F. Reinitzhuber, "Beitrag zur Berechnung der Eigenschwingungszahlen räumlicher Stabwerke," *idem*, v. 8, 1937, p. 349–363.

[3] W. Prager, "Die Beanspruchung von Tragwerken durch schwingenden Lasten," *Ingenieur-Archiv*, v. 1, 1930, p. 527–532; S. Gradstein and W. Prager, "Beanspruchung und Formänderung von Stabwerken bei erzwungenen Schwingungen," *Ingenieur-Archiv*, v. 2, 1931, p. 622–650; "Berichtigungen," v. 3, 1932, p. 434. Tables of $\tan \lambda/2$, $\cot \lambda$, $\csc \lambda$, $\sec \lambda/2$, are given on p. 643–646; and those for the last group of functions listed above, p. 647–650. See also E. Fliegel, "Die Elasticitätsgleichungen zweiter Art der Stabwerksdynamik," *idem*, v. 9, 1938, p. 20–28, where $\phi/(\phi^2 + \psi^2)$, $\psi/(\phi^2 - \psi^2)$, $1/(\phi - \psi)$, and $1/\phi$ are tabulated for $\lambda = [0.00(0.50)5.00; 2D]$; and R. E. Gaskell, "On moment balancing in structural dynamics," *Quarterly of Applied Mathematics*, Brown University, v. 1, 1943, p. 237–249.

RECENT MATHEMATICAL TABLES

108[K].—W. G. Cochran, "The distribution of the largest of a set of estimated variances as a fraction of their total," *Annals of Eugenics*, v. 11, 1941, p. 47–52. 20.2 × 27.5 cm.

In this paper the distribution of

$$g = (\text{largest } u_i)/\sum_{i=1}^{n} u_i$$

is studied for the case where $u_1, u_2, \cdots, u_i, \cdots, u_n$ are mutually independent random variables each distributed as $\chi^2\sigma^2$ for r degrees of freedom, i.e.,

$$p(u_i)du_i = \frac{u_i^{(r-2)/2}e^{-u_i/2\sigma^2}}{(2\sigma^2)^{r/2}\Gamma(r/2)}\,du_i, \qquad (i = 1, 2, \cdots, n).$$

The values of g which have a probability 0.05 of being exceeded in random sampling are tabulated for $n = [3(1)10; 4D]$, $r = 1(1)6(2)10$.

The utility of these "5% points of g" derives from the fact that u may be interpreted as the "sum of squares with r degrees of freedom" corresponding to an unbiased estimate, $s^2 = u/r$, of the variance, σ^2, of a Gaussian population. In consequence, the distribution of g provides the distribution stated in the title, and the "5% points of g" provide a test of whether a variance estimate which appears to be anomalously larger than the others of a set is an estimate of the same σ^2.

The distribution of g for the case of $r = 2$ was obtained by R. A. Fisher, "Tests of significance in harmonic analysis," R. So. London, *Proc.* v. 125A, 1929, p. 180–190, who proposed its use in the harmonic analysis of a series to test the statistical significance of a particular term which is picked out on inspection because of its exceptional magnitude. In this paper Fisher gives a table of the 5% points of g to 5D for $r = 2$ and $n = 5(5)50$. In R. A. Fisher, "The sampling distribution of some statistics obtained from non-linear equations," *Annals of Eugenics*, v. 9, 1939, p. 238–249, the distribution of g is employed to illustrate the disturbance introduced into the analysis of variance test of a regression equation when the equation is 'non-linear' in the sense that the eliminant of the regression coefficients is a non-linear function of the observations which satisfy the regression equation exactly. W. L. Stevens, "Solution to a geometrical problem in probability," *Annals of Eugenics*, v. 9, 1939, p. 315–320, found that the distribution of g for $r = 2$ provides the answer to the following question: "On the circumference of a circle of unit length, n arcs, each of length x, are marked off at random. What is the probability that every point of the circle is included in at least one of the arcs?" This curious coincidence is explained by R. A. Fisher, "On the similarity of the distributions found for the test of significance in harmonic analysis, and in Stevens' problem in geometrical probability," *Annals of Eugenics*, v. 10, 1940, p. 14–17, and the distribution of the second largest ratio, g_2, when $r = 2$ is also discussed. This paper is followed, p. 17, by W. L. Stevens, "A table of the test of significance in harmonic analysis," in which the 5% points of g_1 (our g) and g_2 are tabulated for $r = 2$ and $n = [3(1)10(5)50; 5D]$.

The only full table of the distribution of g which I have come across is Table 2 in H. T. Davis, *The Analysis of Economic Time Series*, Bloomington, Indiana, Principia Press, 1941, where for $r = 2$ the probability of g exceeding k/n in random sampling is tabulated to 5D for $k = 0.1(0.1)10.0$; $n = 10(10)70$ and for $k = 5.1(0.1)10.0$; $n = 80(10)160(20)300$.

C. EISENHART

109[K].—FRIEDA S. SWED and C. EISENHART, "Tables for testing randomness of grouping in a sequence of alternations," *Annals Math. Stat.*, v. 14, 1943, p. 66–87. 16.3 × 23.4 cm.

If two different kinds of objects are arranged along a line, they will form two or more distinct sets of like objects. For example, if 4 a's and 3 b's are arranged as follows: *aaabbab* there are 4 such groups. In general, if there are m objects of one kind and n of another, there are C_m^{m+n} different possible arrangements. The proportion of these arrangements having u distinct sets of like objects is

$$(1) \qquad f_u = \frac{2\, C_{k-1}^{m-1} \cdot C_{k-1}^{n-1}}{C_m^{m+n}}, \quad \text{if} \quad u = 2k, \quad \text{(i.e., } u \text{ even)}$$

$$= \frac{C_{k-1}^{m-1} \cdot C_{k-2}^{n-1} + C_{k-2}^{m-1} \cdot C_{k-1}^{n-1}}{C_m^{m+n}}, \quad \text{if} \quad u = 2k - 1, \quad \text{(i.e., } u \text{ odd).}$$

We can assume $m \leq n$ without loss of generality. Then (1) holds for $k = 1, 2, \cdots, m + 1$. The probability law (1) was derived by W. L. Stevens (*Annals of Eugenics*, v. 9, 1939, p. 10–17). Wald and Wolfowitz (*Annals Math. Stat.*, v. 11, 1940, p. 147–162) have used the function u as a criterion for testing the statistical hypothesis that a sample of m elements and a second independently drawn sample of n elements (both known to be from continuous probability distributions) could have come from identical probability distributions. Wald and Wolfowitz have shown that if m and n are large, then the quantity

$$(2) \qquad x = \left(u - \frac{2m}{1+\alpha}\right) \Big/ 2\left(\frac{\alpha m}{(1+\alpha)^3}\right)^{1/2}$$

where $\alpha = m/n$, is approximately distributed according to the normal probability law

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

Now let us write

$$P\{u \leq u'\} = \sum_{u=2}^{u'} f_u / C_m^{m+n}.$$

The authors have presented in Table I the values of $P\{u \leq u'\}$, to 7D, for $m \leq n \leq 20$ and $2 \leq m \leq 20$.

Now let us denote by u_ϵ the largest integer for which $P\{u \leq u_\epsilon\} \leq \epsilon$ when $\epsilon < 0.50$, and the smallest integer for which $P\{u \leq u_\epsilon\} \geq \epsilon$ when $\epsilon > 0.50$. Table II gives the values of u_ϵ for $\epsilon = 0.005, 0.01, 0.025, 0.05, 0.95, 0.975, 0.99$, and 0.995, and for $m \leq n \leq 20$ and $2 \leq m \leq 20$.

By making use of the approximately normally distributed variable x in (2) the authors have given in Table III values of u_ϵ for $m = n$ from 10 to 100. In making this tabulation, a continuity correction was used in order to improve the approximation to u_ϵ for small values of m and n; in particular the values $10 \leq m = n < 20$. The authors give several examples to illustrate the use of their tables.

These tables should prove extremely useful as a basis for making simple significance tests in a wide variety of statistical problems which can be reduced to a treatment of sequences of two kinds of elements.

S. S. W.

110[K].—E. S. PEARSON and H. O. HARTLEY, "The probability integral of the range in samples of n observations from a normal population," *Biometrika*, v. 32, 1942, p. 301–310. 19.3 × 27.3 cm.

If $x_1, x_2, \cdots, x_n$ denote a random sample of n observations arranged in ascending order of magnitude, the sample range is defined as $x_n - x_1$. The authors consider the quantity

$$w = \frac{x_n - x_1}{\sigma},$$

assuming that the sample has been drawn from a normal or Gaussian population with standard deviation σ. If $f_n(w)dw$ is the probability element of w, the cumulative distribution law of w is defined as

$$P_n(W) = \int_0^W f_n(w)dw.$$

It is shown that

$$P_n(W) = \left(\int_{-\frac{1}{2}W}^{\frac{1}{2}W} z(x)dx\right)^n + 2n\int_{\frac{1}{2}W}^{\infty} z(u)\left(\int_{u-W}^{u} z(x)dx\right)^{n-1} du,$$

where

$$z(x) = (2\pi)^{-1/2}e^{-1/2x^2}.$$

In Table 1 of the present paper, values of $P_n(W)$ are calculated, to 4D, for $W = 0.00$ (0.05)7.25 and for $n = 2(1)20$.

