Complex Zeros of $Y_0(z)$, $Y_1(z)$, and $Y'_1(z)$

One of the authors has obtained conditions for the existence of real and of complex zeros on the various branches of Bessel functions of real order. For the integral order Bessel functions of the second kind, $Y_n(z)$, there are real zeros only on the branch which is real along the positive real axis. (These zeros are positive.) Every other branch has non-real zeros in the right half-plane and all branches have non-real zeros in the left half-plane.

Below are listed the first fifteen (non-real) zeros of $Y_0(z)$, $Y_1(z) = -Y'_0(z)$, and $Y'_1(z)$ in the left half-plane on the branch which has positive real zeros. Approximations to the first few zeros tabulated were obtained from "contour lines" of $Y_0(z)$ and $Y_1(z)$ which are soon to be published. These approximations were refined by Newton’s iteration method. The process was not at all simple since it involved the calculation to maximum accuracy of the $Y$'s and their derivatives at points off the rays for which the functions have been tabulated.

Zeros with absolute values greater than 10 (i.e. those outside the range of the above mentioned volume) were computed from the following asymptotic expansions:

\[
-z_j \sim \beta - \frac{4n^2 - 1}{8\beta} - \frac{112n^4 - 152n^2 + 31}{384\beta^3} - \cdots,
\]

\[
-z'_j \sim \beta_1 - \frac{4n^2 + 3}{8\beta_1} - \frac{112n^4 + 328n^2 - 9}{384\beta_1^3} - \cdots
\]

where the $z_j$ are zeros of $Y_n(z)$, the $z'_j$ are zeros of $Y'_n(z)$, $\beta = (j + \frac{1}{2}n - \frac{1}{4})\pi - i\arctanh \frac{1}{z}$, and $\beta_1 = (j + \frac{1}{2}n + \frac{1}{4})\pi - i\arctanh \frac{1}{z}$.

Table A below lists to 9D the zeros with absolute values less than 10. With the zeros of $Y_0$ are the values of $Y_1$ and $Y'_1$ at the zeros. Similarly for the zeros of $Y_1$ and $Y'_1$. Table B has fifteen zeros to 5D for each of $Y_0$, $Y_1$, and $Y'_1$.

### Table A

Zeros $z_{0,s}$ of $Y_0(z)$ and Values of $Y_1$ and $Y'_1$ at the Zero

<table>
<thead>
<tr>
<th>$z_{0,s}$</th>
<th>$Y_1$</th>
<th>$Y'_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_1$</td>
<td>$Y'_1$</td>
<td></td>
</tr>
<tr>
<td>$Y_0$</td>
<td>$Y_1$</td>
<td>$Y'_1$</td>
</tr>
</tbody>
</table>

Zeros $z_{1,s}$ of $Y_1(z)$ and Values of $Y_0$ and $Y_1'$ at the Zero

<table>
<thead>
<tr>
<th>$z_{1,s}$</th>
<th>$Y_0$</th>
<th>$Y_1'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_0$</td>
<td>$Y_1$</td>
<td>$Y'_1$</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>$Y'_1$</td>
<td></td>
</tr>
<tr>
<td>$Y'_1$</td>
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<td></td>
</tr>
</tbody>
</table>

Zeros $z_{1,s}$ of $Y'_1(z)$ and Values of $Y_0$ and $Y_1$ at the Zero

<table>
<thead>
<tr>
<th>$z_{1,s}$</th>
<th>$Y_0$</th>
<th>$Y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_0$</td>
<td>$Y_1$</td>
<td>$Y'_1$</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>$Y'_1$</td>
<td></td>
</tr>
<tr>
<td>$Y'_1$</td>
<td></td>
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</tr>
</tbody>
</table>

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### Table B
Complex Zeros of $Y_0(z)$, $Y_1(z)$, and $Y'_1(z)$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$z_{u,s}$ Real</th>
<th>$z_{u,s}$ Imag.</th>
<th>$z_{l,s}$ Real</th>
<th>$z_{l,s}$ Imag.</th>
<th>$z_{l',s}$ Real</th>
<th>$z_{l',s}$ Imag.</th>
</tr>
</thead>
<tbody>
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<td>-5.51988</td>
<td>.54718</td>
<td>-8.65367</td>
<td>.54841</td>
</tr>
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<td>.54841</td>
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<td>-27.49348</td>
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<td>.54927</td>
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</tbody>
</table>

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**A Method of Plotting on Standard IBM Equipment**

The advent of automatic computing machinery in research in physics and chemistry has eliminated bulky arithmetic procedures, but in many cases this advantage is lost by the bottleneck of plotting the results; as for example in spectrum analysis where many computed curves have to be compared with the data in the form of a graph. If the results are computed on cards, or are in volume enough to be punched on cards, the following procedure simplifies the problem of plotting a large number of points.

The cards are sorted on abscissa and are fed into an IBM tabulator (preferably a 405) fitted with two digit selectors and preferably six class selectors. Each spacing of the platen (corresponding to each line in a typewritten roll of paper) is taken as unit increase in the abscissa. The range of the abscissa is then infinite—with obvious practical limitations. The unit spacing of the ordinate is that between type bars, of which there are 88 across the paper plus a space between the alphabetical and numerical sections. The range of ordinates is therefore limited to 0 to 88. This, however, is a suitable match for the abscissa scale for a reasonable length of paper. If the numbers to be plotted do not lie between 0 and 88 or lie in only a small fraction of this range, say 0 to 20, they can be machine-multiplied beforehand.

The problem then is, for a given value of the ordinate (punched on the card), say 35, to actuate the thirty-fifth type bar, which prints a symbol...