(a) two equal blunders in successive values or (b) a systematic succession of erroneous values in a table.

It is also proposed to give error patterns, such as that in Table III, for tables of divided differences, for use with tables having certain common arrangements of arguments at unequal intervals, for example, with a table having arguments

\[ 0, \frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{2}, 1\frac{3}{4}, \ldots \]

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1 The introduction of this useful distinction in name between rounding-off and true errors is due to C. R. G. COSSENS.

2 It must be remarked that the sequences of errors discussed here can arise from a cause other than the one indicated, though such causes are comparatively less common. For instance, if differencing is done on a calculating machine, a function value may be correctly recorded, but wrongly set on the machine. Likewise, a different sequence of differences indicates blunders of a different type. It is hoped to discuss some of these in Part II of the paper.

3 In practice, a large blunder shows up well enough for location in earlier orders of differences, in fact, as soon as the largest of the differences due to the blunder sufficiently exceeds the true differences in magnitude, say in the ratio 5 to 1 or 10 to 1. Detection is possible in still earlier differences.

4 A. VAN WIJNGAARDEN & W. L. SCHEEN of the Mathematisch Centrum of Amsterdam, Holland, have developed the theory independently and have obtained an asymptotic expansion. The result given for 9-th differences in our table was obtained by them and communicated to us for inclusion in this paper. Their 1 percent limit for 10-th differences is 303.

An ENIAC Determination of \( \pi \) and \( e \) to more than 2000 Decimal Places

Early in June, 1949, Professor JOHN VON NEUMANN expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of \( \pi \) and \( e \) to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits, suggesting the employment of one of the formulas:

\[
\begin{align*}
\pi/4 &= 4 \arctan 1/5 - \arctan 1/239 \\
\pi/4 &= 8 \arctan 1/10 - 4 \arctan 1/515 - \arctan 1/239 \\
\pi/4 &= 3 \arctan 1/4 + \arctan 1/20 + \arctan 1/1985
\end{align*}
\]

in conjunction with the GREGORY series

\[ \arctan x = \sum_{n=0}^{\infty} (-1)^n (2n + 1)^{-1} x^{2n+1}. \]

Further interest in the project on \( \pi \) was expressed in July by Dr. NICHOLAS METROPOLIS who offered suggestions about programming the calculation.

Since the possibility of official time was too remote for consideration, permission was obtained to execute these projects during two summer holiday week ends when the ENIAC would otherwise stand idle, and the planning and programming of the projects was undertaken on an extra-curricular basis by the author.

The computation of \( e \) was completed over the July 4th week end as a
an eniac determination of \( \pi \) and \( e \) practice job to gain experience and technique for the more difficult and longer project on \( \pi \). The reciprocal factorial series was employed:

\[
e = \sum_{n=0}^{\infty} \frac{1}{n!}.
\]

The first of the above-mentioned formulas was employed for the computation of \( \pi \); its advantage over the others will be explained later. The computation of \( \pi \) was completed over the Labor-Day week end through the combined efforts of four members of the ENIAC staff: CLYDE V. HAUFF (who checked the programming for \( \pi \)), Miss HOME S. McALLISTER (who checked the programming for \( e \)), W. BARKLEY FRITZ and the author, taking turns on eight-hour shifts to keep the ENIAC operating continuously throughout the week end.

While the programming for \( e \) is valid for a little over 2500 decimal places and, with minor alterations, can be extended to much greater range, and while the programming for \( \pi \) is valid for around 7000 decimal places, the arbitrarily selected limit of 2000+ was a convenient stopping point for \( e \) and about all that could be anticipated for a week end's operation for \( \pi \).

While the details of the programming for each project were completely different, the general pattern of procedure was roughly the same, and both projects will be discussed together. In both projects the ENIAC'S divider was employed to determine a chosen number \( i \) of digits of each successive term of the series being computed, the remainder after each division being stored in the ENIAC'S memory and the digits of each term being added to (or subtracted from) the cumulative total. After performing this operation for as many successive terms as practicable, the remainders for these terms were printed on an I.B.M. card (the standard input-output vehicle for the ENIAC), and the process was repeated, continuing through some term beyond which the digits of and remainders for all further terms would be zeros. At this point was printed the cumulative total of the digits of the individual terms, which yielded (after adjustment for carry-over) the actual digits of the series being determined.