L. H. C. Tippett (*Biometrika*, v. 17, 1925, p. 386–387) has tabulated the mean value of w $\left(\text{i.e., the value of } \int_0^\infty wf_n(w)dw\right)$ for $n = 2(1)1000$.

If a constant a_n is chosen so that the mean value of $a_n(x_n - x_1)$ is equal to σ, then $a_n(x_n - x_1)$ may be considered as an estimate of σ, although not a highly efficient one. The efficiency of this estimate of σ has been discussed by O. L. Davies and E. S. Pearson (R. Stat. So., *Jn.*, supp. 1, 1934, p. 76). In Table 2 of the paper under review, values of a_n are tabulated for $n = 2(1)12$. In Table 2 are also tabulated values of W such that $P_n(W) = 0.001$, 0.005, 0.01, 0.025, 0.05, 0.10, 0.90, 0.95, 0.975, 0.99, 0.995, 0.999 for $n = 2(1)12$.

S. S. W.

111[K].—MAXINE MERRINGTON, "Table of percentage points of the t-distribution," *Biometrika*, v. 32, 1942, p. 311. 19.3 × 27.3 cm.

By setting $n_1 = 1$ and $n_2 = n$ in the incomplete beta-function

$$\frac{\Gamma(\frac{1}{2}n_1 + \frac{1}{2}n_2)}{\Gamma(\frac{1}{2}n_1)\Gamma(\frac{1}{2}n_2)} x^{\frac{1}{2}n_1 - 1}(1 - x)^{\frac{1}{2}n_2 - 1}$$

and setting

$$t = \sqrt{n(1 - x)/x}$$

one obtains the distribution function

$$g_n(t)dt = \frac{2\Gamma(\frac{1}{2}n + \frac{1}{2})}{\sqrt{n}\sqrt{\pi}\,\Gamma(n/2)} \frac{dt}{(1 + t^2/n)^{(n+1)/2}} \quad (0 < t < \infty)$$

of the absolute value of the "Student" ratio for n degrees of freedom. Mrs. Merrington has tabulated values of t_P for which

$$\int_{t_P}^{\infty} g_n(t)dt = P,$$

to 5D, for $P = 0.50$,0.25, 0.10, 0.05, 0.025, 0.01, 0.005, and for $n = 1(1)30$, 40, 60, 120, ∞.

The table was derived from Miss Thompson's table RMT **99**.

S. S. W.

112[K].—E. S. PEARSON and H. O. HARTLEY, "Tables of the probability integral of the studentized range," *Biometrika*, v. 33, 1943, p. 89–99. 19.3 × 27.3 cm.

Denote by $x_1, x_2, \cdots, x_n$ a random sample of n observations (in ascending order of magnitude) from a normal population with standard deviation σ. The authors of the present paper have tabulated (RMT **110**) the probability integral of the quantity $w = (x_n - x_1)/\sigma$.

In most practical statistical problems σ is unknown and hence the ratio w is unusable. Hence, it is desirable in problems of this kind to replace σ by an estimate of σ calculated from a sample. In the present paper, the authors consider two independent samples, the first consisting of n observations from which $x_n - x_1$ is calculated, and the second consisting of $\nu + 1$ observations, $X_1, X_2, \cdots, X_{\nu+1}$ from which an estimate s of σ is calculated as follows:

$$s^2 = \frac{1}{\nu} \sum_{i=1}^{\nu+1} (X_i - \bar{X})^2, \tag{1}$$

where $\bar{X}$ is the arithmetic mean of the $\nu + 1$ observations of the second sample. Instead of w the quantity

$$q = \frac{x_n - x_1}{s}$$

is considered. Denoting the probability law of q by $g_n(q)dq$, and the probability integral by

$${}_\nu P_n(Q) = \int_0^Q {}_\nu g_n(q)dq, \tag{2}$$

an approximation of the following form for ${}_\nu P_n(Q)$ is found

$${}_\nu P_n(Q) = P_n(Q) + \frac{1}{\nu} a_n(Q) + \frac{1}{\nu^2} b_n(Q) \tag{3}$$

where $P_n(Q)$ is the probability integral of w and

$$a_n(Q) = \frac{1}{4}\left\{Q^2 \frac{d^2P_n}{dW^2} - Q\frac{dP_n}{dW}\right\}$$

$$b_n(Q) = \frac{1}{16}\left\{\frac{Q^4}{2}\frac{d^4P_n}{dW^4} - \frac{Q^3d^3P_n}{3dW^3} - \frac{Q^2d^2P_n}{2dW^2} + Q\frac{dP_n}{dW}\right\}.$$

$P_n(Q)$, $a_n(Q)$, and $b_n(Q)$ are tabulated to four, two and one places of decimals respectively for $Q = .00(.25)6.00$, $n = 3(1)20$. The authors do not guarantee the last figure in $b_n(Q)$ for values of n in the upper end of the range. Values of Q are tabulated to two places of decimals (in some cases only one) for ${}_\nu P_n(Q) = 0.01, 0.05, 0.95, 0.99$ using approximation (3), for

$$n = 2(1)20, \qquad \nu = 10(1)20, 24, 30, 40, 60, 120, \infty.$$

Harmonic interpolation with respect to ν is made possible by the tabulations for this particular set of values of ν.

S. S. W.

113[B, D, E, G, H, L].—PAUL RUDOLF EUGEN JAHNKE (1863–1921) and FRITZ EMDE (1873–), *Tables of Functions. [Funktionentafeln mit Formeln und Kurven] . . . Published and distributed in the public interest by authority of the Alien Property Custodian under license No. A.57.* New York, Dover Publications, 31 East 27th Street, 1943. [xv], 303, 76 p. 15.4 × 29.8 cm. $3.50.

There is hardly any single volume which could be more useful to the general applied mathematician and the mathematical physicist, than this remarkable work of Jahnke and Emde, which has been developed during the past 34 years. The Alien Property Custodian

is to be congratulated not only upon his choice of such a work for reproduction at the present time, but also upon the publication price which places the volume within the reach of ordinary individuals and small libraries, as well as of those more opulent. That the opportunity was appreciated is illustrated by the fact that a thousand copies were ordered before publication on June 1.

The work started out in modest fashion as one of the series *Mathematisch-Physikalische Schriften für Ingenieure und Studierende*, no. 5, ed. by E. JAHNKE, and with the title

1. *Funktionentafeln mit Formeln und Kurven*, Leipzig, 1909, xii, 176 p. 15.3 × 23.7 cm. With 53 text figures. The text contained the following XIII main headings: I, The functions $x \tan x$ and $\tan x/x$; II, Roots of transcendental equations; III, Transformation of $a + bi$ into $re^{\phi i}$, and conversely; IV, The exponential functions e^x and e^{-x}; V, Hyperbolic functions; VI, Sine, cosine, and logarithmic integral; VII, Fresnel integrals; VIII, Gamma function; IX, The Gauss error integral ϕx; X, The Pearson function $F(r, \nu)$; XI, Elliptic integrals and functions (p. 46–79); XII, Spherical harmonics (p. 79–89); XIII, Bessel functions (p. 90–174, that is, about half of the volume). The work is filled with formulae, and numerical tables and a notable collection of more than fifty interesting text-figures, from which not only the general variation of the functions may be observed, but also their approximate numerical values can often be readily estimated for given arguments. There are many references to the literature, where more extensive material is to be found. The tables given are usually short and to a small number of decimal places. When this volume was published no tables existed for many of the higher functions, but after the first world war great advances had been made in this regard, so that there was an embarrassment of riches to deal with when EMDE (his colleague having died) came to prepare the second edition nearly 25 years after the first one had been published. This volume, nearly double that of the first edition in size, was in two languages throughout—German and English, with two title pages,

2. *Tables of Functions with Formulae and Curves*, second (revised) edition, with 171 figures. *Funktionentafeln mit Formeln und Kurven*, Leipzig, 1933. xviii, 330 p. 16 × 24.2 cm. Compare RMT **32**.

The number of sections is here increased from XIII to XIX, ten of which were devoted to the elementary functions, the number of pages devoted to them being increased from 18 to 78. These sections bore the following titles: I, Table of powers; II, Auxiliary tables for computation with complex numbers; III, Cubic equations (graphical and tabular aids to their solution); IV, Elementary transcendental equations; V, $x \tan x$, $\tan x/x$, and $\sin x/x$; VI, Exponential function; VII, Planck's radiation function; VIII, Source functions of heat conduction; IX, Hyperbolic functions; X, Circular and hyperbolic functions of a complex variable. These sections were concluded with an Index of tables of the elementary transcendentals.

