Appendix II. Glossary.

n the number of component sentences in the complete schema.
p_j the jth component sentence.
a_j, b_j, etc. the jth component schema of order 1, 2, etc.
c the number of connectives in the schema.
T estimated computing time in seconds.
1 (or 01) in the proper context the truth value of a true sentence.
0 (or 00) in the proper context the truth value of a false sentence.
x(B)y y is to replace x wherever x appears.
(h) address in the memory of the number h.
C(M) contents of the memory box M.
$\sim$ It is not the case that . . .
$\wedge$ . . . and . . .
$\vee$ . . . or . . .
$\rightarrow$ If . . . then . . .
$\leftrightarrow$ . . . if and only if . . .
$_x$ For every x . . .

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[1] EDMUND C. BERKELEY, *Giant Brains*, New York, 1949, Ch. 9.
[2] GEORGE W. PATTERSON, *Logical Syntax and Transformation Rules*. Moore School of Electrical Engineering, Research Division Report 50–8, Univ. of Pennsylvania, Philadelphia, 1949.

On the Accuracy of Runge-Kutta's Method

1. Introduction. While the accuracy of the most frequently used methods of integrating differential equations is fairly well known, that of the Runge-Kutta method does not seem to be too well established; except for a formula in Bieberbach's text[1] on differential equations there are no references pertaining to the error inherent in the Runge-Kutta method to be found in the standard textbooks on this subject.

Since this method may be employed quite advantageously in many cases of practical interest it is important to have on hand an estimate of the error. The purpose of the following sections is to provide such an estimate. As a comparison shows, the bound derived for this error seems to be somewhat better than the one cited by Bieberbach.

2. Runge-Kutta's Fourth Order Method. In trying to find that solution of the differential equation

$$dy/dx = f(x, y), \qquad y(x_0) = y_0, \tag{1}$$

at $x_1 = x_0 + h$, which agrees with the exact Taylor expansion about x_0:

$$y(x_1) = y_0 + hy_0' + h^2(y_0''/2) + h^3(y_0'''/6) + h^4(y_0^{\text{iv}}/24) + h^5(y_0^{\text{v}}/120) + \cdots \tag{2}$$

up to the term in h^4, Runge and Kutta developed the following formulae:

$$
\begin{aligned}
(3) \qquad & y(x_1) \approx y_0 + (k_1 + 2k_2 + 2k_3 + k_4)/6, \\
& k_1 = hf_0, \qquad f_0 \equiv f(x_0, y_0), \\
& k_2 = hf(x_0 + h/2, y_0 + k_1/2), \\
& k_3 = hf(x_0 + h/2, y_0 + k_2/2), \\
& k_4 = hf(x_0 + h, y_0 + k_3).
\end{aligned}
$$

To get an estimate of the truncation error inherent in this procedure, one may apply the method first to an interval of length $h_1 = h$, and then integrate over two consecutive intervals of length $h_2 = h/2$. Having the results Y_1, Y_2 of these integrations it is easy to obtain an estimate of the error of the second integration: Since the values Y_1, Y_2 differ from the exact value y_1 by certain errors E_1, E_2:

$$Y_1 = y_1 + E_1, \qquad Y_2 = y_1 + E_2,$$

where

$$E_1 = Ch_1^5, \qquad E_2 \approx 2Ch_2^5 = E_1/16,$$

obviously

$$(4) \qquad E_2 \approx (Y_1 - Y_2)/15.$$

3. Calculation of the Error Term. A more accurate estimate of the error is obtained by a comparison of the exact coefficient $(y_0^{v}/120)$ of h^5, as it occurs in the Taylor expansion (2), with the approximate one resulting from Runge-Kutta's algorithm (3). Suppose equations (3) have been expressed in the form

