Thus the values of $p_n$ for $2100 \leq n \leq 2500$ are still significantly low but higher than the value of $p_n$ at $n = 2000$.

Note that the general size and trend of $p_n$, as well as its sudden deviation at $n = 2000$, indicate a non-random character in the digits of $e$.

More detailed investigations are in progress and will be reported later.

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1 Both $e$ and $1/e$ were computed somewhat beyond 2500 D and the results checked by actual multiplication.

Notes on Numerical Analysis—2  
Note on the Condition of Matrices

1. The object of this note is to establish the following theorem.

**Theorem.** Let $A$ be a real $n \times n$ non-singular matrix and $A'$ be its transpose. Then $AA'$ is more "ill-conditioned" than $A$.

This theorem confirms an opinion expressed by Dr. L. Fox\(^1\) based on his practical experience. The term "condition of a matrix" has been used rather vaguely for a long time. The most common measure of the condition of a matrix has been the size of its determinant, ill-conditioned matrices being those with a "small" determinant. With this interpretation imposed, the theorem is clearly correct. More adequate measures of the condition of a matrix have been proposed recently by John von Neumann & H. H. Goldstine\(^2\) and by A. M. Turing.\(^3\) Their definitions concern all matrices, not just the ill-conditioned ones, characterized by very large condition numbers. The following two of these definitions will form a basis for the proof of the above-mentioned theorem:

The $P$-condition number is $|\lambda_{\max}| / |\lambda_{\min}|$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the characteristic roots of largest and smallest modulus.\(^2\)

The $N$-condition number is $N(A)N(A^{-1})/n$, where\(^3\)

$$N(A) = \left(\sum_{i,k} a_{ik}^2\right)^{1/2}.$$

2. Proof of the theorem in the $P$ case:

Let $\lambda_i$ be the characteristic roots of $A$ and $\mu_i$ those of $AA'$ (which are in general distinct from the squares of the absolute values of $\lambda_i$). E. T. Browne\(^4\) has shown that

$$\mu_{\min} \leq \lambda_i \lambda_i \leq \mu_{\max}.$$

From this it follows that

$$1 \leq \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| \leq \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|^2 \leq \frac{\mu_{\max}}{\mu_{\min}},$$

which implies the required result.

3. Proof of the theorem in the $N$ case:

It is known that $N(A)$ is the square root of the trace of $AA'$ and therefore equal to $(\sum \mu_i)^{1/2}$. The numbers $\mu_i$ are all positive since $AA'$ is symmetric and positive definite. Since the characteristic roots of $A'A$ and $AA'$ are the
same and since the characteristic roots of the inverse of a matrix are the
reciprocals of those of the original matrix, it follows that
\[ N(A^{-1}) = (trA^{-1}(A^{-1})') = (tr(A'A)^{-1}) = (\sum \mu_i)^{-1}. \]

The \(N\)-condition number of \(A\) is therefore
\[ \frac{1}{n} (\sum \mu_i)^{\frac{1}{n}} (\sum \mu_i^{-1})^{-\frac{1}{n}}. \]

In a similar way it can be shown that the \(N\)-condition number of \(AA'\) is
\[ \frac{1}{n} (\sum \mu_i^2)^{\frac{1}{n}} (\sum \mu_i^{-2})^{-\frac{1}{n}}. \]

The theorem follows from the inequality
\[ \sum \mu_i^2 \sum \mu_i^{-2} \geq \sum \mu_i \sum \mu_i^{-1}, \]
which is in fact true for all real and positive numbers. (It is, indeed, true
when the first power on the right is replaced by an arbitrary power \(r\) and
the second power on the left by a power \(s > r\).) The proof of the inequality
is as follows:
\[ \sum \mu_i^2 \sum \mu_i^{-2} - \sum \mu_i \sum \mu_i^{-1} \]
\[ = n + \sum_{i \neq j} \mu_i \mu_j^{-2} - n - \sum \mu_i \mu_j^{-1} \]
\[ = \sum_{i < j} (\mu_i^2 \mu_j^{-2} + \mu_j^2 \mu_i^{-2}) - \sum \mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} \]
\[ = \sum_{i < j} \{ (\mu_i \mu_j^{-1} + \mu_j \mu_i^{-1})(\mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} - 1) - 2 \} \geq 0, \]
since
\[ \mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} \geq 2, \quad \text{and} \quad \mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} - 1 \geq 1. \]

There is equality if and only if
\[ \mu_1 = \mu_2 = \cdots = \mu_n. \]

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1 In a course of lectures given at the British Admiralty by himself and D. H. Sadler
in 1949.
2 J. von Neumann & H. H. Goldstine, "Numerical inverting of matrices of high order," Amer. Math. Soc., Bull., v. 53, 1947, p. 1021-1099. (These authors consider symmetric matrices only, but it is reasonable to apply the definition to the general case.)

Bibliography Z–XI


Controlled from remote stations, this new digital computer of the relay type reduces punched-tape instructions to a minimum. With novel control features similar to those used in recent automatic dial-telephone developments, this "upper-class" computer possesses six "intelligence levels." Sub-