots, \varphi_d) \in (V^*)^d | \langle f, (\varphi_1^n + \cdots + \varphi_d^n)^d \rangle \neq 0 \}$ is nonempty and, hence, Zariski dense. \hfill $\Box$

3 (b). **Basics on highest weight vectors.** We refer to [18] for more details and proofs for the following basic notions and facts.

A partition $\lambda$ is a nondecreasing finite sequence of nonnegative integers $(\lambda_1, \ldots, \lambda_N)$. It can be visualized as a Young diagram, which is a finite collection of boxes, arranged in left-justified rows, with $\lambda_i$ boxes in the $i$th row. Depending on the context, we also write $\lambda$ for the set of boxes of the diagram. Recall that $|\lambda| := \sum \lambda_i$ is the size of $\lambda$, which is the number of the boxes of the Young diagram, and $\ell(\lambda)$ is the length of $\lambda$, which is defined as the number of its nonzero parts $\lambda_i$. We briefly write $\lambda \vdash D$ to express that $\lambda$ is a partition of size $D$. The body $\bar{\lambda}$ of $\lambda$ is obtained from $\lambda$ by removing its first row. We denote by $k \times \ell$ the rectangular diagram with $k$ rows of length $\ell$, so it has size $k\ell$. In partition notation, $k \times \ell = (\ell, \ldots, \ell)$ with $\ell$ appearing $k$ times.

Recall that $\mathfrak{S}_D$ denotes the symmetric group on $D$ symbols. It is well known that the irreducible $\mathfrak{S}_D$-modules can be encoded by partitions $\lambda \vdash D$ of size $D$. They are called Specht modules, and we shall denote them by $[\lambda]$. 
We denote by $U_N \subseteq \text{GL}_N(\mathbb{C})$ the subgroup of upper triangular matrices with ones on the main diagonal. Moreover, let $\text{diag}(\alpha_1, \ldots, \alpha_N)$ denote the diagonal matrix with entries $\alpha_i$ on the diagonal. Suppose that $\mathcal{Y}$ is a rational $\text{GL}_N(\mathbb{C})$-module. A nonzero vector $f \in \mathcal{Y}$ is called a highest weight vector of weight $\lambda \in \mathbb{Z}^N$ iff $f$ is $U_N$-invariant (i.e., $u \cdot f = f$ for all $u \in U_N$) and $f$ is a weight vector of weight $\lambda$ (i.e., $\text{diag}(\alpha_1, \ldots, \alpha_N) \cdot f = \alpha_1^{\lambda_1} \cdots \alpha_N^{\lambda_N} f$ for all $\alpha_i \in \mathbb{C}^\times$). We remark that necessarily $\lambda_1 \geq \cdots \geq \lambda_N$, so that $\lambda$ is a partition if its entries are nonnegative.

We denote by $\text{HWV}_\lambda(\mathcal{Y})$ the vector space of highest weight vectors of weight $\lambda$. An irreducible $\text{GL}_N(\mathbb{C})$-module $\mathcal{Y}$ is called a Schur–Weyl module. It is known that there is a unique $\lambda$ such that $\text{HWV}_\lambda(\mathcal{Y})$ is one dimensional. Moreover, $\text{HWV}_\mu(\mathcal{Y}) = 0$ for all $\mu \neq \lambda$. We call $\lambda$ the type of the Schur–Weyl module $\mathcal{Y}$ and shall abbreviate $\mathcal{Y}$ by the symbol $\{\lambda\}$. Isomorphic $\text{GL}_N(\mathbb{C})$-modules have the same type.

In the following we assume $V := \mathbb{C}^N$ and denote by $e_1, \ldots, e_N$ the standard basis of $V$. The group $\text{GL}(V)$ acts on the $D$th tensor power $\otimes^D V$ by $g(v_1 \otimes \cdots \otimes v_D) = (gv_1) \otimes \cdots \otimes (gv_D)$, and the group $\mathfrak{S}_D$ acts by permuting the factors. Since these actions commute, we have an action of $\text{GL}(V) \times \mathfrak{S}_D$ on $\otimes^D V$.

We next explain how to construct highest weight vectors in $\otimes^d V$. Let $\lambda \vdash D$ and $\mu$ denote the transpose of $\lambda$, so $\mu_i$ denotes the number of boxes in the $i$th column of $\lambda$. For $j \leq N$, we note that $v_{j \times 1} := e_1 \wedge e_2 \wedge \cdots \wedge e_j \in \Lambda^j V$ is a highest weight vector of weight $j \times 1$. We now define

$$(3.3) \quad v_\lambda := v_{\mu_1 \times 1} \otimes \cdots \otimes v_{\mu_N \times 1} \in \otimes^D V. $$

It is easy to check that $v_\lambda$ is a highest weight vector of weight $\lambda$.

**Proposition 3.3.** Let $\lambda \vdash D$. Then the vector space $\text{HWV}_\lambda(\otimes^D V)$ is spanned by the $\mathfrak{S}_D$-orbit of $v_\lambda$.

**Proof.** Schur–Weyl duality provides a $\text{GL}(V) \times \mathfrak{S}_D$-isomorphism

$$\otimes^D V \simeq \bigoplus_{\lambda \vdash D} \{\lambda\} \otimes [\lambda].$$

Recalling that $\text{HWV}_\lambda(\{\lambda\})$ is one dimensional, we see that $\text{HWV}_\lambda(\otimes^D V)$ is isomorphic to $[\lambda]$ as an $\mathfrak{S}_D$-module. It follows that $\text{HWV}_\lambda(\otimes^D V)$ is spanned by the $\mathfrak{S}_D$-orbit of any of its nonzero elements. \hfill $\square$

We analyze now the tensors $v_\lambda$ in more detail. A Young tableau of shape $\lambda \vdash D$ is a filling of the boxes of the diagram $\lambda$ with numbers. We shall assume that each of the numbers $1, \ldots, D$ occurs exactly once, so that we obtain an enumeration of the boxes. The column-standard Young tableau $T^\text{std}_\lambda$ of shape $\lambda$ is the Young tableau of shape $\lambda$ that contains the numbers $1, \ldots, D$ ordered columnwise, from top to bottom and left to right. For example,

$$T^\text{std}_{(4,2)} = \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 \end{bmatrix}$$

is column-standard. The symmetric group $\mathfrak{S}_D$ acts on the set of Young tableaux of the diagram $\lambda$ by replacing each entry $i$ with $\pi(i)$. For example, for $\pi = (2453)$, we obtain

$$(3.4) \quad \pi T^\text{std}_{(4,2)} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 \end{bmatrix}. $$
Describing $\pi v_\lambda$ in terms of the permutation $\pi$ has redundancies that can be read off the tableau $\pi T^\text{std}_\lambda$. Namely, the following is an easy consequence of the definition of $v_\lambda$ in (3.3). For a transposition $\tau = (i \ j)$, we have
\begin{equation}
\tau \pi v_\lambda = -\pi v_\lambda \quad \text{if } i \text{ and } j \text{ are in the same column in } \pi T^\text{std}_\lambda.
\end{equation}
Moreover, if $\tau$ is a permutation that switches two columns of the same length in $\pi T^\text{std}_\lambda$, then
\begin{equation}
\tau \pi v_\lambda = \pi v_\lambda.
\end{equation}

4. Plethysms

We partition the position set $[dn] := \{1, \ldots, dn\}$ into the blocks $B_1, \ldots, B_d$, where $B_u := \{(u-1)n + v \mid 1 \leq v \leq n\}$. The subgroup of $S_{dn}$ of permutations that preserve the partition into blocks is called the wreath product $S_d \wr S_n$. It is generated by the permutations leaving the blocks invariant and the permutations of the form $(u-1)n + v \mapsto (\tau(u)-1)n + v$ with $\tau \in S_d$, which simultaneously permute the blocks. Structurally, the wreath product is a semidirect product $S_d \wr S_n \simeq (S_n)^d \rtimes S_d$. Note that its order equals $d! n!^d$. Symmetrizing over $S_d \wr S_n$, we obtain
\begin{equation}
\Sigma_{d,n} := \frac{1}{d! n!^d} \sum_{\sigma \in S_d \wr S_n} \sigma
\end{equation}
in the group algebra $\mathbb{C}[S_{dn}]$. We obtain the plethysm $\text{Sym}^d \text{Sym}^n V$ as the space of $S_d \wr S_n$-invariants in $\bigotimes^{dn} V$. This space is the image of the projection $\bigotimes^{dn} V \twoheadrightarrow \text{Sym}^d \text{Sym}^n V, w \mapsto \Sigma_{d,n} w$ induced by $\Sigma_{d,n}$. We define the plethysm coefficient, for $\lambda \vdash dn$,
\begin{equation}
a_\lambda(d[n]) := \dim \text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V),
\end{equation}
as the multiplicity of $\{\lambda\}$ in $\text{Sym}^d \text{Sym}^n V$.

4 (a). A system of generators encoded by tableaux. Again let $V = \mathbb{C}^N$ with the standard basis $e_1, \ldots, e_N$. As a consequence of Proposition 3.3, the highest weight vector space of $\text{Sym}^d \text{Sym}^n V$ of weight $\lambda \vdash dn$ is spanned by the projections $\Sigma_{d,n} \pi v_\lambda$, where $\pi$ runs over all permutations in $S_{dn}$. However, this description is highly redundant: we have $\Sigma_{d,n} \pi = \Sigma_{d,n} \pi'$ iff $(S_d \wr S_n) \pi = (S_d \wr S_n) \pi'$, for $\pi, \pi' \in S_{dn}$. We next give an intuitive description of the cosets of $S_d \wr S_n$ in terms of certain tableaux.

Definition 4.1. A tableau $T$ of shape $\lambda \vdash dn$ with content $d \times n$ is a partition of the set of boxes of the Young diagram of $\lambda$ into $d$ classes $C_1, \ldots, C_d$, each of size $n$.

Intuitively, we think of such a tableau $T$ as a filling of the Young diagram of $\lambda$ with $d$ different letters, where all boxes in the class $C_u$ have the same letter (and the order of the letters is irrelevant).