$$y(x_1) = y_0 + C_1h + C_2h^2 + C_3h^3 + C_4h^4 + C_5h^5 + \cdots$$

Then $C_i = y_0^{(i)}/i!$, $i = 1, 2, 3, 4$,

$$(5) \qquad C_5 = (y_0^{v}/5!) + \epsilon,$$

and the following considerations are concerned with the determination of ϵ. In computing the successive total derivatives of a function $u = u(x, y(x))$, $y' = f$, it is helpful to make use of the operator[2]

$$D \equiv \partial/\partial x + f\partial/\partial y.$$

This operator has the following properties:

$$
\begin{aligned}
& D^n = \sum_{k=0}^{n} \binom{n}{k} f^k \partial^n/\partial x^{n-k}\partial y^k, \\
(6) \qquad & D(D^n u) = D^{n+1}u + n(Df)D^{n-1}u_y, \\
& D^0 u = u.
\end{aligned}
$$

Let us now apply the operator D to $f(x, y(x))$ and use the abbreviating notations

$$T^n \equiv D^n f, \qquad S^n \equiv D^n f_y.$$

Then (6) may be expressed as

$$
\begin{aligned}
(7) \qquad & D(T^n) = T^{n+1} + nTS^{n-1}. \\
& DS = S^2 + Tf_{yy}.
\end{aligned}
$$

Applying (7) repeatedly we get

$$\begin{aligned} y' &= f \\ y'' &= Df = T \\ y''' &= D(T) = T^2 + Tf_y \\ y^{iv} &= D(T^2 + Tf_y) = D(T^2) + D(T)f_y + TS \\ &= T^3 + f_yT^2 + 3ST + Tf_y^2. \end{aligned}$$

Similarly,

$$y^v = T^4 + f_yT^3 + 6TS^2 + 4ST^2 + f_y^2T^2 + 3f_{yy}(P)^2 + 7f_yST + f_y^3T,$$

with $(P)^2 = (Df)^2$.

The expansions of the Runge-Kutta's expressions (3) are somewhat more laborious to calculate. First we find that

$$\begin{aligned} k_2 &= hf(x_0 + h/2, y_0 + k_1/2) = h[f + (h/2)f_x + (k_1/2)f_y \\ &\qquad + \tfrac{1}{2}((h^2/4)f_{xx} + 2(h/2)(k_1/2)f_{xy} + (k_1^2/4)f_{yy}) + \cdots] \\ &= k_1 + (T/2)h^2 + (T^2/8)h^3 + (T^3/48)h^4 + (T^4/384)h^5 + O(h^6). \end{aligned} \tag{8}$$

Here, as well as in the following, the arguments of f and $f_{x^iy^j}$ are x_0, y_0.

Next we compute

$$\begin{aligned} k_3 &= hf(x_0 + h/2, y_0 + k_2/h) = h[f + (h/2)f_x + (k_2/2)f_y \\ &\qquad + \tfrac{1}{2}((h^2/4)f_{xx} + 2(h/2)(k_2/2)f_{xy} + (k_2^2/4)f_{yy}) + \cdots] \\ &= h[f + (1/2)(hf_x + k_2f_y) + (1/8)(h^2f_{xx} + 2hk_2f_{xy} + k_2^2f_{yy}) + \cdots]. \end{aligned}$$

It is convenient to introduce here the operators:

$$\begin{aligned} G_i &\equiv h\partial/\partial x + k_i\partial/\partial y, \qquad i = 1, 2, 3, \\ U_i^n &\equiv G_i^n f = (h\partial/\partial x + k_i\partial/\partial y)^n f. \end{aligned}$$

In terms of the U_i^n we may write

$$k_3 = h[f + (U_2/2) + (U_2^2/8) + (U_2^3/48) + (U_2^4/384) + \cdots]. \tag{9}$$

In order to express k_3 in powers of h we notice from (8) that

$$k_2^2 = k_1^2 + fTh^3 + (1/4)(P + fT^2)h^4 + O(h^5).$$

Further,

$$\begin{aligned} k_2^3 &= k_1^3 + (3/2)f^2Th^4 + O(h^5), \\ k_2^4 &= k_1^4 + O(h^5). \end{aligned}$$