Then comes XI, Sine, cosine, and logarithmic integral tables. The Pearson function table is eliminated (see *MTAC*, p. 70), and the Gauss error integral and Fresnel integrals are brought together under XIII, Error integral and related functions. The Gamma function is here treated under XII, Factorial function, much enlarged and greatly enriched with charts and relief drawings. Section XIV, Theta functions, is now separated from the single section on elliptic integrals and functions which has been divided into XV, Elliptic integrals, and XVI, Elliptic functions. The material here has been much expanded and the drawings and reliefs are indeed notable new features. So also with respect to XVII, Legendre functions, and XVIII, Bessel functions (p. 192–318). The latter section has been greatly extended. The Nielsen definition of the functions of the second and third kind (N and H—although Y and H are more commonly used in Great Britain, but this Y is not used by Watson) is now uniformly introduced in all of the tables. The final section, XIX, The Riemann zeta-function, is new. In every section after X the lists of "text-books and other tables," adds largely to the usefulness and value of the volume. The representation of complex functions by surfaces whose ordinates are equal to the modulus of the function, have been referred to above as "reliefs" of the function. This graphical method, adopted here for the first time, is an innovation of outstanding importance.

Even without the introductory article of our last issue special reference would naturally be made to the following quotation from the preface: "As in the first edition great use has been made of the work of the Mathematical Tables Committee of the British Association. Fortunately this committee has decided to publish collections of the very accurate tables which they have calculated in past years. Two volumes have already been published.[1] The mathematicians, physicists, and engineers of the whole world regard with the greatest wonderment and gratitude this colossal undertaking of their English colleagues, who have taken upon themselves almost entirely the heavy load of new computation. It is hardly to be conceived that other countries can continue much longer to look idly on without helping in this work."

EMDE published **3.** third (revised) edition of his work, "with 181 figures," in Leipzig, Teubner, 1938, xii, 305 p. Here again important new revisions and additions were made, especially with a view to providing advanced research workers with an up-to-date desirable tool. Emde summarizes as follows the chief points in which the third edition differs from the second: "In the complete elliptic integrals of the first and second kind, formulae and numerical tables have been added for other than the Legendre standard forms. The numerical calculation is thereby improved in many cases. In the cylinder functions the Debeye series have been brought into a more convenient form by limiting them to real values of the index and by limiting the angle of the argument to values which occur in practice; they have thus been freed from the additional determinations which necessarily added to the difficulties of the general case. Detailed numerical tables are given for the Lommel-Weber and Struve functions of orders zero and one. Formulae and graphical representations are given for the confluent hypergeometric functions and for the Mathieu functions associated with the elliptic cylinder, which will enable these two classes of functions to be employed more widely in scientific calculations."

This new edition had also 40 new figures and reliefs. The elementary functions occupying 76 pages of the second edition are not included in the third.[2] These pages have been elaborated into an independent work,

3A. oF. EMDE, *Tafeln Elementarer Funktionen*, Leipzig, Teubner, 1941, xii, 181 p., 83 figures, text in German and English. I know of no copy of this work which has reached this country. The eleven main topics of no. **3** are now as follows: I, Sine, cosine, and logarithmic integral; II, Factorial function; III, Error integral and related functions; IV, Theta-functions; V, Elliptic integrals; VI, Elliptic functions; VII, Legendre functions; VIII, Bessel functions (p. 126–168); IX, The Riemann zeta function; X, Confluent hypergeometric functions; XI, Mathieu functions.

4. Of this third edition in 1941 G. E. Stechert and Co. of New York, purchased from Teubner the right to print a facsimile edition. Since this reprint (now sold at $3.00) has been constantly available for purchase by scholars since 1941 it caused considerable surprise to learn that in 1943 the Alien Property Custodian had authorized the Dover Publications to produce

5. the fifth edition of JAHNKE and EMDE's work, the volume under review. This volume includes not only no. **3**, but also the first 76 pages (about elementary functions) of no. **2**. In most respects the volume makes an excellent impression.

We have good reason for believing that while in ignorance of the Stechert edition the Alien Property Custodian was fully justified in authorizing the production of no. **5**. As we have indicated above, no. **4** is also for sale, until the edition is exhausted.

References to errors may now be mentioned. Errors in $F(\phi, \alpha)$ include, (a) $F(60°, 60°)$, *for* 1.2125 *read* 1.2126; (b) $F(37°, 75°)$ *for* 0.6913 *read* 0.6919. These are in no. **1**, p. 57, 58; in no. **2**, p. 137, 138; in nos. **3–5**, p. 65, 66. Compare RMT **50** (*Scripta Mathematica*, v. 3, p. 365). Mr. W. D. LAMBERT, of the Coast and Geodetic Survey, discovered an error in no. **2**, p. 131, and in nos. **3–5**, p. 59, in one of three forms of elliptic integrals equal to $\frac{1}{2}F(k, \phi)$. "In the first of these forms, $\int_x^1 \frac{dt}{\sqrt{1+t^4}}$, $\cos\phi = x\sqrt{2}/\sqrt{1+x^4}$. The corre-

sponding modulus should then be $1/\sqrt{2}$ or $\sin 45°$, and not $2(\sqrt{2}-1)\sqrt[4]{2} = 0.985\,171\,4 = \sin 80°.1207$. This latter value of the modulus, k_1, is obtained from $\int_x^1 \frac{dt}{\sqrt{1+t^4}} = (2-\sqrt{2})F(\omega, k_1)$, where ω is determined from

$$\tan\omega = (\sqrt{2}+1)(1-x)/(1+x).$$

These are the familiar Landen relations $k_1 = 2\sqrt{k}/(1+k)$ and $\sin(2\omega - \phi) = k \sin\phi$."

Mr. LAMBERT pointed out to me also that O. S. ADAMS in his *Elliptic Functions Applied to Conformal World Maps*, Washington, 1925, p. 58, recorded two serious errors in the table (to 4D) of Weierstrass functions, no. **1**, p. 75, namely: the value of $\wp(u)$ for $r = 3$ should be 1537.9625 instead of 1468.820; and the value of $\wp'(u)$ for $r = 35$ should be -75.9603 instead of -73.4302. These errors occur also in all later editions of J. and E. as well as in the table computed by A. G. HADCOCK (to 5D) and published by A. G. GREENHILL in R. Artillery Institution, *Minutes and Proc.*, v. 17, 1889, p. 212–213. Mr. Adams verified the accuracy of other values of $\wp(u)$ and $\wp'(u)$ in this table for $r = 5$ to $r = 59$ inclusive; it continues to $r = 240$.

In Amer. Math. So., *Bull.*, v. 3, 1897, p. 153–155, R. W. WILLSON and B. O. PEIRCE give a "Table of the first forty roots of the Bessel equation $J_0(x) = 0$ [to 10D] with corresponding values of $J_1(x)$ [to 8D]." $J_1(17)$ is given as $+0.10999114_5$. In all editions of J. and E. this is abridged to 4D. HARRY BATEMAN pointed out to me that for $J_1(17)$, $+0.1010$ is always given, instead of $+0.1100$. He drew my attention also to the fact that in the table of the roots of $J_0(x)N_0(kx) - J_0(kx)N_0(x) = 0$, on p. 204 of nos. **3–5**, p. 274 of no. **2**, and p. 162 of no. **1**, all 6 roots for each of three values of k (1.2, 1.5, 2.0) are wrong. These 18 values are given correctly in M. MUSKAT, F. MORGAN, M. W. MERES, "The lubrication of plane sliders of finite width," *Jn. Appl. Physics*, v. 11, 1940, p. 212. The table printed in J. and E. first appeared in A. KALÄHNE, *Zeits. f. Math. u. Phys.*, v. 54, p. 68, 1906.

Mr. R. D. BROWN Jr., of Blue Bell, Penna., reported that the value of $J_9(21) = -0.3175$, given on p. 248 of no. **2**, and p. 177 of nos. **3–5**, does not agree with what appears to be the correct value, $-.03175$ (verified by H. Bateman), given on p. 296 of A. GRAY and G. B. MATHEWS, *A Treatise on Bessel Functions*, second ed. by A. GRAY and T. M. MACROBERT, London, 1922.

We shall be happy to receive for publication other reports of errors in this work.

R. C. A.

[1] Emde's reference is doubtless to B.A.A.S., *Mathematical Tables*, v. 1, *Circular and Hyperbolic Functions, Exponential Sine and Cosine Integrals, Factorial (Gamma) and Derived Functions, Integrals of Probability Integral*, London, 1931; and v. 2, *Emden Functions being Solutions of Emden's Equation together with Certain Associated Functions*, London, 1932. The first volume published by the Committee was, however, A. J. C. Cunningham's *A Binary Canon*, London, 1900, viii, 172 p. In 1901 the Committee recommended the publication of the same author's *Quadratic Partitions*, but this work was published otherwise in 1904.—EDITOR.

[2] The "Index of tables of the elementary transcendentals," occupying p. 76–78 of the second edition, has been revised and improved on p. 299–301 of the third edition. The date of publication (1921) of U. Meyer, *Fluchtlinientafeln des Hyperbeltangens einer komplexen Veränderlichen* is, however, still lacking.

114[Q, U].—A. A. AGETON, *Manual of Celestial Navigation*, New York, Van Nostrand, 1943. 103 p. 17.4 × 24.4 cm. $1.85. The first printing of this book was in April 1942; the fifth in February 1943.