The cards bearing the remainders then were fed into the ENIAC reader, and the entire process was repeated for the next \( i \) digits, the ENIAC reading each remainder in turn and placing it before the digits of the appropriate term. Each deck of cards bearing remainders was then employed to determine the "next" \( i \) digits and the "next" deck of "remainder" cards continuing through the first stopping point beyond the 2000th decimal place. The cards bearing the cumulative totals of sets of \( i \) digits of the terms were then adjusted for carry-over into each preceding set of \( i \) digits. In the case of \( e \) this yielded the final result; in the case of \( \pi \) all the above described operations were performed once for each inverse tangent series, so that each set of "cumulative total" cards, adjusted for carry-over, yielded the digits of one of the series, the final result being determined by the combination of these series in appropriate manner.

The number of places \( i \) chosen for each interval of computation, the maximum magnitude of each remainder, the amount of memory space available, and the details of divider operation (the number of places to which division can be performed to yield a positive remainder, and the necessary conditions of relative and absolute positioning of numerator and...
denominator) all were interrelated, and where opportunity for selection existed, that selection was made which provided maximum efficiency of computation. In the case of \( \pi \) there was imposed the additional requirement that identical programming apply for all series employed, and for this reason the formula:

\[
\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}
\]

was superior to the other two.

In order to insure absolute digital accuracy, the programming was arranged so that one half applied to computation and the other half to checking. Before any deck of “remainder” cards was employed to determine the next \( i \) digits, the cards were reversed and employed in the checking sequence to confirm each division by a multiplication and each addition by a subtraction and vice versa, reproducing the previous deck of “remainder” cards and insuring that the cumulative total reduced to zero. (In the case of \( e \) this was a simple inversion of the computation; in the case of \( \pi \) the factor \( (2n + 1)^{-1} \) in each term made it a more complicated affair.) After the correctness of each deck was established through this checking, the “remainder” cards were rereversed, and the computation proceeded for the next \( i \) digits.

Since the determination of each \( i \) digits was not begun until the determination of the previous \( i \) digits had been confirmed by checking, the ENIAC stood idle during the reversals and rereversals and comparisons of the decks in the computation of \( e \); in the case of \( \pi \), however, the ENIAC was never idle, for operation on each series was alternated with operation on the other, card-handling on either being accomplished while the other was being operated upon by the ENIAC. In the case of \( e \), insurance against any undiscovered accidental misalignment of cards was provided by rerunning the entire computation without checking, i.e., without card reversals, confirming the original results; in the case of \( \pi \), the same assurance was provided by a programmed check upon the identification numbers of each successive card in both computation and checking.

In the case of \( e \), there was printed (in addition to each “remainder” card) a card containing the current \( i \) digits of \((\pi!)^{-1}\) for \( n = 20K; K = 1, 2, 3 \ldots \); in the case of \( \pi \) only remainder and final total cards were printed.

The ENIAC determinations of both \( \pi \) and \( e \) confirm the 808-place determination of \( e \) published in *MTAC*, v. 2, 1946, p. 69, and the 808-place determination of \( \pi \) published in *MTAC*, v. 2, 1947, p. 245, as corrected in *MTAC*, v. 3, 1948, p. 18–19.

Only the following minor observation is offered at this time concerning the randomness of the distribution of the digits. Publication on this subject will, however, be forthcoming soon. A preliminary investigation has indicated that the digits of \( e \) deviate significantly from randomness (in the sense of staying closer to their expectation values than a random sequence of this length normally would) while for \( \pi \) no significant deviations have so far been detected.

The programming was checked and the first few hundred decimal places of each constant were determined on a Sunday before each holiday weekend mentioned above, the principal effort being made on the longer weekend. The actual required machine running time for both computation and checking in the case of \( e \) was around 11 hours, though card-handling time approxi-
mately doubled this, and the recomputation without checking added about 6 hours more; actual required machine running time (including card-handling time) for $\pi$ was around 70 hours.

The following values of $\pi$ and $e$ have been rounded off to 2035D and 2010D respectively.

$$\pi = 3.14159265358979323846264338327950288419716939953510,$$

$$e = 2.718281828459045235360287471352662497757256065246264212143243821050.001357838752886587532083814206171866478686857772358$$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Values of the auxiliary numbers \(	ext{arccot } 5\) and \(	ext{arccot } 239\) to 2035D are in the possession of the author and also have been deposited in the library of Brown University and the UMT File \(^1\) of MTAC.

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\(^1\) See MTAC, v. 4, p. 29.

RECENT MATHEMATICAL TABLES


The table gives the first 20 significant figures of \(n!\) for \(n = 1(1) 200\) together with the exponent of the power of 10 by which the figure should be multiplied to give the approximate value of \(n!\). The author was unaware of a previous table by Uhler\(^1\) giving the exact values of these factorials.