We assign to a permutation $\pi \in S_{dn}$ and $\lambda \vdash dn$ a tableau $T^\pi_\lambda$ of shape $\lambda$ with content $d \times n$ as follows: take $d$ different letters and replace in $\pi T^\text{std}_\lambda$ the numbers in the block $B_u$ by the same letter. It is clear that any tableau of shape $\lambda \vdash dn$ with content $d \times n$ can be obtained this way. For example, for the tableau in (3.4) we get for $n = 3, d = 2$, using the letters $a, b$,
\begin{equation}
\begin{array}{c|c|c|c|}
\hline
a & a & a & b \\
\hline
b & b & & \\
\hline
a & a & \\
\hline
\end{array}
= \begin{array}{c|c|c|c|c|}
\hline
b & b & b & a \\
\hline
b & & & \\
\hline
a & a & & \\
\hline
\end{array}
\end{equation}

It should be clear that $T^\lambda(\pi) = T^\lambda(\pi')$ iff $(S_n \wr S_d) \pi = (S_n \wr S_d) \pi'$, for $\pi, \pi' \in S_{dn}$.
By this observation, the following is well-defined.

**Definition 4.2.** Let \( T \) be a tableau of shape \( \lambda \) with content \( d \times n \). We define \( v_T := \Sigma_{d,n} \pi v_\lambda \), where \( \pi \in \mathfrak{S}_{dn} \) is such that \( T = T_\lambda(\pi) \).

By Proposition 3.3, \( v_T \) is a highest weight vector in \( \text{Sym}^d \text{Sym}^n V \) of weight \( \lambda \). We can restrict our attention to certain \( T \) because of the following.

**Lemma 4.3.**

1. Let \( T \) be a tableau of shape \( \lambda \) with content \( d \times n \). If the same letter appears in a column of \( T \) more than once, then \( v_T = 0 \).

2. Let \( T \) and \( T' \) be two tableaux of shape \( \lambda \) with content \( d \times n \) that can be obtained from each other by switching two columns that have the same length. Then \( v_T = v_{T'} \).

**Proof.** For the first assertion let \((r, c)\) and \((r', c)\) be different positions in \( T \) in the same column that have the same letter. Assume \( T = T_\lambda(\pi) \), and let \( i \) and \( j \) denote the entries of \( \pi T_\lambda^{\text{std}} \) at the positions \((r, c)\) and \((r', c)\), respectively. Then \( i \) and \( j \) lie in the same block since they are mapped to the same letter. Hence the transposition \( \tau := (i \, j) \) is an element of \( \mathfrak{S}_d \wr \mathfrak{S}_n \). Using (3.5), we see that symmetrizing \( \pi v_\lambda \) over the 2-element subgroup \( \{\text{id}, \tau\} \subseteq \mathfrak{S}_d \wr \mathfrak{S}_n \) maps \( \pi v_\lambda \) to zero. Hence symmetrizing over the full group \( \mathfrak{S}_d \wr \mathfrak{S}_n \) maps \( \pi v_\lambda \) to zero as well.

The second assertion is shown analogously, but it uses (3.6) instead of (3.5). \( \square \)

By a *singleton column* we understand a column of length 1. According to Lemma 4.3 (2), singleton columns of \( T \) can be permuted without changing the value of \( v_T \). Moreover, according to Lemma 4.3 (1), we can restrict our attention to tableaux, where no letter appears more than once in a column. This leads to the following definition.

**Definition 4.4.** Let \( T \) and \( T' \) be tableaux of shape \( \lambda \) with content \( d \times n \), where no letter appears more than once in a column. We call \( T \) and \( T' \) equivalent if they differ only by a reordering of their singleton columns. In this case we write \( T \simeq T' \).

By Lemma 4.3 (2), \( v_T \) depends only on the equivalence class of \( T \). For example, the following two tableaux with content \( 2 \times 3 \) are equivalent:

\[
\begin{array}{ccc}
  a & a & a \\
  b & b & b \\
\end{array} \quad \simeq \quad \begin{array}{ccc}
  a & a & b \\
  b & b & a \\
\end{array}
\]

Summarizing, we arrive at the following result.

**Proposition 4.5.** The vector space \( \text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V) \) is spanned by the highest weight vectors \( v_T \), where \( T \) ranges over all equivalence classes of tableaux of shape \( \lambda \) with content \( d \times n \), such that no letter appears more than once in a column of \( T \).

**4 (b). Explicit contractions.** Our goal here is to work out an explicit combinatorial rule for contracting a highest weight vector in the plethysm with a tensor of rank 1. This will be our main technical tool for proving that a specific highest weight vector does not vanish on the orbit closure \( \Omega_n \) or \( Z_{n,m} \), respectively.

Again we assume \( V = \mathbb{C}^N \) and denote by \( X_1, \ldots, X_N \) the basis of \( V^* \) dual to the standard basis \( e_1, \ldots, e_N \) of \( V \). The basis vectors \( X_{s(1)} \otimes \cdots \otimes X_{s(dn)} \) of \( (V^*)^{\otimes dn} \) are encoded by maps \( s : [dn] \to [N] \). We view \( \lambda \vdash dn \) as a Young diagram so that \( \lambda \) also denotes the set of boxes of this diagram. Recall that a tableau \( T \) of shape \( \lambda \)
with content \(d \times n\) is given by a partition \(\lambda = C_1 \cup \cdots \cup C_d\) of the set of boxes of \(\lambda\) into classes \(C_n\) of size \(n\). Also, recall the decomposition \([dn] = B_1 \cup \cdots \cup B_d\) into blocks of size \(n\) introduced at the very beginning of Section 4. We consider now bijective assignments \(\vartheta: \lambda \to [dn]\) that map boxes in the same class to numbers in the same block. More formally:

**Definition 4.6.** Let \(\lambda\) be a tableau of shape \(\lambda\) with content \(d \times n\) and \(\vartheta: \lambda \to [dn]\) be a bijection. We say that \(\vartheta\) respects \(T\) iff there exists a permutation \(\tau \in S_d\) such that \(\vartheta(C_i) = B_{\tau(i)}\) for all \(i\).

This notion is closely related to the wreath product \(S_d \wr S_n\) as follows. Let us call standard enumeration \(\vartheta_0: \lambda \to [dn]\) the labeling of the boxes in \(T_{\lambda}^{\text{std}}\), and let \(\pi \in S_{dn}\). Then the assignments \(\vartheta\) respecting \(T_{\lambda}(\pi)\) are given by \(\vartheta = \sigma \circ \pi \circ \vartheta_0\), where \(\sigma \in S_d \wr S_n\).

It is useful to introduce some further notations. Let \(j = (j_1, \ldots, j_k)\) be a list of integers. If \(\{j_1, \ldots, j_k\} = \{1, 2, \ldots, k\}\), then \(\text{sgn}(j)\) denotes the sign of the permutation \(j\); otherwise, we define \(\text{sgn}(j) := 0\). For instance, \(\text{sgn}(2, 1, 3) = -1\) and \(\text{sgn}(2, 1, 2) = 0\).

Suppose \(\vartheta: \lambda \to [dn]\) respects the tableau \(T\) of shape \(\lambda\) with content \(d \times n\), and take a map \(s: [dn] \to [N]\). We define the value \(\text{val}_\vartheta(s)\) of \(\vartheta\) at \(s: [dn] \to [N]\) by

\[
\text{val}_\vartheta(s) := \prod_{\text{column } c \text{ of } \lambda} \text{sgn}(s \circ \vartheta)|_c.
\]

Here, \(\text{sgn}(s \circ \vartheta)|_c\) denotes the sign of the list of integers \((s(\vartheta(1, c)), \ldots, s(\vartheta(c, c)))\) corresponding to the \(c\)th column of the diagram \(\lambda\). It is important to note that if \(\text{val}_\vartheta(s) \neq 0\), then \(s(\vartheta(\square)) = 1\) for all singleton columns \(\square\) of \(\lambda\).

We shall use the following rule throughout the paper.

**Theorem 4.7.** Let \(T\) be a tableau of shape \(\lambda \vdash dn\) with content \(d \times n\), and let \(s: [dn] \to [N]\) be a map. Then

\[
\langle v_T, X_{s(1)} \otimes \cdots \otimes X_{s(dn)} \rangle = \frac{1}{d!n!} \sum_{\vartheta} \text{val}_\vartheta(s),
\]

where the sum is over all bijections \(\vartheta: \lambda \to [dn]\) respecting \(T\).

**Proof.** We first note that for \(\varphi_1, \ldots, \varphi_k \in V^*\), the contraction \(\langle e_1 \wedge \cdots \wedge e_k, \varphi_1 \otimes \cdots \otimes \varphi_k \rangle\) equals the determinant of the top \(k \times k\) minor of the \(N \times k\) matrix whose columns are \(\varphi_1, \ldots, \varphi_k\). In particular, if \(\varphi_i = X_{s(i)}\) are from the dual standard basis, we get

\[
\langle e_1 \wedge \cdots \wedge e_k, X_{s(1)} \otimes \cdots \otimes X_{s(k)} \rangle = \text{sgn}(s(1), \ldots, s(k)).
\]

Let \(\mu\) denote the transpose of \(\lambda\) and put \(r := \lambda_1\). From the definition (3.3) of the highest weight vector \(v_{\lambda} \in \bigotimes^{dn} V\), we get

\[
\langle v_{\lambda}, X_{s(1)} \otimes \cdots \otimes X_{s(dn)} \rangle = \langle e_1 \wedge \cdots \wedge e_{\mu_1}, X_{s(1)} \otimes \cdots \otimes X_{s(\mu_1)} \rangle
\]

\[
\cdots \langle e_1 \wedge \cdots \wedge e_{\mu_r}, X_{s(dn-\mu_r+1)} \otimes \cdots \otimes X_{s(dn)} \rangle.
\]

Combined with the above observation we obtain that

\[
\langle v_{\lambda}, X_{s(1)} \otimes \cdots \otimes X_{s(dn)} \rangle = \text{val}_{\vartheta_0}(s),
\]

where \(\vartheta_0\) is the standard enumeration of \(\lambda\).
Suppose that $T = T_\lambda(\pi)$ for $\pi \in \mathfrak{S}_{dn}$. From the definition of $\Sigma_{d,n}$ in (4.1) and $v_T = \Sigma_{d,n} \pi v_\lambda$, we obtain for any $\Phi \in \text{Sym}^d \text{Sym}^n V^*$ that
\[
\langle v_T, \Phi \rangle = \frac{1}{d! n^d} \sum_{\sigma \in \mathfrak{S}_d \wr \mathfrak{S}_n} \langle \sigma \pi v_\lambda, \Phi \rangle = \frac{1}{d! n^d} \sum_{\sigma \in \mathfrak{S}_d \wr \mathfrak{S}_n} \langle v_\lambda, (\sigma \pi)^{-1} \Phi \rangle.
\]
Applying (4.3) to $\Phi = X_s(1) \otimes \cdots \otimes X_s(du)$, we obtain
\[
\langle v_T, (\sigma \pi)^{-1} (X_s(1) \otimes \cdots \otimes X_s(du)) \rangle = \langle v_\lambda, X_s(1) \otimes \cdots \otimes X_s(\pi(1)) \rangle = \text{val}_{\pi^0}(s \sigma \pi).
\]
Recall that if $\sigma$ runs through the wreath group $\mathfrak{S}_d \wr \mathfrak{S}_n$, then $\vartheta = \sigma \circ \pi \circ \vartheta_0$ runs through the assignments respecting $T_\lambda(\pi)$. Using this, we easily see that $\text{val}_{\pi^0}(s \sigma \pi) = \text{val}_\pi(s)$, which completes the proof. \hfill \qed

As a first application we give a short proof of the following result from [10, Thm. 3.19] on the evaluation of highest weight vectors on power sums. An extension of this will lead us later to the proof of Proposition 2.3.