Consequently,

$$\begin{aligned} U_1 &= hT \\ U_2 &= hf_x + k_2f_y \\ &= U_1 + (1/2)f_yTh^2 + (1/8)f_yT^2h^3 + (1/48)f_yT^3h^4 + O(h^5) \\ U_2^2 &= U_1^2 + TSh^3 + (1/4)(ST^2 + Pf_{yy})h^4 + O(h^5) \\ U_2^3 &= U_1^3 + (3/2)TS^2h^4 + O(h^5) \\ U_2^4 &= U_1^4 + O(h^5). \end{aligned}$$

Therefore, by (9),

$$k_3 = k_1 + \tfrac{1}{2}Th^2 + k_{33}h^3 + k_{34}h^4 + k_{35}h^5 + O(h^6), \tag{10}$$

with

$$\begin{aligned} k_{33} &= (2Tf_y + T^2)/8 \\ k_{34} &= (3f_yT^2 + 6ST + T^3)/48 \\ k_{35} &= (4f_yT^3 + 12(ST^2 + Pf_{yy} + TS^2) + T^4)/384. \end{aligned}$$

Finally we compute

$$\begin{aligned} k_4 &= hf(x_0 + h, y_0 + k_3) \\ &= h[f + hf_x + k_3f_y + \tfrac{1}{2}(h^2f_{xx} + 2hk_3f_{xy} + k_3^2f_{yy}) + \cdots] \\ &= h[f + U_3 + (U_3^2/2) + (U_3^3/6) + (U_3^4/24) + \cdots]. \end{aligned} \tag{11}$$

Since, by (10),

$$\begin{aligned} k_3^2 &= (k_1 + (T/2)h^2 + k_{33}h^3 + \cdots)^2 \\ &= k_1^2 + fTh^3 + (1/4)(P + fk_{33})h^4 + O(h^5), \\ k_3^3 &= k_2^3, \qquad k_3^4 = k_2^4, \end{aligned}$$

there result the following expressions:

$$\begin{aligned} U_3 &= hf_x + k_3f_y \\ &= U_2 + (1/4)Tf_y^2h^3 + (1/16)f_y(f_yT^2 + 2ST)h^4 \\ U_3^2 &= U_2^2 + (1/2)STf_yh^4 \\ U_3^3 &= U_2^3, \qquad U_3^4 = U_2^4. \end{aligned}$$

Thus it follows from (11) that

$$k_4 = k_1 + Th^2 + k_{43}h^3 + k_{44}h^4 + k_{45}h^5 + O(h^6), \tag{12}$$

with

$$\begin{aligned} k_{43} &= (1/2)(Tf_y + T^2) \\ k_{44} &= (1/6)(6k_{33}f_y + 3ST + T^3) \\ k_{45} &= (1/24)(24(k_{34}f_y + Sk_{33}) + 3Pf_{yy} + 6TS^2 + T^4). \end{aligned}$$

In the Runge-Kutta expression (3) the coefficient C_5 of h^5 is thus found to be

$$\begin{aligned} C_5 &= (k_{15} + 2k_{25} + 2k_{35} + k_{45})/6 \\ &= (36Pf_{yy} + 60TS^2 + 72f_yST + 36ST^2 \\ &\qquad + 12f_y^2T^2 + 8f_yT^3 + 10T^4)/1152. \end{aligned}$$

