Note on the Computation of the Bessel Function $I_n(x)$.

The Bessel function of purely imaginary argument

$$(1) \qquad I_n(x) = i^{-n}J_n(ix) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{\Gamma(m+1)\Gamma(m+n+1)}$$

is of frequent occurrence in applied mathematics, statistics, and asymptotic number theory. The calculation of $I_n(x)$ for n and x both large presents an interesting problem to the computer.[1] It is clear that the series for $I_n(x)$, (though everywhere convergent) is of little practical use for large x. Even the well-known asymptotic formula, designed for use with large x

$$(2) \quad I_n(x) = \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left(1 - \frac{4n^2 - 1^2}{1!\,8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!\,(8x)^2} - \cdots\right) + O(e^{-x}x^{-\frac{1}{2}})$$

fails to be of practical use when n is of order $\sqrt{x}$.

Bessel functions of large integral order can, theoretically, be built up from the values of $I_0(x)$ and $I_1(x)$ (given in tables,[2] or found by (2)) by use of the recurrence formula

$$(3) \qquad I_{k+1}(x) = -\,2kI_k(x)/x + I_{k-1}(x).$$

This procedure, however, affords another example in which great accuracy is needed to start with (as has been pointed out in case of $J_n(x)$ by COSENS [*MTAC* p. 99]), since more and more accuracy is lost at each step. The value of $I_{90}(60) = 0.0725775\cdots$ would, if computed this way, require about 40 decimal places in $I_0(60)$ and $I_1(60)$, and 89 applications of (3).

NICHOLSON[3] has given a formula for $I_n(x)$ purporting to be "suitable for rapid tabulation," when either n or x is large. The formula is not given explicitly (except its first term) but in terms of an infinite differential operator with undetermined coefficients operating on a cumbersome function of n and x.

The following formula, which I have not found in the literature, is quite effective in case n is large and x has any real value.

$$(4) \qquad I_n(nx) = \frac{1}{(2\pi nz)^{\frac{1}{2}}}\,\{(z-1)e^z/x\}^n \exp \sum_{m=1} \psi_m(x^2)/(nz^3)^m$$

where $z = (1 + x^2)^{\frac{1}{2}}$, and where the ψ's are the following polynomials:

$$\begin{aligned}
24\,\psi_1(x^2) &= 3x^2 - 2,\\
16\,\psi_2(x^2) &= x^4 - 4x^2,\\
5760\,\psi_3(x^2) &= 375x^6 - 3654x^4 + 1512x^2 - 16,\\
128\,\psi_4(x^2) &= 13x^8 - 232x^6 + 288x^4 - 32x^2,\\
322560\,\psi_5(x^2) &= 67599x^{10} - 1914210x^8 + 4744640x^6 - 1891200x^4\\
&\qquad + 78720x^2 - 256,\\
192\,\psi_6(x^2) &= 103x^{12} - 4242x^{10} + 17493x^8 - 14884x^6 + 2580x^4 - 48x^2,\\
3440640\,\psi_7(x^2) &= 5635995x^{14} - 318291750x^{12} + 1965889800x^{10}\\
&\qquad - 2884531440x^8 + 1135145088x^6 - 99783936x^4\\
&\qquad + 881664x^2 - 2048,\\
4096\,\psi_8(x^2) &= 23797x^{16} - 1765936x^{14} + 15252048x^{12} - 34280896x^{10}\\
&\qquad + 24059968x^8 - 5095936x^6 + 248320x^4 - 1024x^2.
\end{aligned}$$

This formula may be derived from an asymptotic expansion of the solution

$$y = d(\ln I_n(nx))/dx$$

of the differential equation

$$y' + y/x + y^2 = n^2(1 + x^2)/x^2 = (nz/x)^2.$$

Further polynomials $\psi_m(t)$, if needed, may be found expeditiously as follows. Define the polynomial $Q_m(t)$ by

$$Q_0(t) = 1, \qquad Q_1(t) = 5 - t, \qquad Q_2(t) = 60 - 48t + 4t^2,$$

and in general by the recurrence formula [4]

$$Q_m = (6m - (6m - 4)t)Q_{m-1} - 4t(1 - t)Q'_{m-1} + (1 - t)\sum_{\lambda=2}^{m-1} Q_{\lambda-1}Q_{m-\lambda}.$$

Then the mth term T_m of the asymptotic series occurring in (4) may be written

$$T_m = \tfrac{1}{2}(-4n)^{-m}\int_z^\infty Q_m(t^2)t^{-3m-1}dt.$$

In other words, if we denote the coefficient of t^k in $Q_m(t)$ by $A_k^{(m)}$ and define the polynomial $\phi_m(t)$ by

$$\phi_m(t) = \sum_{k=0}^{m} A_k^{(m)}t^k/(3m - 2k),$$

then

$$T_m = (-1)^m 2^{-2m-1}n^{-m}z^{-3m}\phi_m(z^2).$$

As a matter of fact, the polynomial

$$\psi_m(t) = \phi_m(1 + t)$$

has simpler coefficients than $\phi_m(t)$ and since $\phi_m(z^2) = \psi_m(x^2)$ we have given T_m partly in terms of x. In doing this we are following MEISSEL[5] whose formulas for $J_n(nx)$ also involve $\psi_m(t)$ for $m \leqq 6$. A comparison of his extended results reveals four errors in his $\psi_6(\sec^2 \alpha)$. *For* denominators

1, 4, 192, 256, 128, 768, *read* 1, 4, 48, 16, 8, 48.