This volume contains two basic tables which allow one to use an Assumed Position method, similar to but an improvement over that of J. Y. Dreisonstok, *Navigation Tables for Mariners and Aviators*, (H. O. 208, RMT **103**), or a Dead Reckoning Position method identical with that presented by the author in his *Dead Reckoning Altitude and Azimuth*

Table (H. O. 211, RMT **104**). Both methods give an accuracy of solution to within five-tenths of a minute of arc in altitude without interpolation; this is adequate for all ordinary navigation on land, at sea and in the air. The author is a Commander in the U. S. Navy and Head of the Navigation Division of the Department of Seamanship and Navigation at the United States Naval Academy at Annapolis.

In the notation of RMT **103**, in his AP method, Ageton drops a perpendicular from Z on the great circle through P and S. The foot of the perpendicular is X and the sides PX and ZX are called $90° - K$ and R respectively. Table I is a double-entry table in which are tabulated three quantities associated with the polar triangle, B, K, and Z', for integral values of the arguments $L(0°$ to $90°)$ and $t(1°$ to $179°)$. K has already been defined as the declination of X; it is given to the nearest tenth of a minute of arc. B is $10^5 \log \csc R$, and is given to the nearest integer except that one decimal is given for $L(0°$ to $63°)$ and $t(1°$ to $20°)$, and for $L(64°$ to $90°)$ and $t(1°$ to $16°)$. Z' is the angle PZX and is given to the nearest tenth of a degree. All of the values corresponding to a single latitude are presented in two columns on two pages facing each other, and may be reached with a single opening of the book! This marks a great improvement over H. O. 208.

The formulas used to complete the solution are

$$A(h) = B(R) + B(K \sim d), \qquad A(Z'') = A(K \sim d) - B(h), \qquad Z = Z' + Z''.$$

As in H. O. 211, $(K \sim d)$ is used to indicate $|K - d|$, and $A(x)$ and $B(x)$ are used to denote $10^5 \log \csc x$ and $10^5 \log \sec x$ respectively.

Table II is similar to the one contained in H. O. 211 except that only the values of A and B corresponding to angles with an integral number of minutes of arc are given in their entirety; the end figures which differ from the complete values above them are given opposite the half-minutes. This makes the table seem somewhat more open, and perhaps easier to read. It should be mentioned that the printing of the second table is poor; this is especially noticeable if one has been using the Hydrographic Office Tables which are printed at the Government Printing Office.

Both Tables I and II have good thumb indices along the right hand edge of the page. They are easy to use but would be even easier to use if the back cover of the book were more flexible. In the index to Table II, the value of A given beneath 0° is 38400 and should be 384000. This will doubtless be corrected in another printing.

Inasmuch as the DRP method given in this volume is identical with that presented in H. O. 211 (RMT **104**), there seems to be no point in repeating the discussion of its advantages and disadvantages.

A comparison of Table I with a corrected copy of Dreisonstok, *Navigation Tables for Mariners and Aviators* (H. O. 208) indicates that the *Manual* has a much smaller number of large errors. Some of its errors are obviously inherited. Ageton mentions in the Preface that he has studied the works of Souillagouet, Bertin, Ogura, Weems, Dreisonstok, and Gingrich before undertaking the construction and arrangement of Table I. No mention is made of *Hughes' Tables for Sea and Air Navigation* by L. J. Comrie which first appeared in 1938 and which contains material similar in content and arrangement to that in the *Manual;* compare RMT **115**. Ageton's method of determining azimuth is different from that of Comrie.

It is interesting to note that the beautifully simple and rapid *short* method of computing azimuth included in the first printing is not contained in later printings. In a letter, the author states that it "proved to have unpredictable errors in it and so had to be eliminated from the later printings."

This is an excellent book to use in the teaching of celestial navigation. The elegance and simplicity of its two methods reduce the tedious details of a computation. Since it contains the elements for both an AP and a DRP method, it should appeal to many users. One suspects that a student well trained in the use of this volume will prefer it to any other.

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115[Q, U].—L. J. COMRIE, *Hughes' Tables for Sea and Air Navigation*, London, Henry Hughes, 1938. lvi, 182 p. 15 × 24.6 cm. 20 shillings.

Three separate tables and two methods, one AP and the other DRP, are offered for the solution of the fundamental problem of celestial navigation. An accuracy of approximately 0′.4 in computed altitude can be attained without interpolation by the use of the AP method; by using the DRP method with interpolation, an accuracy of 0′.1 can be had. The former is adequate for most navigation, more than adequate for air navigation; the latter will meet the most rigid demands of navigators. Comrie was formerly the Superintendent of the British Nautical Almanac Office and is one of the outstanding authorities of the world on the design, construction and accuracy of tables.

Using again the notation of RMT **103**, Comrie chooses to divide the astronomical triangle into two right triangles by dropping a perpendicular from Z on the great circle through P and S. Call the foot of this perpendicular, X, (Comrie uses Q), and let $PX = 90° - K$ and $ZX = R$. Also let

$$B(x) = 10^5 \log \sec x$$
$$C(x) = 10^5 \log \csc x$$

In Table I are given four quantities associated with the polar right triangles, K, A, D, and Z_1, with integral degree arguments, L(0° to 89°) and t(0° to 180°). As indicated above, K is the declination of the point, X; it is given to the nearest tenth of a minute of arc. A is $B(R)$; it is given to the nearest integer except that for values of A less than 665, and L between 0° and 34° inclusive, one decimal is given. D is $10^{-2}C(R)$; it is given to the nearest integer. Z_1 is the angle PZX; it is given to the nearest tenth of a degree.

Table I is undoubtedly the most accurate table of this type yet available to the general public, although there are approximately 500 insignificant errors of one unit in the last place. Comrie recomputed the values for all latitudes 62° to 89°; this section is remarkably free from error. Since all of the values corresponding to a single latitude are assembled on a single page, the table is extraordinarily convenient to use. No edge-thumb-index of the table is offered, although notched pages allow one to turn at once to each of the various tables.

The formulas used in the AP method are:

$$C(h) = A + B(K \sim d), \quad 10^3 \log \tan Z_2 = D + 10^3 \log \tan (K \sim d); \quad Z = Z_1 + Z_2.$$

Table II is a table of $B(x)$ and $C(x)$ with argument $x = [0°(0'.5)180°]$. B and C are given to the nearest integer except that one decimal is given for all values less than 665. Comrie has carefully eliminated rounding-off errors and the values given are correct to the last place.

Table III is a table of values of $10^3 \log \tan x$, each given to the nearest integer, with argument $x = [0°.1(0°.1)89°.9]$. The number -10000 is omitted for values of x less than 45°. This entire table is printed on two pages facing each other.

The formulas used in Comrie's DRP method are:

$$C(R) = B(L) + C(t), \qquad C(K) = C(L) - B(R),$$
$$C(h) = B(R) + B(K \sim d), \qquad 10^3 \log \tan Z_1 = 10^{-2}C(L) - 10^3 \log \tan h,$$
$$10^3 \log \tan Z_2 = 10^{-2}C(R) + 10^3 \log \tan (K - d).$$

Some persons will object to the elaborate and detailed explanations which occupy 56 pages at the beginning of the book. Others will appreciate the security afforded by having a complete explanation permanently attached to the tables.

In the Preface, Comrie modestly claims originality only in arrangement, accuracy, typography and explanation; it is precisely the high quality of these features that makes this book so outstanding in its field. In fact, if one looks for something to criticize, there are only these minor defects: three separate tables are required for each of Comrie's methods; no edge-thumb-index is offered for Tables I and II; some shifting of the decimal point is required at two places in his DRP method.

In addition to the three principal tables, a table of meridional parts to one decimal for latitudes [0°(1′)80°] and a traverse table giving Difference in Latitude, and Departure, each to one decimal, with arguments Bearing [1°(1°)89°] and Distance [0(1)100(100)1000] are given. Some twelve other auxiliary tables are presented which will meet most of the requirements of ordinary navigation.

C. H. SMILEY

116[C].—THEODOR WITTSTEIN, *Addition and Subtraction Logarithms to Seven Decimal Places*, Chicago, Ill., Charles T. Powner Co., P.O. Box 796, 1943. viii, 126 p. 17.2 × 22.9 cm. $2.00.

The name of the real author of this work is nowhere mentioned in the volume, but instead, after the above title, is merely given "Edited by William W. Johnson, Instructor in Applied Mathematics John Huntington Polytechnic Institute." This so-called editor contributed nothing but three and one half brief pages of illustrative material, partly copied, partly highly misleading, and quite inadequate for instructing the novice in the use of the volume. Pages 2–126 are complete facsimile reproductions of exactly these pages in the following well-known work of WITTSTEIN, with title pages and introductory matter in French and German: *Logarithmes de Gauss à sept Décimales pour servir à trouver le Logarithme de la Somme ou de la Différence de deux Nombres, leurs Logarithmes étant donnés, arrangés d'après une nouvelle méthode . . . Ouvrage servant de supplément à toute table ordinaire de Logarithmes à Sept Décimales. Siebenstellige Gaussische Logarithmen zur Auffindung des Logarithmus der Summe oder Differenz zweier Zahlen, deren Logarithmen gegeben sind . . . Ein Supplement zu jeder gewöhnlichen Tafel siebenstelliger Logarithmen*, Hannover, 1866, XVI, 127 p. 17.9 × 25.2 cm.