**Corollary 4.8.** Let $n$ be even, and let $T$ be the tableau of shape $d \times n$ with content $d \times n$, where in each row all boxes have the same letter. Then
\[
\langle v_T, (c_1 X_1^n + \cdots + c_d X_d^n)^d \rangle = d! c_1 \cdots c_d \quad \text{for } c_1, \ldots, c_d \in \mathbb{C}.
\]

**Proof.** We have $(c_1 X_1^n + \cdots + c_d X_d^n)^\otimes d = \sum_{i_1, \ldots, i_d} c_1 \cdots c_d X_{i_1}^{\otimes n} \otimes \cdots \otimes X_{i_d}^{\otimes n}$, where the sum is over all $1 \leq i_1, \ldots, i_d \leq d$.

Assume first that $(i_1, \ldots, i_d) = (1, \ldots, d)$. We apply Theorem 4.7 to compute $\langle v_T, (X_1^{\otimes n} \otimes \cdots \otimes X_d^{\otimes n}) \rangle$. Using a row-wise enumeration of the boxes of $\lambda$, the bijections $\vartheta: \lambda \to [dn]$ respecting $T$ are in one-to-one correspondence with the elements of the wreath product $\mathfrak{S}_d \wr \mathfrak{S}_n$. They are given by permutations $\tau \in \mathfrak{S}_d$ of the rows, and permutations of the numbers within the rows. Such a bijection $\vartheta$ contributes $\text{val}_\vartheta(s) = \text{sgn}(\tau)^n$, where $s = (1, \ldots, 1, \ldots, d, \ldots, d)$ (each index occuring $n$ times). Hence we obtain, by the assumption that $n$ is even,
\[
\langle v_T, c_1 \cdots c_d X_{1}^{\otimes n} \otimes \cdots \otimes X_d^{\otimes n} \rangle = \frac{c_1 \cdots c_d}{d!} \sum_{\tau \in \mathfrak{S}_d} \text{sgn}(\tau)^n = c_1 \cdots c_d.
\]

We turn now to the contributions of an arbitrary sequence $(i_1, \ldots, i_d)$. If it is a permutation of $1, \ldots, d$, then, by the same argument as before, we see that we get the same contribution. On the other hand, if the sequence is not a permutation of $1, \ldots, d$, we get zero. This proves the assertion. \hfill \qed

As a further application, we prove the following general result, which immediately implies Theorem 2.1.

**Theorem 4.9.** Let $V$ be a finite-dimensional $\mathbb{C}$-vector space, and let $n \in \mathbb{N}$.

1. Assume $\pi: V \to W$ is a projection and $p \in \text{Sym}^n W^*$, and let $Z$ denote the $\text{GL}(V)$-orbit closure of $\pi^*(p) \in \text{Sym}^n V^*$. If $\lambda \vdash nd$ occurs in $\mathbb{C}[Z]_d$, then $\ell(\lambda) \leq \dim W$.

2. Let $q = X^{n-m} p$ where $X \in V^*$, and let $p \in \text{Sym}^m V^*$ for $m \leq n$. Let $Z$ denote the $\text{GL}(V)$-orbit closure of $q$. If $\lambda \vdash nd$ occurs in $\mathbb{C}[Z]_d$, then $|\lambda| \leq md$.

**Proof.** (1) For the first assertion, choose a basis $e_1, \ldots, e_N$ of $V$ such that $e_1, \ldots, e_M$ is a basis of $W$. Then $W^*$ is the span of $X_1^*, \ldots, X_M^*$, where $X_1, \ldots, X_N$ denotes the dual basis of $V^*$. Assume $\ell(\lambda) > M$. We need to prove that $\lambda \vdash dn$ does not occur
in \( \mathbb{C}[Z] \). By Proposition \[4.5\] this means that \( \langle v_T, \pi^*(p)^\otimes d \rangle = 0 \) for all tableau \( T \) of shape \( \lambda \) with content \( d \times n \). For this, it is enough to show that \( \langle v_T, \Phi \rangle = 0 \) for all tensors \( \Phi = X_{s(1)} \otimes \cdots \otimes X_{s(dn)} \), where \( s: [dn] \to [M] \). The first column \( c \) of \( \lambda \) has \( \ell \) boxes. We compose \( s \) with a bijection \( \vartheta: \lambda \to [dn] \). Then the restriction of \( s \circ \vartheta \) to the first column \( c \) is not injective since \( \ell > M \). Hence \( \text{sgn}(s \circ \vartheta)|_c = 0 \), and Theorem \[4.7\] implies that indeed \( \langle v_T, \Phi \rangle = 0 \).

(2) We now choose the basis \((e_i)\) of \( V \) such that \( X = X_1 \) is the first element of the dual basis of \((e_i)\). Assume \( |\lambda| > md \), so that \( \lambda_1 < (n-m)d \). We need to prove that \( \langle v_T, q^\otimes d \rangle = 0 \) for all tableau \( T \) of shape \( \lambda \) with content \( d \times n \). We can express \( q^\otimes d \) as a linear combination of tensors \( \Phi = X_{s(1)} \otimes \cdots \otimes X_{s(dn)} \), where \( s: [dn] \to [N] \) maps at least \((n-m)d\) elements to 1. Fix such a tensor \( \Phi \) and consider a bijection \( \vartheta: \lambda \to [dn] \). Since \( \lambda_1 < (n-m)d \), \( \lambda \) has less than \((n-m)d\) columns. By the pigeonhole principle, there is a column \( c \) in which \( s \circ \vartheta \) puts a 1 in at least two boxes. Therefore, \( \text{sgn}(s \circ \vartheta)|_c = 0 \) and Theorem \[4.7\] implies that indeed \( \langle v_T, \Phi \rangle = 0 \).

\[\Box\]

5. Lifting highest weight vectors in plethysms

In the following we analyze two ways of lifting highest weight vectors in \( \text{Sym}^d \text{Sym}^m V \) by raising either the inner degree \( m \) or the outer degree \( d \). If \( d \) and \( m \) are sufficiently large in comparison with \( |\mu| \) for \( \mu \vdash dm \), then these liftings provide isomorphisms of the spaces of highest weight vectors. In particular, the multiplicity \( a_\mu(dm) \) does not increase, which is known as the stability property of the plethysm coefficients \[14\,34\,47\]. A detailed understanding of the lifting in terms of highest weight vectors and the tableaux encoding them is central for the proof of our main result. We believe the results of this section are of independent mathematical interest. As a side result, we also obtain new proofs for the stability properties.

5 (a). The multiplication maps and their duals. Again assume \( V = \mathbb{C}^N \) with the standard basis \( e_1, \ldots, e_N \). Let \( n \geq m \) and consider the multiplication with \( e_1^{n-m} \),

\[ M: \text{Sym}^m V \to \text{Sym}^n V, \ p \mapsto pe_1^{n-m}. \]

(5.1)

Clearly, this is an injective linear map. Moreover, \( M \) is \( U_N \)-equivariant, since \( U_N \) acts on \( \text{Sym} V \) via algebra automorphisms and leaves \( e_1 \) invariant.

We determine now the dual \( M^*: \text{Sym}^n V^* \to \text{Sym}^m V^* \) of the linear map \( M \). It turns out to be proportional to the \((n-m)\)-fold directional derivative \( \partial_{e_1}^{n-m} \) in the direction \( e_1 \); see Section 3 (a).

Lemma 5.1. For \( q \in \text{Sym}^n V^* \), we have \( M^*(q) = \frac{m!}{n!} \partial_{e_1}^{n-m} q \).

Proof. We have to show that for all \( f \in \text{Sym}^m V \) and \( q \in \text{Sym}^n V^* \),

\[ \langle M(f), q \rangle = \frac{m!}{n!} \langle f, \partial_{e_1}^{n-m} q \rangle. \]

By bilinearity, it suffices to check the equality for the basis elements \( f = e^\alpha \) and \( q = X^\beta \), where \( |\alpha| = m \) and \( |\beta| = n \). We put \( \delta := (n-m, 0, \ldots, 0) \). By (3.1) we have

\[ \langle e^{\alpha + \delta}, X^\beta \rangle = \frac{1}{\binom{m}{\beta}} \delta_{\alpha + \delta, \beta}. \]

\[ (5.2) \]
We assume $\beta_1 \geq n - m$; otherwise, this expression vanishes. We calculate
\[
\partial_{e_1^{n-m}} q = \beta_1 (\beta_1 - 1) \cdots (\beta_1 - n + m + 1) X^{\beta - \delta}
\]
and obtain
\[
\frac{m!}{n!} (e^\alpha, \partial_{e_1^{n-m}} q) = \frac{m!}{n!} \frac{1}{(m_\alpha)^n} \beta_1 (\beta_1 - 1) \cdots (\beta_1 - n + m + 1) \delta_{\alpha, \beta - \delta}.
\]
It is easily verified that this equals (5.2). \qed

The map $M^*$ is clearly surjective. We now show that when restricting it to the subspace $\Sym_1^{n-m} \Sym^m V^*$, we get an isomorphism.

**Lemma 5.2.** The map $\Delta_{m,n} : \Sym^m V^* \to \Sym^n V^*$, $p \mapsto M^*(X_1^{n-m} p)$ is a linear automorphism.