This incorrect polynomial has produced an error in the corresponding term in the formula as quoted by WATSON[6]

for 3072, 768, 41280, *read* 192, 48, 2580.

WATSON has another error (not due to MEISSEL) in the polynomial where 98720 is a misprint for 78720; noted by W. G. BICKLEY, *Phil. Mag.*, s. 7, v. 34, 1943, p. 45.

To illustrate the behavior of (4) for a moderate value of x we take $x = \frac{3}{4}$ so that $z = \frac{5}{4}$. In this case (4) becomes:

$$\begin{aligned}\ln I_n(3n/4) = {} & (\tfrac{5}{4} - \ln 3)n - \tfrac{1}{2}\ln n - \tfrac{1}{2}\ln(5\pi/2) - 1/150n - 99/3125n^2 \\ & - 1044097/175781250n^3 + 868626/48828125n^4 \\ & + 857735827/76904296875n^5 \\ & - 4925585088/152587890625n^6 \\ & + 4590654771247901/1001358032226562 50n^7 \\ & \qquad + 278080398059472/2384185791015625n^8 + \cdots.\end{aligned}$$

For $n = 100$ these various terms are, to 20 decimals,

$$\begin{array}{rl}
 & 15.13877\ 11331\ 89030\ 86048 \\
- & 2.30258\ 50929\ 94045\ 68402 \\
- & 1.03051\ 03088\ 84078\ 40331 \\
- & .00006\ 66666\ 66666\ 66667 \\
- & .00000\ 31680\ 00000\ 00000 \\
- & .00000\ 00059\ 39751\ 82222 \\
+ & .00000\ 00001\ 77894\ 60480 \\
+ & .00000\ 00000\ 01115\ 32887 \\
- & .00000\ 00000\ 00032\ 28031 \\
- & .00000\ 00000\ 00000\ 45844 \\
+ & .00000\ 00000\ 00000\ 01166,
\end{array}$$

and their sum is

$$11.80560\ 58908\ 83465\ 49\cdots.$$

Thus

$$I_{100}(75) = 134001.44880\ 18810.$$

D. H. L.

[1] Such Bessel functions have been used recently at the statistical laboratory of the University of California in preparation of certain statistical tables to appear in *Annals of Math. Statistics*. See also: J. WISHART, "A note on the distribution of the correlation ratio," *Biometrika*, v. 24, 1932, p. 454, formula (27).

[2] The most extensive tables of $I_n(x)$ are in B.A.A.S., *Math. Tables*, v. 6, *Bessel Functions*, part I, Cambridge, 1937, Tables VI and VIII.

[3] J. W. NICHOLSON, *Phil. Mag.* s. 6, v. 20, 1910, p. 938–943.

[4] The Q's may be checked by the relation $Q_m(1) = 4^m$; also $m(m+1) \int_0^1 t^{2m-1} Q_m(t^{-2})dt = 2^{2m+1}|B_{m+1}|$, where B_k is the kth Bernoulli number, in the notation of Lucas.

[5] D. F. E. MEISSEL, *Astr. Nach.* v. 130, 1892, cols. 363–4.

[6] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge 1922, p. 228.

Mathematical Tables in Phil. Mag.

The important scientific periodical *London, Edinburgh & Dublin Philosophical Magazine and Journal of Science*, has been published under different titles since 1798, in 7 series of volumes (varying in number from 11 to 68). In recent years the usual print abbreviation of the title of this periodical has been *Phil. Mag.* In what follows it is proposed to list, with a few added notes, mathematical tables in *Phil. Mag.*, s. 3, v. 18 to s. 7, v. 34, no. 235, inclusive, 1841–1943. It will be observed that more than half of the Tables listed were published after the last world war started in 1914, a period of unparalleled scientific development. Many mathematical tables have been already published as the result of problems arising in the prosecution of the present war; the number is likely to be greatly increased in the next few years.

S. 3

1. G. B. AIRY "On diffraction of an annular aperture," v. 18, 1841, p. 7. Table of $E[= J_0(e)] = (1/2\pi) \int_0^{2\pi} \cos(e \cos \theta)d\theta$, and of E^2, for $e = [0.0(0.2)10.0; 4D]$. Airy had an earlier table of $2J_1(e)/e$ of the same scope in Cambridge Phil. So., *Trans.*, v. 5, 1835, p. 291. In B.A.A.S., *Math. Tables*, v. 6, *Bessel Functions*, Cambridge, 1937, J_0 is given, among other values, for $e = [0.0(0.2)25.0; 8D]$. The value of $J_0(40)$, to $7D$, was computed by W. R. Hamilton, *Phil. Mag.*, s. 4, v. 14, 1857, p. 381.