On the back of the title page of the work under review is "Copyright, 1943 P.R.C. Publications, Inc." For reasons suggested above, it is hardly conceivable that this notice implies any real equity on the part of the copyrighter, any more than it did in the case of William Charles Mueller who put his own name and the following original title on the cover: *A New Manual of Natural Trigonometrical Functions to Seven Places of Decimals of Sines and Cosines of the Angles from 0 to 10 000 seconds* [*sic*] of a work wholly prepared by W. JORDAN (see *MTAC*, p. 8), before having it copyrighted in 1907. The title proves that "editor" Mueller also did not really understand what was in his volume. America's record of pirating mathematical tables is getting to be unenviable. We learn that the fourth edition of the Barlow-Comrie Tables (RMT 82), not copyrighted in this country, has been pirated by the Chemical Publishing Co. of Brooklyn, N. Y., some time American agent for the sale of this English work. These are but three illustrations.

Addition and Subtraction Logarithms are used to obtain the logarithm of the sum or difference of two numbers when only their logarithms are known. Thus the result is found in one table entry, instead of three entries necessary when the ordinary table of logarithms is employed. If a and b are two positive numbers, $a > b$, then in the notation of the book under review

$$\log a - \log b = \log (a/b) = B = \log x;$$
$$\log (a - b) = \log b + A \quad \text{or} \quad A = \log (x - 1).$$

The table gives values for B or A from which the corresponding A or B are found. Since $x > 1$, $B > 0$, and $A \leqq 0$ when $x \leqq 2$. For this range B lies between 0 and $\log 2 = 0.301\,0300$; and A lies between $-\infty$ and 0. But the mantissas of A and B would be the same to about 7D at -7; hence the range for A can be taken from -7 to 0, or 3 to 10 if 10 is added to these values of A. It is these latter values for A which are given, p. 1–45, where A = [3.0(0.1)4.00(0.01)6.000(0.001)8.0000(0.0001)10.0000; 7D].

If $\log (a/b) = A = \log x$; and $\log (a + b) = \log b + B$, then $B = \log (x + 1)$. Here $A > 0$, $B > 0.3010300$, and when A reaches 7.0, B has become 7.000 0000. Here A = [0.0000(0.0001)4.0000(0.01)6.00(0.1)7; 7D], p. 46–126.

Wittstein's original page presented an excellent appearance which is naturally somewhat marred in reproduction, slightly reduced in size, with curtailed margins.

Another seven-place table of Addition and Subtraction Logarithms, by JULIUS ZECH, appeared as Table XII (p. 636–836, introduction p. XXI–XXIII) in the 1849 edition of Vega's *Sammlung mathematischer Tafeln*, edited by J. A. Hülsse. This was reprinted with an independent title page, but its original page numbers, at Leipzig and Berlin in 1849, 1892 (third) and 1910 (fourth ed.). The Zech form of table is now generally preferred to that of Wittstein so that a reproduction of this table would have been decidedly preferable to that of Wittstein. A modern 6-place form of the Zech table, is B. COHN, *Tables of Addition and Subtraction Logarithms with Six Decimals*, second ed., London, Scientific Computing Service, 1939; there is an excellent preface by L. J. Comrie.

After the above-mentioned 7-place tables of this type, there is a single table with a larger number of places, namely: M. H. ANDOYER, "Tables fondamentales pour les logarithmes d'addition et de soustraction," *Bull. Astronomique*, s. 2, v. 2, 1922, p. 5–32; the final Table III (p. 32) is of 10^x for $x = [0.00(0.01)1.00; 15D]$. As in the Zech table, if $\log a - \log b = d$, $\log (a + b) = \log a + A$, $\log (a - b) = \log a - S$, $A = \log (1 + 10^{-d})$, $S = -\log (1 - 10^{-d})$, and the table is for the Addition and Subtraction Logarithms A and S in terms of the parameter $d = [0.00(0.01)9.00; 16D]$.

A truly American table, original in form, is
L. M. BERKELEY'S *Addition-Subtraction Logarithms to Five Decimal Places*, New York, White Book and Supply Co., 1930.

The idea of a table of Addition and Subtraction Logarithms was first suggested by ZECCHINI LEONELLI (1776–1847) who for a time taught mathematics and architecture at Bordeaux. Here in 1802 or 1803 [*an onze*] he published his
o *Supplément Logarithmique, contenant La décomposition des grandeurs numériques quelconques en facteurs finis; . . . et La théorie des logarithmes additionels et déductifs, ou de certains logarithmes qui donnent directement les logarithmes des sommes et des différences des valeurs naturelles, dont on ne connoît que les logarithmes.*[1] This edition is not listed in the published catalogues of the Bibliothèque Nationale, or British Museum, or Library of Congress. Leonelli considered that a table of this kind to 14D was desirable.

Apparently it was this particular edition of Leonelli which first drew Gauss's attention to the subject. In a paper [2] of 1812 Gauss wrote about the importance of such logarithms in practical computing and gave a five-place table where $\log (1 + 1/x)$ and $\log (1 + x)$ are given with their differences for argument $\log x = 0.000(0.001)2.000(0.01)3.40(0.1)5.0; 5D]$. Leonelli gave a similar table, $\log x = [0.00000(0.00001)0.00104; 14D]$.

While Gauss simplified the method of presenting Addition and Subtraction Logarithms, there was, of course, no adequate ground for later German writers to refer to these as Gauss's, rather than Leonelli's Logarithms. Gauss suggested that his five-place table be enlarged to a seven-place table, "10 to 100 times more extensive." Such a table was published by E. A. MATTHIESSEN [3] in 1817 or 1818, but the arrangement of the table was very bad and it contained numerous errors.

When PETER GRAY was preparing his six-place
Gauss's Tables, re-computed and re-arranged . . . [Private and temporary issue], London, 1848, published in the following year in his *Tables and Formulae for the Computation of Life Contingencies*, his original calculations had thoroughly checked the inadequacies of Matthiessen. This work of Gray was materially extended in the second edition of *Tables and Formulae . . .*, London, 1870. He was the first to perceive the utility of Leonelli's logarithms "in the calculation of life contingencies, and to him is due their introduction as well as the calculation of the necessary tables, which it is evident are valuable mathematically, apart from the particular subject for which they were undertaken."

Comrie indicates the methods for applying a table of Addition and Subtraction Logarithms to finding latitude and longitude, $(a^2 - b^2)^{1/2}$, $(a^2 + b^2)^{1/2}$, and the solution of Kepler's equation. It may also be applied to many problems in trigonometry, surveying, and engineering.

And finally, among works in English, a reference may be given to one of the numerous editions of

C. BREMIKER. *Tables of the Common Logarithms of Numbers* . . ., London, Nutt, 1887, with "Gauss's Logarithms for Addition and Subtraction," p. 535–593. Of the three types illustrated by tables of Leonelli-Gauss, Bremiker or Wittstein, and Zech, that of Zech is the best.

For at least fifteen years J. T. PETERS has had a completed manuscript of Eight-place Addition and Subtraction Logarithms, but the opportunity for publication has not been offered.

Information about a number of other tables of the types here considered may be found in J. W. L. Glaisher's *Report* . . . *on Mathematical Tables*, 1873.

R. C. A.

[1] There was a second edition of this work, *Supplément Logarithmique par Leonelli; précédé d'une notice sur l'auteur* par J. HOÜEL, Paris 1875, x, 75 p. This was reprinted from *Publications de la Société des Bibliophiles de Guyenne.* It was translated into German: o *Leonelli's logarithmische Supplemente* . . . *aus dem Französischen nebst einigen Zusätzen* von G. W. LEONHARDI . . ., Dresden, 1806, 88 p. There was also an Italian edition, *Teorica e Pratica dei Logaritmi di Addizione e di Sottrazione* . . . *esposta con Tavole Calcolate a sette decimali dall' ingegnere*, PIETRO CAMINATI, Novara, 1879, 79 p. This pamphlet was reprinted from o *Rivista di Matematica Elementare*, s. 2, v. 1, 1879.

[2] C. F. Gauss, "Tafel zur bequemern Berechnung des Logarithmen der Summe oder Differenz zweier Grössen, welche selbst nur durch ihre Logarithmen gegeben sind," *Monatliche Correspondenz* (Zach), v. 26, 1812, p. 498–528. The table, which fills 27 small octavo pages, was constructed "many years ago," suggesting 1802 rather than 1806 when the German edition was published. The text, but not the table, was reprinted in Gauss, *Werke*, v. 3, Göttingen, 1866 and 1876, p. 244–246. Gauss's table is exactly reprinted in J. Pasquich, *Abgekürtzte logarithmisch-trigonometrische Tafeln* . . ., Leipzig, 1817; see Gauss, *Werke*, v. 3, p. 246–250. It was also added to J. DE LA LANDE'S *Tables de Logarithmes* . . ., in an edition edited by H. G. Köhler in o 1827; third ed., Leipzig, 1849, p. 213–250. 8.8×12.8 cm. The text (p. 251–260) exhibits applications of such a table.