**Proof.** We expand the polynomial $p = \sum_{i=0}^m a_i X_1^i$ with respect to the variable $X_1$. It is easy to check that
\[
\frac{\partial^{n-m}}{\partial X_1^{n-m}} (X_1^i p) = \sum_{i=0}^m (i + n - m)(i + n - m - 1) \cdots (i + 1) a_i X_1^i,
\]
hence $p \mapsto \frac{\partial^{n-m}}{\partial X_1^{n-m}} (X_1^{n-m} p)$ is injective. Now we apply Lemma 5.1. \qed

5 (b). Inner degree lifting. We denote by $\kappa_{m,n}^d := \Sym^d M$ the $d$-fold symmetric product of the map $M : \Sym^m V \to \Sym^n V$, and call this the inner degree lifting by $n - m$:
\[
(5.3) \quad \kappa_{m,n}^d : \Sym^d \Sym^m V \to \Sym^d \Sym^n V, \quad p^d \mapsto M(p)^d = (p e_1^{n-m})^d.
\]
Clearly, $\kappa_{m,n}^d$ is an injective linear map. This map behaves nicely regarding highest weight vectors.

**Lemma 5.3.** The inner lifting $\kappa_{m,n}^d$ maps highest weight vectors of weight $\mu \vdash dm$ to highest weight vectors of weight $\mu^{d \cdot n}$.

**Proof.** The $U_N$-equivariance of $M$ immediately extends to its $d$-fold symmetric product $\kappa_{m,n}^d$. Therefore, $\kappa_{m,n}^d$ maps $U_N$-invariants to $U_N$-invariants.

To conclude the proof, it suffices to check that $\kappa_{m,n}^d$ raises the weight of weight vectors by $(d(n-m), 0, \ldots, 0)$. For this, we note that $\kappa_{m,n}^d$ is obtained by restriction from the $d$th tensor power $\otimes^d M : \otimes^d \Sym^m V \to \otimes^d \Sym^n V$ of $M$, so that it suffices to verify this property for $\otimes^d M$.

Let $\Sym^m V = \bigoplus_\alpha \mathbb{C} e^\alpha$ be the weight space decomposition, where the sum is over all $\alpha \in \mathbb{N}^m$ and $e^\alpha := e_1^\alpha \cdots e_N^\alpha$ has the weight $\alpha$. We obtain the decomposition
\[
\otimes^d \Sym^m V = \bigoplus_{\alpha_1, \ldots, \alpha_d} \mathbb{C} e^{\alpha_1 \otimes \cdots \otimes \alpha_d},
\]
where $e^{\alpha_1 \otimes \cdots \otimes \alpha_d}$ has the weight $\alpha_1 + \cdots + \alpha_d$. The multiplication map $M$ sends $e^\alpha$ to $e^{\alpha + \delta}$, where $\delta := (n-m, 0, \ldots, 0)$. Hence the tensor product $\otimes^d M$ of $M$ sends $e^{\alpha_1 \otimes \cdots \otimes \alpha_d}$ to $e^{\alpha_1 + \delta \otimes \cdots \otimes \alpha_d + \delta}$, which has the weight $\alpha_1 + \cdots + \alpha_d + d\delta$. \qed

The dual of the inner degree lifting
\[
(\kappa_{m,n}^d)^* : \Sym^d \Sym^n V^* \to \Sym^d \Sym^m V^*
\]
is easily described: since \( \kappa_{m,n}^d = \text{Sym}^d M \), we have \((\kappa_{m,n}^d)^* = \text{Sym}^d M^* \). This immediately implies the following important observation that we state as a theorem.

**Theorem 5.4.** Let \( n \geq m \), \( f \in \text{Sym}^d \text{Sym}^m V \), and \( q \in \text{Sym}^n V^* \). Then

\[
\langle \kappa_{m,n}^d(f), q^d \rangle = \langle f, M^*(q)^d \rangle.
\]

This allows us to express the evaluation of the lifted \( \kappa_{m,n}^d(f) \) at \( q \), viewed as a polynomial function on \( \text{Sym}^n V^* \), as the evaluation of \( f \) at \( M^*(q) \). Recall that \( M^*(q) \) equals (up to a normalizing factor) the \((n - m)\)-fold directional derivative of \( q \) in direction \( e \); see Lemma 5.1. (We remark that in the conference version of our paper from FOCS 2016 \[11\], there is an error regarding the definition of the lifting map and the statement of Theorem 5.4).

Recall the generators \( v_T \) of \( \text{Sym}^d \text{Sym}^m V \) labeled by tableaux \( T \) (Definition 4.2). We show now that \( \kappa_{m,n}^d \) maps \( v_T \) to a generator \( v_{T'} \), whose tableau \( T' \) arises from \( T \) in a simple way.

**Theorem 5.5.** Let \( T \) be a tableau of shape \( \mu \) with content \( d \times m \), and let the tableau \( T' \) of shape \( \mu' := \mu^dn \) with content \( d \times n \) be obtained from \( T \) by adding \( n - m \) copies of each of the \( d \) letters in the first row (in some order). Then \( \kappa_{m,n}^d(v_T) = v_{T'} \).

**Proof.** Since we can decompose \( \kappa_{m,n}^d = \kappa_{n-1,n} \circ \cdots \circ \kappa_{m,m+1} \) into liftings by one, it suffices to prove the assertion for those. We thus focus on \( \kappa := \kappa_{m,m+1} \).

Suppose that \( T = T_\mu(\tau) \) for \( \tau \in \mathfrak{S}_{dm} \) so that \( v_T = \Sigma_{d,m}(\tau v_\mu) \) (see Definition 4.2). By definition, the tableau \( T' \) of shape \( \mu' := \mu + (d) \) is obtained from \( T \) by adding one copy of each of the \( d \) letters in the first row (in some order). Let us first determine a permutation \( \pi' \in \mathfrak{S}_{d(m+1)} \) such that \( T' \simeq T_{\mu'}(\pi') \) (see Definition 4.4). We denote by \( \rho \in \mathfrak{S}_{d(m+1)} \) the permutation that merges the last \( d \) entries into the first \( d \) blocks, each at the end of the block, respectively. More specifically,

\[
\rho(v_1 \otimes \cdots \otimes v_{dm} \otimes w_1 \otimes \cdots \otimes w_d) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes v_{m+1} \otimes \cdots \otimes w_2 \\
\quad \otimes \cdots \otimes v_{(d-1)m+1} \otimes \cdots \otimes v_{dm} \otimes w_d.
\]

(For example, for \( m = 3, d = 2 \), we have \( \rho = (4567) \)). We view \( \pi \in \mathfrak{S}_{dm} \subseteq \mathfrak{S}_{d(m+1)} \) and define now \( \pi' := \rho \tau \in \mathfrak{S}_{d(m+1)} \). The reader should verify that we indeed have \( T' \simeq T_{\mu'}(\pi') \).

We have \( v_{\mu'} = v_\mu \otimes e_1^\otimes d \) and hence

\[
\pi' v_{\mu'} = \rho \pi v_{\mu'} = \rho \pi (v_\mu \otimes e_1^\otimes d) = \rho((\pi v_\mu) \otimes e_1^\otimes d).
\]

We can write the symmetrizer \( \Sigma_{d,m} \) of the wreath group \( \mathfrak{S}_d \wr \mathfrak{S}_m \) (see (4.1)) as the composition \( \Sigma_{d,m} = \Pi_{d,m} \circ \otimes^d \Pi_m \), where \( \Pi_{d,m} \) denotes the symmetrizer of the subgroup isomorphic to \( \mathfrak{S}_d \), consisting of the permutations in \( \mathfrak{S}_{dn} \) of the form \( (u - 1)n + v \mapsto (\tau(u) - 1)n + v \) with \( \tau \in \mathfrak{S}_d \), (simultaneous permutations of the blocks). We define

\[
w_T := (\otimes^d \Pi_m)(\pi v_\mu), \quad w_{T'} := (\otimes^d \Pi_{m+1})(\pi' v_{\mu'}),
\]

so that we can write \( v_T = \Pi_{d,m}(w_T) \) and \( v_{T'} = \Pi_{d,m+1}(w_{T'}) \).
Let $M: \text{Sym}^m V \to \text{Sym}^{m+1} V$, $p \mapsto p e_1 = \Pi_n(p \otimes e_1)$ denote the multiplication by $e_1$ (see (5.1)) and note that the following diagram is commutative:

\[
\begin{array}{ccc}
\otimes^d \text{Sym}^m V & \xrightarrow{\Pi_{d,m}} & \otimes^d \text{Sym}^{m+1} V \\
\downarrow & & \downarrow \\
\text{Sym}^d \text{Sym}^m V & \xrightarrow{\text{Sym}^d M} & \text{Sym}^d \text{Sym}^{m+1} V.
\end{array}
\]

Therefore, it is sufficient to prove that

\[(5.5) \quad (\otimes^d M)(w_T) = w_{T'}.\]

To show this, we first note that for $p \in \otimes^m V$, we have

\[(5.6) \quad \Pi_{m+1}(p \otimes e_1) = \Pi_{m+1}(\Pi_m(p) \otimes e_1) = M(\Pi_m(p)).\]

We claim now that for $w \in \otimes^d \otimes^m V$, we have

\[(5.7) \quad (\otimes^d \Pi_{m+1})(\rho(w \otimes e_1^\otimes d)) = (\otimes^d (M \circ \Pi_m))(w).\]

To verify this, we may assume that $w = w_1 \otimes \cdots \otimes w_d$ with $w_i \in \otimes^m V$. By the definition of $\rho$, we have $\rho(w \otimes e_1^\otimes d) = w_1 \otimes e_1 \otimes \cdots \otimes w_d \otimes e_1$, and hence, using (5.6) for the second equality,

\[(\otimes^d \Pi_{m+1})(\rho(w \otimes e_1^\otimes d)) = \otimes_{i=1}^d \Pi_{m+1}(w_i \otimes e_1) = \otimes_{i=1}^d M(\Pi_m(w_i)),\]

and (5.7) follows.

Using (5.4) and (5.7) with $w = \pi v_\mu$, we argue now as follows:

\[
w_{T'} = (\otimes^d \Pi_{m+1})(\pi' v'_{\mu'}) = (\otimes^d \Pi_{m+1})\rho(\pi v_\mu \otimes e_1^\otimes d)
= (\otimes^d (M \circ \Pi_m))(\pi v_\mu) = (\otimes^d M)(w_T),
\]

which shows (5.5), and this completes the proof.