It follows that

$$\begin{aligned} \epsilon &= C_5 - (y_0^{\mathrm{v}}/120) \\ &= (-12Tf_y^3 + 9Pf_{yy} + 3TS^2 + 6STf_y - 3ST^2 \\ &\qquad + 3f_y^2T^2 - 2f_yT^3 + (T^4/2))/1440, \end{aligned}$$

or, more explicitly,

$$\begin{aligned} \epsilon = [&-12Tf_y^3 + 3(f_y^2 - S)f_{xx} + 3(2Tf_y + 2ff_y^2 - 2fS - f^2f_{yy})f_{xy} \\ &+ 3(3P + 2Tff_y + f^2f_y^2 - f^3f_{yy})f_{yy} - 2f_yf_{xxx} + 3(f_x - ff_y)f_{xxy} \\ &+ 6ff_xf_{xyy} + (T + 2f_x)f^2f_{yyy} + (T^4/2)]/1440. \end{aligned} \tag{13}$$

4. Estimate of the Error. Let it be assumed, now, that in a certain

region $B(x, y)$ containing (x_0, y_0)

$$(14)\qquad \begin{aligned} &|f(x, y)| \leq M, \\ &|f_{x^i y^j}| \leq L^{i+j}/M^{j-1}, \end{aligned}$$

where M, L are positive constants independent of x, y. In that case clearly

$$|T^n| \leq \sum_{j=0}^{n} \binom{n}{j} M^j (L^n/M^{j-1}) = M(2L)^n$$

$$|S^n| \leq \sum_{j=0}^{n} \binom{n}{j} M^j (L^{n+1}/M^j) = L(2L)^n, \quad |P| \leq (2ML)^2.$$

By (13), then

$$(15)\qquad |E| \equiv |\epsilon h^5| \leq (73/720) ML^4 h^5.$$

To calculate a suitable value of L one may first compute bounds $L_{i,k}$ for $|f_{x^i y^k}|$, then define quantities

$$L_{i+j} = \max\left[(L_{i+j,0}/M)^{1/(i+j)}, (L_{i+j-1,1})^{1/(i+j)}, (ML_{i+j-2,2})^{1/(i+j)}, \cdots, (M^{i+j-1}L_{0,i+j})^{1/(i+j)}\right],$$

for $i + j \leq 4$. Then one may put

$$(16)\qquad L = \max(L_1, L_2, L_3, L_4).$$

5. Example. Let us consider the differential equation

$$y' = x + y, \qquad y(0) = 0,$$

and restrict x, y to $0 \leq x \leq 0.2$, $0 \leq y \leq 0.2$. The exact solution is $y = e^x - x - 1$, so that $y(0.2) = 0.02140276$. For $h_1 = 0.2$ Runge-Kutta's method gives $Y_1 = 0.0214$, for $h_2 = 0.1$ it gives $Y_2 = 0.02140257$. Now for the above example we may take $M = 1$, $L = 1$, so that for $h = 0.2$, by formula (15),

$$|E| \approx (1/10)(2/10)^5 = 0.000032;$$

actually the error is less than 0.000003. According to the expression exhibited by BIEBERBACH[1]

$$(17)\qquad |E| < 6MNh^5|N^5 - 1|/|N - 1|,$$

where $|f| < M$, $|f_{x^i y^j}| < N/M^{i-1}$, and, further, $hN < 1$, $aM < b$, where $|x - x_0| < a$, $|y - y_0| < b$. With $N = 1$ formula (17) leads to

$$|E| < 0.0096,$$

which is considerably larger than the error estimate obtained by (15).

6. Systems of Equations. For a system of differential equations

$$y_i' = f_i(x, y_1, y_2), \qquad y_i(x_0) = y_{i0}, \qquad i = 1, 2,$$

the Runge-Kutta formulae become:

$$(18)\qquad \begin{aligned} k_{i1} &= hf_{i0}, \quad f_{i0} \equiv f_i(x_0, y_{10}, y_{20}) \\ k_{i2} &= hf_i(x_0 + h/2, y_{10} + k_{11}/2, y_{20} + k_{21}/2) \\ k_{i3} &= hf_i(x_0 + h/2, y_{10} + k_{12}/2, y_{20} + k_{22}/2) \\ k_{i4} &= hf_i(x_0 + h, y_{10} + k_{13}, y_{20} + k_{23}) \\ y_i(x_1) &\equiv y_{i0} + (1/6)(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4}). \end{aligned}$$