[3] *Tafel zur bequemern Berechnung des Logarithmen der Summe oder Differenz zweyer Grössen welche selbst nur durch ihre Logarithmen gegeben sind*, Altona 1817. The Latin title on the opposite page is dated 1818, while the Introduction has the date March 1815. The work was reviewed by Gauss; see his *Werke*, v. 3, p. 250–252.

117[N].—FINANCIAL PUBLISHING COMPANY, *Monthly Bond Values showing accurate monthly values on bonds and other redeemable securities paying interest semiannually. For Quarter Coupon Rates from* 1% *to* 6%. Second ed., Boston, Financial Publ. Co.; London, George Routledge & Sons, Ltd., 1941, xx, 1156 p. 25 × 36.7 cm. $150.00.

It is fifteen years since this great work was originally published. To the first edition a supplement was added in 1937. The decline in levels of coupon rates and yield rates made necessary the appreciably enlarged second edition.

The table now occupying pages 1–925 is one from which we can at once read off the monthly values of bonds paying interest semiannually, with varying Coupon Rates, Yields, and Periods of Maturity. The Coupon Rates = 1(1/4)6%. The Yields for coupons (*a*) $1 - 2\frac{3}{4}\% = .00(.05)5.70\%$; (*b*) $3 - 4\frac{1}{4}\% = 1.00(.05)6\%$, also 8ths $2.50 - 6\%$; (*c*) $4\frac{1}{2} - 6\% = 1.50(.05)6\frac{1}{2}\%$, also 8ths $3 - 6\frac{1}{2}\%$. The Maturities vary from 1 month to 40 years and 3 months by months, from $40\frac{1}{2}$ years to 50 years semiannually, from 50 years to 65 years annually, and from 70 years to 110 years by 5 years. As it is unusual to find many bonds of more than 40 year maturity this covers most practical needs. Ample explanation is given in the introductory pages to cover any maturities beyond 40 years up to the 110 year limit. The table is to six places of decimals throughout. Convenient thumb indexes enable one to turn immediately to the pages with the desired coupon rate.

On every page there are 12 columns (one for each month) and 115 lines making 1380 entries. But these are so spaced in groups of fives that the page is singularly easy to use in practice. The columns of entries are facsimilies of type-script, to which many checks had been applied, and errors in type-setting being avoided in this way, the number of errors already

reported is exceedingly small; see MTE **18**. In black-face type, on each side of every page and every line, are the 115 yields considered. At the bottom of each page are given under appropriate columns the obviously desirable amounts of accrued interest on $100 for 1, 2, $\cdots$, 5 months.

Pages 926–1108 are really part of the preceding table, filling in the gaps in the bond values in the lower yields [= .00(.05)1.00 or 1.50] for coupon rates from 3% to 6%, at the same intervals of time.

Following this (p. 1109–1153) is a set of tables for non-interest bearing securities (0%) to show their values at yields = 0(.05)5.70%, intervals for the same time period as in the preceding tables. And finally there is a "Table of factors" (p. 1154–1156), to 8D, to be used in computing values for one to thirty days between the even months for yields from 0% to 5.7% at .05% intervals.

The manner of preparing the tables was almost wholly dictated by trade practice and by what would be most intelligible to bond dealers. The method of computation involved the addition of the constant difference between values at successive coupon rates. This constant difference was computed to a minimum of two decimal places beyond the six decimal places contained in the table, and in most cases to more than two extra decimals. The mathematician notes with interest that trade practice calls for the addition to every invoice, of simple interest at the coupon rate. Hence it became necessary to compute the theoretically correct "flat" value and from this to subtract simple accrued interest. Thus when an invoice is made for bonds sold at a yield basis the sum which changes hands will be the theoretically correct amount, but it will be itemized in two distinct elements neither of which may be considered mathematically exact. This feature is most readily observed in the case of any bond selling to yield its coupon rate. The value is shown as 100 on coupon dates only, and for intermediate dates at something less than 100; but if the accrued interest at the bottom of the column be added a correct result will be obtained.

Another curious feature is that in spite of solicitude for correct semiannual compound interest in the body of the book, values for less than six months' maturity are shown at true discount, in accord with trade custom of dealers and banks. Again, the editors believed that labelling a present-worth table as the value of a 0% bond evokes a clearer concept to the average bond dealer, than a proper labelling would.

Without a volume of this character, dealing in municipal securities would be a somewhat difficult process. For example, a dealer offers a 3½% coupon municipal having 20 years and 4 months to run on a 1.95% yield basis. What dollar price must the purchaser pay? Turning to the 3½% coupon table, find the page "20 years, column 4 months and on the margin the yield 1.95%" and the price is given as 125.913848 or $1259.14 per $1000 bond. Also at the bottom of the column is given the accrued interest in monthly intervals, $.583333 per $100 or $5.83 per $1000 bond. The methods of adjusting for fractional parts of a month, and for many other special cases of practice, are explained clearly in the introductory pages. These pages contain also quite explicit instructions involving the use of adding or accounting machines where these are available.

This volume of tables, which is the most extensive of the kind, is of value particularly to dealers and those handling state and municipal securities where price is usually quoted on a yield basis, also for insurance companies where exact computations of yield are essential. The tables are obviously of importance also to the actuary or accountant in making up amortization schedules. The volume is strongly bound in full pig skin, and lies open easily to the table desired. It is too bulky for use outside of an office or a library. For the convenience of salesmen and others there is *Pocket Edition of Monthly Bond Values* with tables of coupon values from ½% to 5% at ¼% intervals. These tables are in condensed form and show values to two decimal places only and cover values at monthly intervals up to 12 years for coupon rates of ½% and ¾% and for the remaining coupon values at monthly intervals up to 28 years, quarterly 28–30 years, and semiannually 30–50 years.

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118[L].—ZAKI MURSI, *Tables of Legendre Associated Functions*, (*Fouad I University, Faculty of Science*, no. 4). [Cairo], E. & R. Schindler, 1941, xii, 286 p. 22.8 × 29.5 cm. Price unknown. The Egyptian University at the Zaafaren Palace in Cairo, was founded as a free university in 1908 and adopted as a state university in 1925.

Out of Egypt in the midst of war and issued at a time when the enemy was not far from the gates of Cairo, there comes a set of unusual tables, values of the Legendrian associated functions, $P_n^m(x)$, n, $m = 1, 2, 3, \cdots, 10$, over the range $x = 0.000(0.001)1.000$. Before describing in detail the contents of this volume it might be well to recall some of the historical facts relating to these functions.

The polynomials, $P_n(x)$, associated with the name of A. M. LEGENDRE (1752–1833), were first employed in 1784 by LEGENDRE, who applied them to the determination of the attraction of solids of revolution, in "Recherches sur l'attraction des sphéroides homogènes," *Mémoires des Savants Étrangers*, v. 10, 1785. Inspired by this memoir LAPLACE (1749–1827) published in this same year one of the most remarkable memoirs ever written. It is here that Legendre's spherical harmonics enter into his discussion and Laplace's equation first appears, though only in polar coordinates. Legendre's name is thus properly ascribed to the new harmonic functions; nevertheless some writers still refer to the polynomials as Laplace's coefficients.

The functions, $P_n(x)$, are the polynomial solutions of Legendre's differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0,$$

which arises in connection with the solution of Laplace's equation in spherical coordinates. For this reason the solutions of the equation are frequently referred to as surface zonal harmonics, or spherical functions. The second solution of Legendre's equation is usually designated by the symbol $Q_n(x)$, and is called a zonal harmonic of second kind.

The Legendrian associated functions of mth order and nth degree are the functions $P_n^m(x)$, which are derived from the Legendrian polynomials by the relationship:

$$P_n^m(x) = (1 - x^2)^{\frac{1}{2}m}\frac{d^m}{dx^m}P_n(x).$$

Associated functions of second kind are designated by the symbol $Q_n^m(x)$, and are defined by the formula just written down where $P_n(x)$ is replaced by $Q_n(x)$.

The associated functions of both kinds are sometimes designated by the symbols $P_{n,m}(x)$ and $Q_{n,m}(x)$, although the symbols used above are the more common.

In some applications it is more convenient to replace x by $\cos\theta$, and extensive tables have been computed for the Legendrian functions in their trigonometric form. When θ is the variable, the functions are customarily represented by the symbols $P_n(\cos\theta)$, $Q_n(\cos\theta)$, $P_n^m(\cos\theta)$, and $Q_n^m(\cos\theta)$.

In addition to their use in problems of attraction, the Legendrian polynomials have had an extensive application in numerical quadrature, a use first indicated by K. F. GAUSS. In this application the zeros of $P_n(x)$ become of great significance and considerable attention has been devoted to their computation. Compare RMT **92**.

The volume under review provides tables of the associated functions, $P_n^m(x)$, for values of m and $n \leqq 10$ over the range $x = 0.000(0.001)1.000$ to an accuracy which varies from 8 decimal places in the beginning of the table to 3 decimal places at the end of the table. For those who may contemplate using the tabulated values, the following schedule shows the number of decimal places, D, corresponding to values of m and n:

$m = 1, n = 1(1)10, D = 8$. $m = 2, n = 2(1)9, D = 8; n = 10, D = 7$. $m = 3, n = 3(1)6, D = 8$; $n = 7(1)9, D = 7$; $n = 10, D = 6$. $m = 4, n = 4(1)6, D = 8$; $n = 7, 8, D = 7$; $n = 9, D = 6$; $n = 10, D = 5$. $m = 5, n = 5, D = 8$; $n = 6, 7, D = 7$; $n = 8, D = 6$; $n = 9, 10, D = 4$. $m = 6, n = 6, D = 8$; $n = 7, D = 7$; $n = 8, D = 6$; $n = 9, 10, D = 4$.