Theorem 5.5 now easily implies the stability property of plethysms with respect to inner degree lifting.

**Proposition 5.6.**

1. Suppose $\mu \vdash m d$ is such that $\mu_2 \leq m$, and let $n \geq m$. Then the inner degree lifting $\kappa_{m,n}^d$ defines an isomorphism

\[\text{HWV}_\mu(\text{Sym}^d \text{Sym}^m V) \to \text{HWV}_{\mu^\otimes d n}(\text{Sym}^d \text{Sym}^n V), \ f \mapsto \kappa_{m,n}^d(f).\]

2. Suppose that $\lambda \vdash n d$ satisfies $\lambda_2 \leq m$ and $\lambda_2 + |\lambda| \leq m d$. Then every highest weight vector of weight $\lambda$ in $\text{Sym}^d \text{Sym}^n V$ is obtained by lifting a highest weight vector in $\text{Sym}^d \text{Sym}^m V$ of weight $\mu$, where $\mu \vdash m d$ such that $\mu_2 = \lambda$.

**Proof.** (1) We need to show the surjectivity of the map. Let $T'$ be a tableau of shape $\mu^\otimes d n$ with content $d \times n$ such that no letter appears more than once in a column. Then each of the $d$ letters appears at least $n - \mu_2 \geq n - m$ times in singleton columns. Hence $T'$ is obtained from a tableau $T$ of shape $\mu$ with content $d \times m$ as in Theorem 5.3 (the order of the letters in the singleton columns is irrelevant; see Definition 1.11. Since $\kappa_{m,n}^d(v_T) = v_{T'}$ by Theorem 5.5, the assertion follows with Proposition 1.9.

(2) Note that $\lambda_2 + |\lambda| \leq m d$ is the number of boxes of $\lambda$ that appear in columns that are not singleton columns. We can therefore shorten the given $\lambda$ to a partition
\( \mu \vdash md \) by removing singleton columns. Then \( \bar{\mu} = \bar{\lambda} \) and \( \lambda = \mu^{\sharp dn} \), and we conclude with part one. \( \square \)

5(c). Outer degree lifting. We keep the notation from before. Let \( k \leq d \). By multiplying with the \( (d - k) \)th power of \( \nu^m_1 \), we obtain the injective linear map
\[
\text{Sym}^k \text{Sym}^m V \to \text{Sym}^d \text{Sym}^m V, \ f \mapsto (\nu^m_1)^{d-k} \cdot f.
\]
Since \( \text{GL}(V) \) acts on \( \text{Sym} V \) by algebra automorphisms, it follows that \( 5(8) \) maps a highest weight vector of weight \( \nu \vdash mk \) to a highest weight vector of weight \( \nu^{\sharp dm} \).

**Lemma 5.7.** Let \( T \) be a tableau of shape \( \nu \) with content \( k \times m \), and let the tableau \( T'' \) of shape \( \nu^{\sharp dm} \) be obtained from \( T \) by adding \( m \) copies of \( d - k \) new letters to the first row (in some order). Then \( \nu T'' \) is obtained as the image of \( \nu T \) under the map \( 5(8) \).

**Proof.** The lifting from degree \( k \) to \( d \) can be obtained as a composition of liftings that increase the degree by one only. Hence we may assume without loss of generality that \( d = k + 1 \). So we assume that \( T'' \) is obtained from \( T \) by adding \( m \) letters of a new letter to the first row. In order to show that \( \nu T'' = \nu^m T \), it suffices to prove that
\[
\langle \nu T'', \Phi \rangle = \langle \nu^m T, \Phi \rangle
\]
for all tensors \( \Phi = \varphi_1 \otimes \cdots \otimes \varphi_{k+1}, \) where \( \varphi_i \in \otimes^m V^* \) is of rank 1. We shall evaluate the inner products with Theorem 4.7. For this, consider the decomposition \( [(k+1)m] = B_1 \cup \cdots \cup B_{k+1} \) into the blocks \( B_i \) of size \( m \) as in the beginning of Section 4. For \( 1 \leq i \leq k + 1 \), let \( \tau_i \in S_{(k+1)m} \) be the permutation that exchanges the blocks \( B_i \) and \( B_{k+1} \) and preserves the order within the blocks (note \( \tau_{k+1} = \text{id} \)).

Suppose that \( \vartheta: \nu \to [km] \) respects \( T \), and put \( \nu'' := \nu^{\sharp dm} \). We extend \( \vartheta \) to a map \( \tilde{\vartheta}: \nu'' \to [km + m] \) by sending the \( m \) boxes of \( T'' \setminus T \) bijectively to the numbers in the block \( B_{k+1} \). Clearly, \( \tilde{\vartheta} \) respects \( T'' \). Let \( 1 \leq i \leq k + 1 \), and let \( \pi \) be a permutation of \( B_{k+1} \). Composing \( \tilde{\vartheta} \) with \( \tau_i \pi \), we obtain a map \( \vartheta'': \nu'' \to [km + m] \) respecting \( T'' \). Moreover, any assignment \( \vartheta'' \) respecting \( T'' \) arises this way from uniquely determined \( \vartheta \), \( i \), and \( \pi \).

Taking this observation into account, we deduce from Theorem 4.7 after some thought that
\[
\langle \nu T'', \Phi \rangle = \frac{1}{k + 1} \left( \langle \nu T, \varphi_1 \otimes \cdots \otimes \varphi_k \rangle \cdot \langle \nu^m_1, \varphi_{k+1} \rangle \right) + \sum_{i=1}^k \langle \nu T, \varphi_1 \otimes \cdots \otimes \varphi_i \otimes \varphi_{k+1} \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_k \rangle \cdot \langle \nu^m_1, \varphi_i \rangle.
\]
(Note that the first summand corresponds to \( i = k + 1 \).) This equals \( \langle \nu^m T, \Phi \rangle \) by the definition of the symmetric product. \( \square \)

The outer degree lifting \( 5(8) \) behaves nicely with respect to highest weight vectors.

**Proposition 5.8.**

(1) Suppose \( \nu \vdash mk \) such that \( \nu_2 + |\tilde{\nu}| \leq k \), and let \( d \geq k \). The lifting \( 5(8) \) defines an isomorphism
\[
\text{HWV}_\nu(\text{Sym}^k \text{Sym}^m V) \to \text{HWV}_{\nu^{\sharp dm}}(\text{Sym}^d \text{Sym}^m V).
\]
Suppose that \( f \) is a highest weight vector in \( \text{Sym}^d \text{Sym}^m V \) of weight \( \mu \vdash dm \), and assume that \( m_2 + |\bar{\mu}| \leq k \leq d \) for some \( k \). Then \( \mu = \nu^{d \bar{m}} \) for some \( \nu \vdash mk \) and \( f = (e_1^n)^{d-k} \cdot g \) for some \( g \in \text{HWV}_\nu(\text{Sym}^k \text{Sym}^n V) \).

**Proof.** (1) We need to show the surjectivity. Let \( T'' \) be a tableau of shape \( \mu := \nu^{d \bar{m}} \) with content \( d \times m \). Note that \( m_2 + |\bar{\nu}| \) is the number of boxes in \( \mu \) that are not singleton columns. Hence there are at least \( d - (m_2 + |\bar{\nu}|) \geq d - k \) many letters appearing in singleton columns of \( T'' \) only. Removing the \( (d - k)m \) many boxes with these letters from the first row of \( \mu \) leads to a tableau \( T \) of shape \( \nu \) with content \( k \times m \).

We conclude with Lemma \ref{mainlemma} and Proposition \ref{mainproposition}.

(2) There exists a tableau \( T'' \) of shape \( \mu \) with content \( d \times m \) by assumption and Proposition \ref{mainproposition}. As before, we see that there are at least \( d - k \) many letters appearing in singleton columns of \( T'' \) only. In particular, we have \( \mu = \nu^{d \bar{m}} \) for some \( \nu \vdash mk \). Now we apply part one.

We remark that the stability of plethysm in Proposition \ref{mainproposition}(1), for the slightly weaker condition \( |\bar{\nu}| \leq k \leq d \), was first shown in \cite{34} with a geometric method.

## 6. PROOF OF THEOREM \ref{mainthm}

6(a). Small degrees or extremely long first rows. In the following we put \( V := \mathbb{C}^{n \times n} \). To warm up, we first show that \((n)\) occurs in \( \mathbb{C}[\Omega_n] \). Indeed, we have \( X_i^n \in \Omega_n \), and \( e_1^n \in \text{Sym}^n V \) is a highest weight vector of weight \((n)\) such that \( \langle e_1^n, X_i^n \rangle = 1 \).

**Proof of Proposition \ref{mainproposition}**. We assume that \( \lambda \vdash nd \) is such that there exists a positive integer \( m \) satisfying \( |\lambda| \leq md \) and \( md^2 \leq n \). Further, let \( h \in \text{Sym}^d \text{Sym}^n V \) be any highest weight vector of weight \( \lambda \). We need to prove that there exists \( q \in \Omega_n \) such that \( \langle h, q^d \rangle \neq 0 \). The case \( d = 1 \) is trivial as noted before. So suppose \( d \geq 2 \).

We have \( \lambda_2 \leq |\lambda| \leq md \) and \( \lambda_2 + |\bar{\lambda}| \leq 2|\bar{\lambda}| \leq 2md \leq md \cdot d \). Therefore, we are in the setting of Proposition \ref{mainproposition}(2) with respect to the lifting (note that the inner degree on the left-hand side is \( md \) and not \( d \)),

\[
\text{Sym}^d \text{Sym}^{md} V \to \text{Sym}^d \text{Sym}^n V.
\]

We conclude that \( h \) arises by an inner degree lifting from a highest weight vector \( f \in \text{Sym}^d \text{Sym}^{md} V \) of weight \( \lambda \); so we have \( h = k_{md,n}^d(f) \). Recall the linear automorphism \( \Delta := \Delta_{md,n} : \text{Sym}^{md} V^* \to \text{Sym}^{md} V^* \) from Lemma \ref{linearlemma}.

