D. F. E. MEISSEL 1890

Hilbert's Double Series Theorem and Principal Latent Roots of the Resulting Matrix

The inequality $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{m+n-1} \leqq \pi \sum_p a_p^2$, was proved by HILBERT and published by WEYL.[1] Various proofs were given by HARDY, LITTLEWOOD & PÓLYA.[2] In this inequality π is the best possible constant; that is, the maximum value of $\sum_m \sum_n \frac{a_m a_n}{m+n-1} \Big/ \sum_p a_p^2$ for arbitrary $\{a_p\}$ is π. It is no longer the best possible sum when the summation is finite; from 1 to N, say. In this case FRAZER[3] has shown that $(N+1) \sin [\pi/(N+1)]$ is better. But this result is not the best possible, and COPSEY, FRAZER, & SAWYER have published investigations [4] based on empirical values of the constant λ for $N = 1(1)5,10,20$, computed by the Royal Aircraft Establishment. Further computations for $N = 2(1)20$ are being made by the National Physical Laboratory.

The ordinary method for maximizing this quadratic form shows [5] that the best possible value of the constant is the greatest latent root of the matrix $\|1/(m+n-1)\|$; $n \leqq N$, and it was in this way that the values were computed.

The roots and vectors of the segments of the Hilbert matrices of N rows were derived from the iterated multiplication of the matrix into an arbitrary column vector, the procedure and accelerating processes being essentially those of AITKEN.[6] Operations were carried out on British Hollerith machines following the cycle, tabulator-reproducer-sorter-multiplier-tabulator. Machines were checked in the usual way by check sums. Final checking was done on a Brunsviga 20 machine, and roots and vectors for $n = 6$ and 8 were added at this stage.

An approximate relation between N and λ of the form $1/(\pi - \lambda) = a \ln (N + b) + c$ has been found,[4] but it is evidently incomplete as the error increases rapidly with N. Dr. OLGA TODD, in a recent unpublished communication, has shown that though λ tends to π asymptotically as N tends to ∞, π is not a latent root of the infinite matrix. Further, she shows that

$$\lambda = \pi\{1 + O(1/\log N)\} \text{ and } \sum_1^N \sum_1^N \frac{a_m a_n}{m+n-1} \Big/ \sum_1^N a_p^2 \sim \pi\{1 + O(1/\log N)\},$$

when $\{a_p\} = \{1, 1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{4}, \cdots 1/\sqrt{N}\}$.

Principal Latent Root and Vector of Segments of the Hilbert Matrix

N	2	3	4	10	20
Latent Root	1.26759 188	1.40831 893	1.50021 428	1.75191 967	1.90713 472
Latent Vector	1	1	1	1	1
	0.53518 376	0.55603 256	0.57017 208	0.60899 191	0.63153 893
		0.39090 795	0.40677 899	0.45313 830	0.48170 552
			0.31814 097	0.36528 601	0.39577 939
				0.30775 305	0.33864 052
				0.26672 518	0.29732 839
				0.23580 131	0.26579 806
				0.21156 396	0.24080 108
				0.19200 513	0.22041 627
				0.17586 003	0.20342 569
					0.18901 536

N	5	6	8	20 *concl.*
Latent Root	1.56705 069	1.61889 986	1.69593 900	0.17661 823
				0.16582 577
				0.15633 540
				0.14791 772
Latent Vector	1	1	1	0.14039 536
	0.58056 692	0.58862 854	0.60050 425	0.13362 876
	0.41880 095	0.42832 928	0.44267 155	0.12750 652
	0.33006 105	0.33966 189	0.35437 045	0.12193 851
	0.27325 824	0.28252 359	0.29691 858	0.11685 095
		0.24233 781	0.25618 093	
			0.22562 937	
			0.20179 019	

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[3] H. Frazer, "Note on Hilbert's inequality," London Math. Soc., *Jn.*, v. 21, 1946, p. 7–9.
[4] E. H. Copsey, H. Frazer, & W. W. Sawyer, "Empirical data on Hilbert's inequality," *Nature*, v. 161, 6 Mar. 1948, p. 361.
[5] R. Courant & D. Hilbert, *Methoden der mathem. Physik*, second ed., v. 1, Berlin, 1931; U.S.A. photo-lithoprint, 1943.
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Piecewise Polynomial Approximation for Large-Scale Digital Calculators

1. Introduction. Most large-scale digital calculating machines are equipped to perform automatically the arithmetic operations of addition, subtraction, multiplication, division, and in some cases of extracting the square root. All arithmetic processes must be carried out by suitably combining these given operations. But many functions whose evaluation is frequently required, such as the elementary transcendental functions, for example, cannot be represented exactly by any combination of a finite number of the given operations. In order to evaluate such functions, it is necessary to resort to some sort of approximation.

A method frequently employed may be called "piecewise polynomial approximation." This method consists of dividing the interval upon which the required function is to be approximated into a number of sub-intervals upon each of which the function is represented by a polynomial. The coefficients of these polynomials are stored within the machine or external to it in a manner consistent with the machine's construction. When the value of the independent variable is given, the proper sub-range is selected by the machine itself. The operations of addition and multiplication applied to the value of the independent variable and to the stored coefficients are then sufficient to evaluate the appropriate polynomial and hence to obtain an approximation to the required function.

In practice, the range over which the approximation is to hold and the maximum allowable error are usually known in advance. We shall assume that the maximum allowable degree of the approximating polynomials is