Proceeding in the manner described in the foregoing sections for the case of a single differential equation, and making use of the abbreviations

$$R^n \equiv D^n f_z, \quad \tau^n = D^n g, \quad \sigma^n \equiv D^n g_y, \quad \rho^n \equiv D^n g_z, \quad \pi \equiv [\tau]^2$$

one obtains for $\epsilon_1 = C_{15} - y^{\mathrm{v}}(x_0, y_{10}, y_{20})/120$ the expression

$$\begin{aligned} 1440\epsilon_1 = (T^4/2) &- 2(f_y T^3 + f_z \tau^3) + 3[T^2(f_y^2 + f_z g_y) + \tau^2(f_y f_z + f_z g_z)] \\ &- 3(ST^2 + R\tau^2) + 12[S(Tf_y + \tau f_z) + R(Tg_y + \tau g_z)] \\ &- 6[T(Sf_y + \sigma f_z) + \tau(Rf_y + \rho f_z)] + 3(TS^2 + \tau R^2) \\ &+ 9(Pf_{yy} + 2T\tau f_{yz} + \pi f_{zz}) \\ &- 12[T(f_y^3 + f_z g_y(2f_y + g_z)) + \tau f_z(f_y^2 + f_y g_z + f_z g_y)] \\ &- 15 f_{yz}(T^2 g + \tau^2 f). \end{aligned}$$

Assumptions similar to (14) now permit one to get a bound for $\epsilon_i h^5$.

If, namely, near (x_0, y_{10}, y_{20}),

$$|f_i| \leq M, \qquad |\partial^{p+q+r} f_i / \partial x^p y_1^q y_2^r| \leq L^{p+q+r}/M^{q+r-1}$$

for $0 \leq p + q + r \leq 4$, it is found that

$$|E_i| \leq (973/720) M L^4 h^5. \tag{19}$$

For the differential equation $y'' - y = 1$, $y(0) = 0$, $y'(0) = 1$, which is equivalent to the system $y_1' = y_2$, $y_2' = 1 + y_1$, $y_1(0) = 0$, $y_2(0) = 1$, the solution is $y = e^x - 1$. Thus $y(0.1) = .1051709$. With $h = 0.1$, R.K.'s method gives $y(0.1) = .1051707$, whence $E = 2 \cdot 10^{-7}$. In the region $0 \leq x \leq 0.1$, $0 \leq y_1 \leq 0.11$, $0 \leq y_2 \leq 1.11$; above estimate (19) asserts that $|E_i| \leq 1.5 \cdot 10^{-5}$.

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[1] L. BIEBERBACH, *Theorie der Differentialgleichungen*. New York, 1944, p. 54.
[2] C. RUNGE & H. KÖNIG, *Vorlesungen über numerisches Rechnen*. Berlin, 1924.

RECENT MATHEMATICAL TABLES

878[F].—H. CHATLAND & H. DAVENPORT, "Euclid's algorithm in real quadratic fields," *Canadian Jn. Math.*, v. 2, 1950, p. 289–296.

The tables in this paper are rather special and are used to prove a result of INKERI[1] that the field $k(m^{\frac{1}{2}})$ is not Euclidean for

$$m = 193{,}241{,}313{,}337{,}457{,}601.$$

There are six tables corresponding to these values of m. Each gives a complete period of a periodic algorithm showing certain inequalities which establish the non-existence of Euclid's algorithm in each field considered.

[1] K. INKERI, "Über den Euklidischen Algorithmus in quadratischen Zahlkörpern," Acad. Sci. Fennicae, *Annales*, v. 41, 1947, p. 5–34.

879[F].—A. GLODEN, "Factorisation de nombres $N^{16} + 1$," *Euclides*, v. 10, 1950, p. 157.