We view \( f \) as a degree \( d \) homogeneous polynomial map \( \text{Sym}^{md} V^* \to \mathbb{C} \) and apply Proposition \ref{mainproposition} to the composition \( \text{Sym}^{md} V^* \to \mathbb{C}, p \mapsto \langle f, \Delta(p)^d \rangle \). Hence there is a power sum

\[
p = \varphi_1^{md} + \cdots + \varphi_d^{md}
\]

with at most \( d \) terms such that \( \langle f, \Delta(p)^d \rangle \neq 0 \) for some \( \varphi_1, \ldots, \varphi_d \in V^* \). We apply now Theorem \ref{mainthm} with \( q := X_1^{n-md} p \) and obtain

\[
\langle h, q^d \rangle = \langle k_{md,n}^d(f), q^d \rangle = \langle f, M^*(q)^d \rangle = \langle f, \Delta(p)^d \rangle \neq 0.
\]

(Note that \( M^*(q) = \Delta(p) \) by the definition of \( \Delta \), see Lemma \ref{linearlemma}.) By Theorem \ref{mainthm} we have \( q \in \Omega_n \) since \( n \geq md \cdot d \). The assertion follows.

While Proposition \ref{mainproposition} is meant to deal with the case of partitions \( \lambda \vdash nd \) where \( d \) is small, the next result deals with the extreme situation, where the body \( \bar{\lambda} \) is
very small compared with the size of $\lambda$. (In this situation, the splitting strategy in the proof of Proposition 6.3 below would fail.)

**Proposition 6.1.** Let $\lambda \vdash nd$ and assume there exist positive integers $s,m$ such that $\ell(\lambda) \leq m^2$, $\lambda_2 \leq s$, $m^2s^2 \leq n$, and $m^2s \leq d$. Then every highest weight vector $h \in \text{Sym}^d \text{Sym}^n V$ of weight $\lambda$, viewed as a degree $d$ polynomial function on $\text{Sym}^n V^*$, does not vanish on $\Omega_n$.

**Proof.** We first consider the inner degree lifting $\text{Sym}^d \text{Sym}^* V \to \text{Sym}^d \text{Sym}^n V$; see (6.3). Since $$\lambda_2 \leq s, \quad \lambda_2 + |\lambda| \leq \lambda_2 + (\ell(\lambda) - 1)\lambda_2 = \ell(\lambda)\lambda_2 \leq m^2s \leq d \leq ds,$$ the assumptions of Proposition 5.8(2) are satisfied, and we conclude that $h$ arises by lifting some $f \in \text{HW}_{\mu}(\text{Sym}^d \text{Sym}^* V)$ with $\mu \vdash ds$ and $\bar{\mu} = \bar{\lambda}$.

By assumption, $k := m^2s \leq d$. We continue with the outer degree lifting map $\text{Sym}^k \text{Sym}^* V \to \text{Sym}^d \text{Sym}^n V$; see (5.8). We have, using the above, $$\mu_2 + |\bar{\mu}| = \lambda_2 + |\lambda| \leq m^2s = k,$$ hence the assumptions of Proposition 5.8(2) are satisfied, and we have $f = (e_1^s)^{d-k}g$ for some highest weight vector $g \in \text{Sym}^s \text{Sym}^n V$ of weight $\nu \vdash k\bar{s}$ such that $\bar{\nu} = \bar{\mu}$.

Recall the linear automorphism $\Delta := \Delta_{s,n} : \text{Sym}^s V^* \to \text{Sym}^n V^*$ from Lemma 5.2, and consider power sums $p = \varphi_1^s + \cdots + \varphi_k^s$. By Proposition 3.2 there are $\varphi_1, \ldots, \varphi_k \in V^*$ such that $\langle g, \Delta(p)^{k} \rangle \neq 0$ and $\langle e_1^s, \Delta(p) \rangle \neq 0$. Applying Theorem 5.4 we obtain with $q := X_1^{n-s}p$ that $$\langle h, q^d \rangle = \langle f, M^s(q)^d \rangle = \langle f, \Delta(p)^d \rangle = \langle e_1^s, \Delta(p) \rangle^{d-k}(g, \Delta(p)^{k}) \neq 0.$$ On the other hand, by Theorem 2.5 the padded polynomial $X_1^{n-s}(\varphi_1^s + \cdots + \varphi_k^s)$ is contained in $\Omega_n$, as $n \geq sk$ by assumption. Therefore, $h$ does not vanish on $\Omega_n$. □

6(b). **Building blocks and splitting technique.** We construct as building blocks certain partitions that occur in $\mathbb{C}[\Omega_n]$. We achieve this by providing explicit tableau constructions and showing that the corresponding highest weight vectors do not vanish on certain tensors describing padded power sums. Then we apply Theorem 2.5. In particular, we prove in this way that certain plethysms coefficients are nonzero.

We first provide the proof of Proposition 2.3 which deals with the case of even partitions and which was already stated in Section 2.

**Proof of Proposition 2.3.** Let $T$ denote the tableau of shape $k \times \ell$ with content $k \times \ell$ from Corollary 4.8. Suppose $n \geq k\ell$, and let $h := \kappa_{k,\ell,n}(v_T) \in \text{Sym}^k \text{Sym}^n V$ denote the lifting of $v_T \in \text{Sym}^k \text{Sym}^\ell V$. Hence $h$ is a highest weight vector of weight $(k \times \ell)^{2nk}$. Recall the corresponding linear automorphism $\Delta := \Delta_{\ell,n} : \text{Sym}^\ell V^* \to \text{Sym}^n V^*$ from Lemma 5.2. Put $p := X_1^{\ell} + \cdots + X_\ell^{\ell}$ and note that $\Delta(p) = aX_1^{\ell} + bX_2^{\ell} + \cdots + bX_k^{\ell}$ for some $a,b \neq 0$. Applying Theorem 5.4 we obtain with $q := X_1^{n-\ell}p$ that $\langle h, q^k \rangle = \langle v_T, M^s(q)^k \rangle = \langle v_T, \Delta(p)^{k} \rangle$. On the other hand, Corollary 4.8 implies that $\langle v_T, \Delta(p)^{k} \rangle$ is nonzero. By Theorem 2.5 we have $X_1^n \in \Omega_n$ since $n \geq k\ell$. Therefore $h$, viewed as a degree $d$ homogeneous polynomial function on $\text{Sym}^n V^*$, does not vanish on $\Omega_n$, and the assertion follows. □

In order to handle partitions with odd parts, we use as further building block partitions obtained from rectangles by adding a single row and a single column.
We postpone the proof of the following technical result to Section 7 (It is based on an explicit construction of a highest weight vector.)

**Theorem 6.2.** Let \(2 \leq b, c \leq m^2\), and let \(n \geq 24m^6\). Then there exists an even \(i \leq 2m^4\), such that
\[
\lambda = b \times 1 + c \times i + 1 \times j
\]
occurs in \(\mathbb{C}[\Omega_n]_{3m^4}\) for \(j = 3m^4n - b - ic\).

The splitting strategy in the following proof is a refinement of the one in [26]. The proof relies on Theorem 6.2 and on the semigroup property (Lemma 2.2).

**Proposition 6.3.** Given a partition \(\lambda\) with \(|\lambda| = nd\) such that there exists \(m \geq 2\) with \(\ell(\lambda) \leq m^2\), \(m^{10} \leq |\lambda| \leq md\), \(n \geq 24m^6\), and \(d > 4m^6\). Then \(\lambda\) occurs in \(\mathbb{C}[\Omega_n]_{d}\).

**Proof.** Let \(L := \ell(\lambda)\), and let \(c_k\) denote the number of columns of length \(k\) in \(\lambda\) for \(1 \leq k \leq L\). Let \(K\) be the index \(k \geq 2\), for which \(c_k\) is maximal; i.e., \(c_k = \max(c_k; k = 2, \ldots, L)\). By assumption, we have \(2 \leq K \leq m^2\) and
\[
m^{10} \leq |\lambda| = \sum_{k=2}^{L} (k - 1)c_k \leq c_K \sum_{k=2}^{L} (k - 1) \leq c_K \frac{L^2}{2} \leq c_K \frac{m^4}{2},
\]
hence \(c_K \geq 2m^6\).

The columns of odd length of \(\lambda\) need a special treatment. Let \(S\) denote the set of integers \(k \in \{2, \ldots, L\}\) for which \(c_k\) is odd. For \(k \in S\) we define the partition
\[
\omega_k := k \times 1 + K \times i_k,
\]
where the even integer \(i_k \leq 2m^4\) is taken from Theorem 6.2 so that \(\omega_k^{3m^4}\) occurs in \(\mathbb{C}[\Omega_n]_{3m^4}\). (Here we have used the assumption \(n \geq 24m^6\).)

Assume first that \(K \not\in S\), that is, \(c_K\) is even. Then we can split \(\lambda\) vertically in rectangles as follows:
\[
\lambda = 1 \times c_1 + \sum_{k \in S \cup \{K\}} k \times c_k + \sum_{k \in S} k \times c_k + K \times c_K
\]
\[
= 1 \times c_1 + \sum_{k \in S \cup \{K\}} k \times c_k + \sum_{k \in S} k \times (c_k - 1) + \sum_{k \in S} \omega_k + K \times \left(c_K - \sum_{k \in S} i_k\right).
\]
If, for \(k \leq L\), we set \(d_k := c_k\) if \(k \not\in S \cup \{K\}\) and \(d_k := c_k - 1\) if \(k \in S\), and define \(d_K := c_K - \sum_{k \in S} i_k\), then the above can be briefly written as
\[
\lambda = 1 \times c_1 + \sum_{k=2}^{L} k \times d_k + \sum_{k \in S} \omega_k.
\]
By construction, all \(d_k\) are even. It is crucial to note that, using \(i_k \leq 2m^4\),
\[
d_{K} = c_{K} - \sum_{k \in S} i_k \geq c_{K} - (m^2 - 1) \cdot 2m^4 \geq c_{K} - 2m^6 \geq 0.
\]
The last inequality is due to our observation at the beginning of the proof.

In the case where \(K \in S\), we achieve the same decomposition as in (6.1) with the modified definition \(d_{K} := c_{K} - 1 - \sum_{k \in S} i_k\). Here as well, \(d_{K} \geq 0\) and all \(d_{k}\) are even.
We need to round down rational numbers to the next even number, so for \( a \in \mathbb{Q} \), we define \( \|a\| := 2\lfloor a/2 \rfloor \). Note that \( \|a\| \geq a - 2 \) for all \( a \in \mathbb{Q} \). Hence \( \|n/k\| \geq n/k - 2 \geq 2 \) for all \( 2 \leq k \leq m^2 \), since \( n \geq 4m^2 \).