$m = 7$, $n = 7$, $D = 7$; $n = 8$, $D = 6$; $n = 9$, 10, $D = 4$. $m = 8$, $n = 8$, $D = 6$; $n = 9$, $D = 4$; $n = 10$, $D = 3$. $m = 9$, $n = 9$, $D = 4$. $m = 10$, $n = 10$, $D = 3$.

The usefulness of the table is considerably enhanced by the inclusion of values of the modified second difference

$$M'' = \Delta'' - 0.184\Delta^{(4)},$$

where Δ'' and $\Delta^{(4)}$ are the second and fourth central differences. The theory of this useful difference has been set forth by L. J. COMRIE in "Interpolation and allied tables" (reprinted from *Nautical Almanac* for 1937), London, H. M. Stationery Office, 1936, p. 928.

The modified differences are used in Everett's interpolation formula as follows:

$$f(x_0 + \xi/1000) = (1 - \xi)f(x_0) + \xi f(x_1) - \mu(\xi)M_1'' - \mu(1 - \xi)M_0'',$$

where x_0 and x_1 are consecutive values of x, and where $\mu(\xi) = \frac{1}{6}\xi(1 - \xi^2)$.

The tables contain (pp. 282–283) values of $\mu(\xi)$ to 7 decimal places over the range $\xi = 0.000(0.001)1.000$. There is no indication as to the source of these values, although $\mu(\xi)$, and other functions used in Everett's formula when higher differences are employed, have been tabulated by A. J. THOMPSON in *Tracts for Computers*, no. V, 1921; second ed., 1943.

In the construction of the tables, the values corresponding to $n = m = 1$ and $n = m = 2$ were first computed from the formula

$$P_m^m(x) = \frac{(2m)!}{2^m m!}(1 - x^2)^{\frac{1}{2}m}.$$

Then, for other equal values, there was used

$$P_{m+2}^{m+2}(x) = (2m + 1)(2m + 3)(1 - x^2)P_m^m(x).$$

In the computation of the functions for the other cases where $n > m$, there was employed the following iteration formula:

$$P_n^m(x) = \frac{2n - 1}{n - m} xP_{n-1}^m(x) - \frac{n + m - 1}{n - m} P_{n-2}^m(x),$$

which, when $n = m + 1$, reduces to

$$P_{m+1}^m(x) = (2m + 1)xP_m^m(x).$$

With respect to the computations the author makes the following comments:

"In view of the fact that the calculation depends on the repeated use of the recurrence formulae involving the accumulation of small errors, it has been found sufficient to calculate $P_1^1(x) = (1 - x^2)^{\frac{1}{2}}$ to 16 decimals and $P_2^2(x)$ exactly in order to obtain the other functions accurate to 8 decimals except those for which $n = 9$ or 10, $m \geqq 5$. The latter are accurate to 4 decimals only, the values of the entries being large."

The author believes that the accuracy of his tables is high. He states that the original calculations were submitted to L. J. COMRIE, who differenced 6,000 entries. Of the ten errors thus revealed, nine were due to transcription. The values were also compared with those computed by H. TALLQVIST over the range 0.00(0.01)1.00. This comparison "shows agreement as far as his figures go," but see MTE **17**. All of the calculations were carried through twice either by the author, or by others.

In view of recent interest in Legendrian functions it may not be out of place to append a short account of the origin and present status of tables relating to them. See also UMT **14**.

The first extensive set of values of $P_n(x)$ was computed by J. W. L. GLAISHER and published in the *Report* of the British Association for the Advancement of Science in 1879. This table gives the complete values for $P_n(x)$, $n = 1, 2, \cdots, 7$, over the range $x = 0.00(0.01)1.00$. Four-decimal abbreviations are to be found in JAHNKE and EMDE, *Funktionentafeln*, first ed., 1909, p. 83–84; second ed., 1933, p. 185–186; third to fifth ed., p. 119–120, as well as in W. E. BYERLY, *An Elementary Treatise on Fourier's Series* . . ., 1895, p. 280–281. Five-decimal abbreviations are given by J. B. DALE in *Five-Figure Tables of Mathematical Functions* . . ., London, 1903, p. 78–79.

The most extensive calculations of the Legendrian functions are due to H. TALLQVIST, who, in addition to a treatise: *Grunderna af Teorin för Sferiska Funktioner, jämte Användningar inom Fysiken*, Helsingfors, 1905, xv, 436 p., which is rich in special tables, has produced the following monographs:

(1) "Tafeln der Kugelfunctionen $P_n(x)$ und ihrer abgeleiteten Functionen," *Acta Societatis Scientiarum Fennicae*, v. 32, no. 6, 1906, 27 p.

(2) "Tafeln der Kugelfunktionen $P_n(\cos\theta)$," *ibid.*, v. 33, no. 4, 1908.

(3) "Tafeln der abgeleiteten und zugeordneten Kugelfunktionen erster Art," *ibid.*, v. 33, no. 9, 1908.

(4) "Tafel der 24 ersten Kugelfunktionen $P_n(\cos\theta)$," *Societas Scientiarum Fennica, Commentationes Physico-Mathematicae*, v. 6, no. 3, 1932, 11 p.

(5) "Tafel der Kugelfunktionen $P_{25}(\cos\theta)$ bis $P_{32}(\cos\theta)$," *ibid.*, v. 6, no. 10, 1932, 5 p.

(6) "Sechsstellige Tafeln der 16 ersten Kugelfunktionen $P_n(x)$," *Acta Societatis Scientiarum Fennicae*, n.s. A, v. 2, no. 4, 1937, 43 p.

(7) "Sechsstellige Tafeln der 32 ersten Kugelfunktionen $P_n(\cos\theta)$," *Acta Societatis Scientiarum Fennicae*, n.s. A, v. 2, no. 11, 1938, 43 p.

In monograph (1) TALLQVIST added to the computations of GLAISHER the exact values of $P_8(x)$ over the range $x = 0.00(0.01)1.00$. He also tabulated the derivatives of $P_n(x)$, namely $P_n^{(m)}(x)$, for m and n from 1 to 8 exactly over the same range. In monograph (6) he gave six-place values of $P_n(x)$ for $n = 1(1)16$ over the range $x = 0.000(0.001)1.000$.

The values of $P_n(\cos\theta)$ were computed by A. LODGE in "Values of Legendre's functions from P_0 to P_{20} at intervals of 5 degrees," R. So. London, *Phil. Trans.*, v. 203A, 1904, p. 100–101, to 7 decimal places. These were later extended to 10 decimal places, $n = 1(1)8$, over the range $\theta = 0°(1°)90°$ by TALLQVIST (2). To these were added the computations to $n = 32$, items (4) and (5), and in 1937 there appeared the six-place tables (7) computed at intervals of 10′. We also note a table for $n = 1(1)7$ computed to 4 decimal places at intervals of 1° by C. E. HOLLAND, P. R. JONES, and C. G. LAMB, *Phil. Mag.*, (5), v. 32, 1891, p. 512–523; also, Phys. So., London, *Proc.*, v. 11, 1892, p. 221–233. This table, according to TALLQVIST, contains "many and not wholly unimportant errors." It was reprinted by BYERLY, p. 278–279, and was the first table of this type.

To TALLQVIST we are also indebted for the derivatives $P_n^{(m)}(\cos\theta)$ for m and n from 1 to 8 to 10 and 11 significant figures over the range $\theta = 0°(1°)90°$. The computation of the first derivative was first given by C. C. FARR, R. So. London, *Proc.*, v. 64, 1898–99, p. 192–202, to 4 decimal places for $n = 1$ to 7 and to 3 decimal places for $\theta = 1°$ to 90°. This table is reprinted by JAHNKE and EMDE, first ed., p. 88–89; second ed., p. 190–191. Since we have $dP_n(\cos\theta)/d\theta = -P_n^1(\cos\theta)$, this table is also included in tables of the associated functions.

The values of the associated functions, $P_n^m(x)$, were first given by TALLQVIST in his treatise mentioned above, and later in monograph (3) over the range $x = 0.00(0.01)1.00$. For even orders (m) these values are either exact or computed to 10 or 11 significant figures; for odd orders they are given to 10 or 11 significant figures. A companion table for $P_n^m(\cos\theta)$ gives values to 10 or 11 significant figures at intervals of 1° for values of m and n from 1 to 8.

