Thus the values of p_n for $2100 \leqslant n \leqslant 2500$ are still significantly low but higher than the value of p_n at $n = 2000$.

Note that the general size and trend of p_n, as well as its sudden deviation at $n = 2000$, indicate a non random character in the digits of e.

More detailed investigations are in progress and will be reported later.

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[1] Both e and $1/e$ were computed somewhat beyond 2500 D and the results checked by actual multiplication.

Notes on Numerical Analysis—2
Note on the Condition of Matrices

1. The object of this note is to establish the following theorem.

THEOREM. *Let A be a real $n \times n$ non-singular matrix and A' be its transpose. Then AA' is more "ill-conditioned" than A.*

This theorem confirms an opinion expressed by Dr. L. FOX[1] based on his practical experience. The term "condition of a matrix" has been used rather vaguely for a long time. The most common measure of the condition of a matrix has been the size of its determinant, ill-conditioned matrices being those with a "small" determinant. With this interpretation imposed, the theorem is clearly correct. More adequate measures of the condition of a matrix have been proposed recently by JOHN VON NEUMANN & H. H. GOLDSTINE[2] and by A. M. TURING.[3] Their definitions concern all matrices, not just the ill-conditioned ones, characterized by very large condition numbers. The following two of these definitions will form a basis for the proof of the above-mentioned theorem:

The P-condition number is $|\lambda_{max}|/|\lambda_{min}|$, where λ_{max} and λ_{min} are the characteristic roots of largest and smallest modulus.[2]

The N-condition number is $N(A)N(A^{-1})/n$, where[3]

$$N(A) = (\sum_{i,k} a_{ik}^2)^{\frac{1}{2}}.$$

2. Proof of the theorem in the P case:

Let λ_i be the characteristic roots of A and μ_i those of AA' (which are in general distinct from the squares of the absolute values of λ_i). E. T. BROWNE[4] has shown that

$$\mu_{min} \leqslant \lambda_i\bar{\lambda}_i \leqslant \mu_{max}.$$

From this it follows that

$$1 \leqslant \left|\frac{\lambda_{max}}{\lambda_{min}}\right| \leqslant \left|\frac{\lambda_{max}}{\lambda_{min}}\right|^2 \leqslant \frac{\mu_{max}}{\mu_{min}},$$

which implies the required result.

3. Proof of the theorem in the N case:

It is known that $N(A)$ is the square root of the trace of AA' and therefore equal to $(\sum \mu_i)^{\frac{1}{2}}$. The numbers μ_i are all positive since AA' is symmetric and positive definite. Since the characteristic roots of $A'A$ and AA' are the same and since the characteristic roots of the inverse of a matrix are the reciprocals of those of the original matrix, it follows that

$$N(A^{-1}) = (trA^{-1}(A^{-1})')^{\frac{1}{2}} = (tr(A'A)^{-1})^{\frac{1}{2}} = (\sum \mu_i^{-1})^{\frac{1}{2}}.$$

The N-condition number of A is therefore

$$\frac{1}{n}(\sum \mu_i)^{\frac{1}{2}}(\sum \mu_i^{-1})^{\frac{1}{2}}.$$

In a similar way it can be shown that the N-condition number of AA' is

$$\frac{1}{n}(\sum \mu_i^2)^{\frac{1}{2}}(\sum \mu_i^{-2})^{\frac{1}{2}}.$$

The theorem follows from the inequality

$$\sum \mu_i^2 \sum \mu_i^{-2} \geqslant \sum \mu_i \sum \mu_i^{-1},$$

which is in fact true for all real and positive numbers. (It is, indeed, true when the first power on the right is replaced by an arbitrary power r and the second power on the left by a power $s > r$.) The proof of the inequality is as follows:

$$\begin{aligned}\sum \mu_i^2 \sum \mu_i^{-2} - \sum \mu_i \sum \mu_i^{-1} \\ &= n + \sum_{i\neq j} \mu_i^2 \mu_j^{-2} - n - \sum_{i\neq j} \mu_i\mu_j^{-1} \\ &= \sum_{i<j} (\mu_i^2\mu_j^{-2} + \mu_j^2\mu_i^{-2}) - \sum_{i<j} (\mu_i\mu_j^{-1} + \mu_j\mu_i^{-1}) \\ &= \sum_{i<j} \{(\mu_i\mu_j^{-1} + \mu_j\mu_i^{-1})(\mu_i\mu_j^{-1} + \mu_j\mu_i^{-1} - 1) - 2\} \geqslant 0,\end{aligned}$$

since

$$\mu_i\mu_j^{-1} + \mu_j\mu_i^{-1} \geqslant 2, \qquad \text{and} \qquad \mu_i\mu_j^{-1} + \mu_j\mu_i^{-1} - 1 \geqslant 1.$$

There is equality if and only if

$$\mu_1 = \mu_2 \cdots = \mu_n.$$

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[1] In a course of lectures given at the British Admiralty by himself and D. H. SADLER in 1949.

[2] J. VON NEUMANN & H. H. GOLDSTINE, "Numerical inverting of matrices of high order," Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 1021–1099. (These authors consider symmetric matrices only, but it is reasonable to apply the definition to the general case.)

[3] A. M. TURING, "Rounding-off errors in matrix processes," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 287–308.

[4] E. T. BROWNE, "The characteristic equation of a matrix," Amer. Math. Soc., *Bull.*, v. 34, 1928, p. 363–368.

BIBLIOGRAPHY Z–XI

1. E. G. ANDREWS, "The Bell Computer, Model VI," *Electrical Engineering*, v. 68, 1949, p. 751–756, 7 figs., 5 tables. 22.2 × 29.5 cm.

Controlled from remote stations, this new digital computer of the relay type reduces punched-tape instructions to a minimum. With novel control features similar to those used in recent automatic dial-telephone developments, this "upper-class" computer possesses six "intelligence levels." Sub-