Using division with remainder, let us write \( d_k = q_k\|n/k\| + r_k \) with \( 0 \leq r_k < \|n/k\| \). Then we split \( k \times d_k = q_k(k \times \|n/k\|) + k \times r_k \). Since \( d_k \) is even and \( \|n/k\| \) is even, \( r_k \) is even as well. From (6.1) we obtain that the partition

\[
\mu := \sum_{k=2}^L q_k((k \times \|n/k\|)^\omega_{nk}) + \sum_{k=2}^L (k \times r_k)^\omega_{nk} + \sum_{k \in S} \omega_{k}^{3nm^4}
\]

coincides with \( \lambda \) in all but possibly the first row.

Since \( \|n/k\| \leq n/k, r_k \leq n/k \), and both \( n/k \) and \( r_k \) are even, Proposition 2.3 implies that \( (k \times \|n/k\|)^\omega_{nk} \) and \( (k \times r_k)^\omega_{nk} \) occur as highest weights in \( \mathbb{C}[\Omega_n]_k \).

Moreover, Theorem 6.2 tells us that \( \omega_{k}^{3nm^4} \) occurs as a highest weight in \( \mathbb{C}[\Omega_n]_{3m^4} \). The semigroup property implies that \( \mu \) occurs in \( \mathbb{C}[\Omega_n] \).

**Claim.** \( |\mu| \leq dn \).

Let us finish the proof assuming the claim. If \( |\mu| \leq dn \), we can obtain \( \lambda \) from \( \mu \) by adding boxes to the first row of \( \mu \). Note that \( |\lambda| - |\mu| \) is a multiple of \( n \). Since \( (n) \in \mathbb{C}[\Omega_n] \), the semigroup property implies that \( \lambda \) occurs in \( \mathbb{C}[\Omega_n]_d \).

It remains to verify the claim. From (6.2) we get

\[
|\mu| \leq \sum_{k=2}^L (q_k nk + nk + 3nm^4).
\]

We have, using \( \|a\| \geq a - 2 \),

\[
q_k \leq \frac{d_k}{\|n/k\|} \leq \frac{kd_k}{n - 2k}.
\]

This implies

\[
|\mu| \leq n \sum_{k=2}^L \left( \frac{k^2 d_k}{n - 2k} + k + 3m^4 \right).
\]

Using \( d_k \leq c_k \) and \( L \leq m^2 \), we get

\[
|\mu| \leq n \sum_{k=2}^L \frac{m^2}{n - 2m^2} k c_k + n \sum_{k=2}^L k + 3m^4(m^2 - 1).
\]

Noting that \( \sum_{k=2}^L k c_k = |\lambda| + \lambda_2 \leq 2|\lambda| \), we continue with

\[
|\mu| \leq \frac{nm^2}{n - 2m^2} \cdot 2|\lambda| + n \left( \frac{m^2(m^2 + 1)}{2} + 3m^4(m^2 - 1) \right)
\]

\[
\leq \frac{ nm^2}{12m^6 - m^2} \cdot |\lambda| + n \left( 3m^6 - \frac{5}{2} m^4 + \frac{1}{2} m^2 \right),
\]

where we have used \( n > 24m^6 \) for the second inequality. Plugging in the assumptions \( |\lambda| \leq dm \) and \( d > 4m^6 \), we obtain

\[
|\mu| \leq \frac{dnm^3}{11m^6} + 3nm^6 \leq \frac{dn}{11} + 3nm^6 \leq \frac{dn}{11} + \frac{3dn}{4} < dn,
\]

which shows the claim and completes the proof. \( \square \)
We can now finally complete the proof of our main result.

**Proof of Theorem 1.4** We may assume that $m \geq 2$, as the case $m = 1$ is trivial. Suppose that $\lambda \vdash nd$ occurs in $\mathbb{C}[Z_{n,m}]$ and $n \geq m^{25}$. Theorem 2.1 implies that $|\lambda| \leq md$ and $\ell(\lambda) \leq m^2$.

In the case of small degree, where $n \geq md^2$, Proposition 2.4 implies that $\lambda$ occurs in $\mathbb{C}[\Omega_n]$. So we may assume that $d > \sqrt{n/m}$. In this case we have $d \geq \sqrt{m^{25}/m} = m^{12}$. We conclude by two further case distinctions.

If $|\lambda| < m^{10}$, we can apply Proposition 6.4 with $s := m^{10}$ since $\lambda_2 \leq |\lambda| \leq s$, $m^2s^2 = m^{22} \leq n$, and $m^2s = m^{12} \leq d$. Thus $\lambda$ occurs in $\mathbb{C}[\Omega_n]_d$.

Finally, if $|\lambda| \geq m^{10}$, then the above Proposition 6.3 tells us that $\lambda$ occurs in $\mathbb{C}[\Omega_n]_d$. □

7. **Explicit constructions of tableaux and positivity of plethysms**

The goal of this last section is to provide the proof of Theorem 6.2.

In order to motivate the construction, we begin with a general reasoning. Let $T$ be a tableau of shape $\lambda$ with content $d \times n$. For the sake of readability, we will use the natural numbers $1, \ldots, d$ as letters. The set of boxes of $T$ is partitioned as $C_1 \cup \cdots \cup C_d$, where $C_u$ denotes the set of boxes with the letter $u$. Note that $|C_u| = n$ for all $u$. We denote by $C_u^1$ the subset of $C_u$ consisting of the boxes in singleton columns. On the other hand, the position set $[dn] = B_1 \cup \cdots \cup B_d$ is partitioned into the blocks $B_u := \{(u-1)n + v \mid 1 \leq v \leq n\}$, where $|B_u| = n$ for all $u$.

We fix a map $s: [dn] \to [N]$, which defines the rank 1 tensor $\Phi = X_{s(1)} \otimes \cdots \otimes X_{s(dn)} \in \mathbb{C}^N$. Depending on $s$, we denote by $B_u^1 := B_u \cap s^{-1}(1)$ the set of positions in block $B_u$ that are mapped to 1 under the map $s$. (Hence the positions in $B_u^1$ are the ones mapped to the basis vector $X_1$.)

Recall from Theorem 1.7 that $\langle \tau_T, \Phi \rangle = (n^{l(d)}d!)^{-1} \sum_{\vartheta} \text{val}_\vartheta(s)$, where the sum is over all bijections $\vartheta: \lambda \to [dn]$ respecting $T$. The bijection $\vartheta$ respects the tableau $T$ if there exists $\tau_\vartheta \in \mathfrak{S}_d$ such that $\vartheta(C_u) = B_{\tau_\vartheta(u)}$ for all $u$; see Definition 1.6.

If $\text{val}_\vartheta(s) \neq 0$, then $s \circ \vartheta$ must map all boxes in singleton columns to 1. This means

\begin{equation}
\forall u \quad \vartheta(C_u^1) \subseteq B_{\tau_\vartheta(u)}^1.
\end{equation}

For proving that $a_\lambda(d[n]) > 0$, we shall design $T$ and $s$ in such a way that $\text{val}_\vartheta(s) \geq 0$ for all $\vartheta$ and there are only a few $\vartheta$ with $\text{val}_\vartheta(s) > 0$ (there must be at least one).

Part of the strategy for realizing this can be described as follows.

**Claim 7.1.** Let $T$ be tableau of shape $\lambda$ and content $d \times n$, where $C_u^1$ denotes the set of boxes with letter $u$ in the singleton columns of $T$. Let $s: [dn] \to [N]$ and recall $B_u^1 := B_u \cap s^{-1}(1)$. Assume there is an integer $D$ with $\ell(\lambda) \leq D \leq d$ such that $|C_1^1| = |B_1^1|, \ldots, |C_D^1| = |B_D^1| \leq n-2$ are pairwise distinct numbers and $|C_u^1| > n-2$ for $D < u \leq d$. Then for any $\vartheta: \lambda \to [dn]$ respecting $T$ with $\text{val}_\vartheta(s) \neq 0$, we have $\tau_\vartheta(u) = u$ for all $1 \leq u \leq D$. [Proof]

Assume $\text{val}_\vartheta(s) \neq 0$ and write $\tau := \tau_\vartheta$. Then (7.1) implies $|C_u^1| \leq |B_{\tau(u)}^1|$ for all $u$. For $u > D$ we have $|C_u^1| > n-2$, hence $\tau(u) > D$, since $|B_u^1| \leq n-2$ for $u' \leq D$. We conclude that $\tau$ permutes the set $[D]$. For $u \leq D$, by assumption,
the cardinalities \( w(u) : = |C^1_u| = |B^1_u| \) are pairwise distinct and (7.1) gives \( w(u) \leq w(\tau(u)) \). This implies that \( \tau(u) = u \) for \( 1 \leq u \leq D \). \( \square \)

By a concrete choice of a tableau \( T \) and map \( s \), we prove now the following.

**Proposition 7.2.** Let \( t \geq r, i \geq 2t + 3 \) be positive integers, and let \( n \geq i \) and \( d \geq 2t + i + 1 \). Let \( \nu = (t+1) \times i + (r+1) \times i + 1 + (j) \), where \( j = dn - (t+1)i - (r+1) \). Then \( a_\nu(d[n]) > 0 \).

**Proof.** We may assume that \( n = i \) and \( d = 2t + i + 1 \).

Let \( T \) be a tableau of shape \( \nu \) labeled with the integers \( 1, 2, 3, \ldots, d \), each appearing \( n \) times, as explained in Figure 1 for the case \( t = 5, r = 3 \) and \( i = 13 \). Formally, if \( 1 \leq k \leq r \), the row \( k+1 \) of \( T \) has \( i + 1 \) boxes: \( k + 1 \) boxes are labeled \( k \), and the remaining \( i - k \) boxes are labeled \( 2t + 1 - k \). If \( r < k \leq t \), then the row \( k + 1 \) of \( T \) has \( i \) boxes: \( k + 1 \) boxes are labeled \( k \), and the remaining \( i - k - 1 \) boxes labeled \( 2t + 1 - k \). The first row of \( T \) starts with the first \( i + 1 \) boxes labeled with \( 2t + 1, \ldots, d = 2t + i + 1 \), respectively, and all the remaining \( j \) labels are put in the singleton columns of \( T \) such that each integer in \( 1, \ldots, d \) appears exactly \( n \) times. Note that each integer \( 1, \ldots, d \) appears in at least one singleton column since \( n \geq i \geq 2t + 3 \).