It is unfortunate that Tallqvist's treatise mentioned above is not more readily available in American libraries, since it contains 50 pages of tables, some of which are not found elsewhere. In addition to abbreviated tables of the polynomials and the associated functions in both forms to 4 decimal places through $n = 8$, and the derivatives of $P_n(x)$ through $n = 8$, there appear tables of the roots of the derivatives and of $P_n(x)$ to 7 decimal places through $n = 8$. Tables are also provided to 4 decimal places over the range $x = 0.00(0.01)1.00$ of the conjugate zonal harmonic of second kind, $\bar{Q}_n(x)$, and its companion function $Z_n(x)$, $n = 1(1)8$, defined by the equations

$$\bar{Q}_n(x) = P_n(x)\log\frac{1+x}{1-x} - Z_n(x),$$

$$Z_n(x) = 2\left\{\frac{2n-1}{1\cdot n}P_{n-1}(x) + \frac{2n-5}{3(n-1)}P_{n-3}(x) + \frac{2n-9}{5(n-2)}P_{n-5}(x) + \cdots\right\}.$$

In a work entitled *Tables de Fonctions Sphériques et de leurs Intégrales*, Paris, 1933, xxx, 95, 160* p., G. PRÉVOST provided tables of integrals of the Legendrian polynomials and the associated functions, as well as values of the functions themselves, over the range $x = 0.00(0.01)1.00$, to from 5 to 6 significant figures, for n and $m = 1(1)8$.

Very extensive calculations of the zeros of $P_n(x)$ have been made, a partial bibliography of these being given in this *MTAC*, p. 51–52. The most conspicuous omission is the important computations of E. R. SMITH, "Zeros of the Legendre polynomials," Iowa State College, *Jn. Sci.*, v. 12, 1938, p. 263–274, where 6-place values are given through $n = 40$. Interesting new asymptotic approximations were used in the determination of these values.

The most extensive collection of values pertaining to the Legendrian functions is contained in an unpublished manuscript in excess of 400 pages in the possession of the writer of this review, which, in addition to the tables mentioned above, contains the following new computations:

Values of $P_n(x)$ and $P_n(\cos\theta)$, $n = 1(1)8$, to 10D over the range $x = 0.000(0.001)1.000$, together with first and second differences. Values of the integrals $I_n(x) = \int_0^x P_n(t)dt$, and $I_n^m(x) = \int_0^x P_n^m(t)dt$, m, $n = 1(1)8$, mainly to 10D, but in every case in excess of 10 significant figures, over the range $x = 0.00(0.01)1.00$, the derivatives $P_n^{(m)}(\cos\theta)$, to 6 or more significant figures, over the range $\theta = 0°0'(10')90°$, and $P_n^m(\cos\theta)$, to 6 or more significant figures over the same range, n, $m = 1(1)6$. The last two tables are from an unpublished manuscript of TALLQVIST kindly furnished the writer of this review.

Since the heroic computations of TALLQVIST have secured for him a permanent place in the history of zonal harmonics, it might not be out of place to include a short notice about his work. AXEL HENRIK HJALMAR TALLQVIST was born February 21, 1870. He entered the University of Helsingfors in 1885, and became Magister and Doctor Phil. in 1890. He was appointed Professor of Mechanics at the Polytechnicum in 1891, Professor of Physics at the University of Helsingfors, 1907, and Promotor, in 1935. His first publication was a two-volume text on technical mechanics, published in 1895 and 1896. His bibliography contains more than 200 items covering many branches of mathematics and mathematical physics. His treatise on spherical harmonics, referred to above, is an outstanding work on the subject with many original features. Since the appearance of this book in 1905, TALLQVIST has labored long, more than a third of a century, to supply the tables needed in the application of zonal harmonics to mathematical physics. His contribution is certainly one of the most extensive calculations ever undertaken by a single individual.

H. T. D.

119[E, L, S].—J. SHERMAN and R. B. EWELL, "A six-place table of the Einstein functions," *Jn. Phys. Chemistry*, v. 46, 1942, p. 641–662.

It is stated that the calculation of thermodynamic quantities from spectroscopic data has become a recognized method for determining the energy, heat capacity, entropy, and similar properties of simple molecules. In this connection Einstein functions are extensively used in calculating vibrational contributions. These functions are $E = x/(e^x - 1)$, e^xE^2, and $-\ln(1 - e^{-x})$ and they are tabulated for $x(= h\nu_0/kT) = [0(0.005)3(0.01)8(0.05)15;$ 6D]. Brief tables were available earlier in Landolt-Börnstein, *Physikalisch-Chemische Tabellen*, 5th ed., 1st Suppl., Berlin, 1927, p. 702–707; and W. Nernst, *The New Heat Theorem: its Foundations in Theory and Experiment*, transl. G. Barr, New York, 1926, p. 246–251. In the construction of the table use was made of F. W. Newman's tables of the exponential functions (1883, 1889) and of three different expansions for the logarithm. A summary of the application of the table to the calculation of thermodynamic quantities is given by L. S. Kassel, in his "The calculation of thermodynamic functions from spectroscopic data," *Chemical Reviews*, v. 18, 1936, p. 277–313.

H. B.

120[L, S].—M. KOTANI, A. AMEMIYA and T. SIMOSE, I. "Tables of integrals useful for the calculations of molecular energies," Phys. Math. So., Japan, *Proc.*, s. 3, v. 20, 1938, extra no. 1, 70 p. II. *idem.*, v. 22, 1940, extra no. 1, 28 p. 18.8 × 25.8 cm.

To some extent the Tables of I form a continuation of the work of S. Ikehara who compiled the tables of the functions of Zener and Guillemin [1] used in the work of Rosen.[2] The tables were used by T. Invi in his work [3] on the hydrogen molecule and he cooperated with the authors. He says that the integrals occurring in his work can be reduced to some basic integrals tabulated by Bartlett, Furry and Rosen [4] but their tables are not sufficiently extensive. Values for the functions $A_n(x)$, $B_n(x)$ in which $\exp(-xt)t^n$ is integrated over the ranges, 1, ∞ and -1, 1 respectively, are given for $x = [0.25(0.25)8; 10D]$ this range being extended in the first case by 8.0(0.5)13(1)16(2)20(1)22, 24. These tables go far beyond the tables with 6S used by Rosen though these latter go up to $n = 20$ instead of $n = 15$ but for $n > 10$ are for $x = 1(1)10, 12, 14$ only. In the 4-place tables of A. Unsöld [5] n has only the values 1, 2, 3 and $x = 0(.1).2, .5(.5)2, 3$. In the work of R. D. Misra [6] $2n = -9(1)6$ and x ranges from π to 4π, 7D being given while in a second table $2n = -1(1)6$ and $4x$ runs from 3π to 11π. Values of $A_n(x)$ for $n = -3, -2, -1$ were given by E. Gold [7] in 1909.

The paper also contains a table of the exponential integral $-Ei(-x)$, for $x = [0.5(0.5)18(1)23, 25; 18D]$, and other values of x, 24–48, p. 15 of II. The integrals involving Legendre functions, Table 4, are given, for $x = [0.5(0.5)1(0.25)7; 8D]$. In the tables of double or triple entry, 5–6, the order of the Legendre functions ranges from 0 to an integral value not exceeding 8 while the other integers have no more than 4 values. In the Tables of II 13 integrals for hydrides are tabulated. P. 16–28 are occupied with lists of errata in I.

H. B.

[1] C. ZENER and V. GUILLEMIN, "The B-state of the hydrogen molecule," *Phys. Rev.*, s. 2, v. 34, 1929, p. 999–1009.

[2] N. ROSEN, "Normal state of the hydrogen molecule," *Phys. Rev.*, s. 2, v. 38, 1931, p. 2099–2114 (see especially p. 2109–2110).

[3] T. INVI, "A contribution to the theory of the hydrogen molecule," Phys. Math. So. Japan, *Proc.*, s. 3, v. 20, 1938, p. 770–779.

[4] J. H. BARTLETT and W. H. FURRY, "Valence forces in lithium and beryllium," *Phys. Rev.*, s. 2, v. 38, 1931, p. 1615–1622.

[5] A. UNSÖLD, "Zur Deutung der Intensitätsverteilung in den Fraunhoferschen Linien, II. Teil: Die Intensität der Linienflügel-approximative Lösung der Schwarzschildschen Integralgleichung für eine beliebig geschichtete Atmosphäre," *Zeit. f. Astrophysik*, v. 4, 1932, p. 339–357.

[6] R. D. MISRA, "On the stability of crystal lattices," Cambridge Phil. So., *Proc.*, v. 36, 1940, p. 173–182.

[7] E. GOLD, "The isothermal layer of the atmosphere and atmospheric radiation," R. So. London, *Proc.*, 82A, 1909, p. 62.

MATHEMATICAL TABLES—ERRATA

On the pages indicated Errata are listed for tables by the following: J. R. AIREY (p. 72), A. GRAY and G. B. MATHEWS (p. 74), D. F. E. MEISSEL (p. 74), A. G. WEBSTER (p. 70), and R. W. WILLSON and B. O. PEIRCE (p. 74). In this issue we have referred to Errata in RMT **113** (JAHNKE and EMDE).

13. L. M. MILNE-THOMSON, *Standard Table of Square Roots. The square roots to eight significant figures of all four-figure numbers, with printed differences*, London, Bell, 1929.

The following error was discovered by Mr. T. WHITWELL:

Difference following 239.7, *for* 3329, *read* 3229.

This error was found because Mr. Whitwell uses a precaution that I have always advocated when doing linear interpolation with a calculating machine. The preceding tabular value is