![Figure 1](https://www.ams.org/journal-terms-of-use)

**Figure 1.** Proposition 7.2, \( t = 5, r = 3, i = 13, d = 24, n = 13, D = 10, dn = 312, j = 230 \).

Put \( D := 2t \). By construction, for any \( 1 \leq u \leq D \) in \( T \), \( u \) appears in row 1 and in a unique row \( k_u + 1 \) for some \( 1 \leq k_u \leq t \). Let \( \beta(u) \) denote the number of occurrences of \( u \) in row \( k_u + 1 \). Note that \( 2 \leq \beta(1) < \beta(2) < \cdots < \beta(D) \) by construction. Using the notation introduced before the proof, we have by construction

\[
(7.2) \quad |C^1_u| = n - \beta(u) \leq n - 2 \quad \text{for} \quad 1 \leq u \leq D \quad \text{and} \quad |C^1_u| = n - 1 \quad \text{for} \quad D < u \leq d.
\]

We consider now the tensor

\[
\Phi := \bigotimes_{u=1}^{D} (X^{\otimes \beta(u)}_{k_u+1} \otimes X^{\otimes (n-\beta(u))}_{1}) \otimes \bigotimes_{u=D+1}^{d} X^{1}_{u},
\]

which, more precisely, is defined by the map, \( s : [dn] \to [N] \),

\[
s_{(u-1)n+v} = \begin{cases} \ k_u + 1 & \text{if} \ 1 \leq u \leq D \ \text{and} \ 1 \leq v \leq \beta(u), \\ 1 & \text{otherwise}. \end{cases}
\]

Using the notation introduced before the proof, we have \( |C^1_u| = |B^1_u| = n - \beta(u) \) for \( 1 \leq u \leq D \). Hence, by (7.2), the assumptions of Claim 7.1 are satisfied. So if
Let $\vartheta: \lambda \to [dn]$ respects $T$ and gives a nonzero contribution to $\langle \nu_T, t \rangle$, then $\vartheta$ bijectively maps $C_{i_u}^1$ to $B_u^1$ for all $1 \leq u \leq D$ since $\tau_0(u) = u$; see (7.1). Hence a box $\square$ with the label $u$, which is not in the first row of $T$, is mapped to a position in the $u$th block $B_u$. By the definition of $s$, this implies that $s(\vartheta(\square))$ equals the row number of $\square$. This also holds true for boxes in singleton columns of $T$. We conclude from Theorem 4.7 that $\vartheta$ contributes the value 1 to $\langle \nu_T, \Phi \rangle$.

To complete the proof, it is sufficient to show the existence of some map $\vartheta: \lambda \to [dn]$ that respects $T$, which is now obvious. In fact, there are

$$(n!)^{n-D}(n-D)! \prod_{u=1}^{D} \beta(u)! \cdot (n-\beta(u))!$$

that all contribute the value 1 to $\langle \nu_T, \Phi \rangle$, since $\vartheta$ can permute within every label the $X_{k_u+1}$ terms and the $X_1$ terms, and $\vartheta$ can as well permute the labels $D + 1, \ldots, n$. $\square$

By generalizing this construction in the following, we can show the following.

**Proposition 7.3.** Let $t, r$ be positive integers, let $i \in \left[\frac{(r+2t)^2}{2t}, \frac{(r+2t)^2}{2t} + r + t + 1\right]$, and let $n > 6t + 2r$ and $d > r + 2t + i$. Let $\nu = (t+1) \times i + (r+1) \times 1 + (j)$, where $j = dn - (r+1) - (t+1)i$. Then $a_\nu(d[n]) > 0$.

**Proof.** If $r < t$, then we can directly apply Proposition 7.2 noticing that

$$2t + 2 \leq \frac{(r+2t)^2}{2t} \leq i \leq \frac{(t+2t)^2}{2t} + r + t + 1 = \frac{11}{2}t + r + 1 \leq 6t + r \leq n.$$  

Let now $r \geq t$. The proof is similar to the proof of Proposition 7.2 so we describe a more general construction which applies in the case $r < t$ as well. Define $e := 2(\lfloor (r-1)/(2t) \rfloor + 1)$, so that $r \leq te \leq r + 2t - 1$ and $e$ is even. Put

$$i' := (te + 1)\frac{e}{2} \leq (r+2t)\frac{e}{2} \leq (r+2t)[\left\lfloor \frac{r-1}{2t} \right\rfloor + 1] \leq (r+2t)\frac{(r+2t-1)}{2t} \leq i.$$  

We will prove the statement for $i = i'$. When $i > i'$, the tableau construction below can be modified by increasing the number of appearances of the $t$ largest labels by $i - i' \leq r + t$ in the subtableau $T'$ as defined below. By assumption, $n > 6t + 2r \geq te + 2$ and $d > r + 2t + i \geq te + i + 1$. Indeed, we will prove the statement for the more general case in which we do not require $n > 6t + 2r$ and $d > r + 2t + i$, but only $n \geq te + 2$ and $d \geq te + i + 1$. It suffices to prove the statement with $n = te + 2$ and $d = te + i + 1$.

Let $T$ be a tableau of shape $\nu$ filled with the labels $1, 2, 3, \ldots, d = te + i + 1$, each number appearing $n = te + 2$ times, as in Figure 2 for the case $t = 2, r = 8, e = 4, i = 18, n = 10, d = 27$.

In the first row and in the first $i+1$ columns, we have the labels $te+1, \ldots, te+i+1$. In the first column and in the rows $2$ to $r+1$, we have the labels $te, te-1, \ldots, te-r+1$. The remaining rectangular $t \times i$ subshape of $T$, denoted $T'$, consisting of the columns $2$ to $i+1$ and the rows $2$ to $t+1$, is filled with the remaining labels $1, \ldots, te$, so that each label appears a different number of times. More precisely, for each $1 \leq s \leq te$, let the label $s$ appear in $T'$ exactly $s$ times and only in row $\min(\ell, 2t - \ell + 1)$, where $s \equiv \ell \pmod{2t}$, $1 \leq \ell \leq 2t$. (Note that the first row in $T'$, which we are referring to, is actually the second row in $T$.) So the row $k$ of $T'$ contains the $e$ different
labels $k, 2t + 1 - k, 2t + k, 4t + 1 - k, \ldots, t(e - 2) + k, te + 1 - k$, each appearing that many times, adding up to the row length of

$$\sum_{\alpha=1}^{e/2} \left( (2(\alpha - 1)t + k) + (2\alpha t + 1 - k) \right) = (te + 1)\frac{e}{2} = i.$$  

The remaining labels of each kind are then put in the singleton boxes of $T$.

As in Proposition 7.2, we show that the corresponding highest weight vector $v_T$ in $\text{HWV}_\nu(\text{Sym}^d \text{Sym}^n V)$ is nonzero by contracting it with a particular monomial tensor $\Phi$. For each label $u$, $1 \leq u \leq d$, let the associated monomial be

$$m_u = \bigotimes_{\square \in T, \text{label}(\square) = u} X_{\text{row}(\square)},$$

where the product goes over all boxes of $T$ labeled $u$ and for each such box we take the variable $X$ whose index is the row the box is in. Again, let $\Phi := \bigotimes_{u=1}^d m_u$ be the tensor. We compute the contraction $\langle v_T, \Phi \rangle$ with Theorem 4.7. The crucial observation is again that the labels of $T$ below row 1 all appear a different number of times, so each $X_k$ appears in distinct nonzero degrees in the monomials $m_u$. The rest of the proof of Proposition 7.2 applies almost verbatim, and we see that $\langle v_T, \Phi \rangle \neq 0$. Again, as in the case of $r \leq t$, we see that all nonzero summands in Proposition 4.7 have the value 1 and there exists a nonzero summand that is trivial to construct. \hfill \Box

Finally we can complete the proof of the promised technical result.

Proof of Theorem 6.2 We apply Proposition 7.3 with $r = b - 1 \leq m^2 - 1$ and $t = c - 1 \leq m^2 - 1$. We have

$$\frac{(r + 2t)^2}{2t} = \frac{(b + 2c - 3)^2}{2(c - 1)} \leq \max \left( \frac{(b + 1)^2}{2}, \frac{(b + 2m^2 - 3)^2}{2(m^2 - 1)} \right) \leq m^4,$$

where we use the fact that $(b + 2c - 3)^2/2(c - 1)$ is a convex function of $c$ and so attains its maximum at the endpoints of the interval $[2, m^2]$. We can then find an even integer $i$ in the interval $\left[ \frac{(r + 2t)^2}{2t}, \frac{(r + 2t)^2}{2t} + r + t + 1 \right] \subseteq [1, m^4 + 2m^2]$. By
Proposition 7.3 there exists a highest weight vector \( f \) of weight \( \nu = b \times 1 + c \times i + 1 \times j' \) in \( \text{Sym}^d \text{Sym}^N V \) for
\[
d := 3m^4 > 3m^2 + 2m^2 + m^4 \geq r + 2t + i, \quad N := 8m^2 > 6t + 2r.
\]

We proceed now as for the proofs given in Section 6 (a). Recall the linear automorphism \( \Delta_{N,n} : \text{Sym}^N V^* \to \text{Sym}^N V^* \) from Lemma 5.2. By Proposition 5.1 we have \( \langle f, \Delta_{N,n}(p)^d \rangle \neq 0 \) for the power sum \( p := \varphi_{N,1}^N + \cdots + \varphi_{N,d}^N \) and generic \( \varphi_i \in V^* \). Moreover, by Theorem 2.5, \( q := X_{1-N}^n p \) is contained in \( \Omega_n \) for all \( n \geq dN \), in particular for \( n \geq 24m^6 \). Consider the lifting \( h \in \text{Sym}^d \text{Sym}^N V \) of \( f \); it has the weight \( \lambda = \nu^{2dn} \) with \( dn = 3m^4n \). By Theorem 5.4 we have \( \langle h, q^d \rangle = \langle f, M^*(q)^d \rangle = \langle f, \Delta_{N,n}(p)^d \rangle \neq 0 \). Therefore, \( \lambda \) occurs in \( \mathbb{C}[\Omega_n]^{3m^4} \).

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