On the Accuracy of Runge-Kutta's Method

1. Introduction. While the accuracy of the most frequently used methods of integrating differential equations is fairly well known, that of the Runge-Kutta method does not seem to be too well established; except for a formula in Bieberbach's text on differential equations there are no references pertaining to the error inherent in the Runge-Kutta method to be found in the standard textbooks on this subject.

Since this method may be employed quite advantageously in many cases of practical interest it is important to have on hand an estimate of the error. The purpose of the following sections is to provide such an estimate. As a comparison shows, the bound derived for this error seems to be somewhat better than the one cited by Bieberbach.

2. Runge-Kutta's Fourth Order Method. In trying to find that solution of the differential equation

\[ \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \]

at \( x_1 = x_0 + h \), which agrees with the exact Taylor expansion about \( x_0 \):

\[ y(x_1) = y_0 + hy_0' + h^2(y_0''/2) + h^3(y_0'''/6) + h^4(y_0''''/24) + \cdots \]

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up to the term in \( h^4 \), Runge and Kutta developed the following formulae:

\[
\begin{align*}
    y(x_1) &\approx y_0 + (k_1 + 2k_2 + 2k_3 + k_4)/6, \\
    k_1 &= hf_0, \quad f_0 = f(x_0, y_0), \\
    k_2 &= hf(x_0 + h/2, y_0 + k_1/2), \\
    k_3 &= hf(x_0 + h/2, y_0 + k_2/2), \\
    k_4 &= hf(x_0 + h, y_0 + k_3).
\end{align*}
\]

To get an estimate of the truncation error inherent in this procedure, one may apply the method first to an interval of length \( h_1 = h \), and then integrate over two consecutive intervals of length \( h_2 = h/2 \). Having the results \( Y_1, Y_2 \) of these integrations it is easy to obtain an estimate of the error of the second integration: Since the values \( Y_1, Y_2 \) differ from the exact value \( y_1 \) by certain errors \( E_1, E_2 \):

\[
Y_1 = y_1 + E_1, \quad Y_2 = y_1 + E_2,
\]

where

\[
E_1 = C h^6, \quad E_2 = 2C h^6 = E_1/16,
\]

obviously

\[
E_2 \approx (Y_1 - Y_2)/15.
\]

3. Calculation of the Error Term. A more accurate estimate of the error is obtained by a comparison of the exact coefficient \((y_0^*/120)\) of \( h^6 \), as it occurs in the Taylor expansion (2), with the approximate one resulting from Runge-Kutta’s algorithm (3). Suppose equations (3) have been expressed in the form

\[
y(x_i) = y_0 + C_1 h + C_2 h^2 + C_3 h^3 + C_4 h^4 + C_5 h^5 + \cdots
\]

Then

\[
C_i = y_0(i)/i!, \quad i = 1, 2, 3, 4,
\]

and the following considerations are concerned with the determination of \( \epsilon \).

In computing the successive total derivatives of a function \( u = u(x, y(x)) \), \( y' = f \), it is helpful to make use of the operator

\[
D = \partial/\partial x + f\partial/\partial y.
\]

This operator has the following properties:

\[
D^n = \sum_{k=0}^{n} \binom{n}{k} f^k \partial^k/\partial x^{n-k} \partial y^k,
\]

\[
D(D^n u) = D^{n+1} u + n(Df)D^{n-1} u_v,
\]

\[
D^n u = u.
\]

Let us now apply the operator \( D \) to \( f(x, y(x)) \) and use the abbreviating notations

\[
T^n = D^n f, \quad S^n = D^n f_v.
\]

Then (6) may be expressed as

\[
D(T^n) = T^{n+1} + nT S^{n-1}.
\]

\[
DS = S^n + Tf_{vv}.
\]
Applying (7) repeatedly we get

\[
\begin{align*}
y' &= f \\
y'' &= Df = T \\
y''' &= D(T) = T^2 + Tf_y \\
y^{iv} &= D(T^2 + Tf_y) = D(T^2) + D(T)f_y + TS \\
&= T^3 + f_yT^2 + 3ST + Tf_y^2.
\end{align*}
\]

Similarly,

\[
y^{iv} = T^4 + f_yT^3 + 6TS^2 + 4ST^2 + f_y^2T^2 + 3f_y^2(P)^2 + 7f_yST + f_y^3T,
\]

with \((P)^2 = (Df)^2\).

The expansions of the Runge-Kutta’s expressions (3) are somewhat more laborious to calculate. First we find that

\[
\begin{align*}
k_2 &= hf(x_0 + h/2, y_0 + k_1/2) = h[f + (h/2)f_x + (k_1/2)f_y] \\
&\quad + \frac{1}{2}((h^2/4)f_{xx} + 2(h/2)(k_x/2)f_{xy} + (k^2_x/4)f_{yy}) + \cdots ] \\
&= k_1 + (T/2)h^2 + (T^2/4)h^4 + (T^3/48)h^6 + O(h^8).
\end{align*}
\]

Here, as well as in the following, the arguments of \(f\) and \(f_{xy}\) are \(x_0, y_0\).

Next we compute

\[
\begin{align*}
k_3 &= hf(x_0 + h/2, y_0 + k_3/h) = h[f + (h/2)f_x + (k_3/2)f_y] \\
&\quad + \frac{1}{2}((h^2/4)f_{xx} + 2(h/2)(k_x/2)f_{xy} + (k^2_x/4)f_{yy}) + \cdots ] \\
&= h[f + (1/2)(hf_x + k_2f_y) + (1/8)(h^2f_{xx} + 2hk_2f_{xy} + k_2^2f_{yy}) + \cdots ].
\end{align*}
\]

It is convenient to introduce here the operators:

\[
\begin{align*}
G_i &= \partial^i/\partial x + k_i\partial/\partial y, \quad i = 1, 2, 3, \\
U_i^n &= G_i^n f = (\partial^i/\partial x + k_i\partial/\partial y)^n f.
\end{align*}
\]

In terms of the \(U_i^n\) we may write

\[
(9) \quad k_3 = h[f + (U_3/2) + (U_2^2/48) + (U_2^3/384) + \cdots ].
\]

In order to express \(k_3\) in powers of \(h\) we notice from (8) that

\[
k_3^2 = k_3^3 + fTh^3 + 1/4)(P + fT^2)h^4 + O(h^6).
\]

Further,

\[
k_3^3 = k_1^3 + (3/2)f^2Th^4 + O(h^6),
\]

\[
k_3^4 = k_1^4 + O(h^6).
\]

Consequently,

\[
\begin{align*}
U_1 &= hT \\
U_2 &= hf_x + k_2f_y \\
&= U_1 + (1/2)f_yT^2h^2 + (1/8)f_yT^2h^3 + (1/48)f_yT^3h^4 + O(h^6) \\
U_2^3 &= U_3^3 + TSh^3 + (1/4)(ST^2 + Pf_{yy})h^4 + O(h^6) \\
U_3^3 &= U_3^3 + (3/2)TS^2h^4 + O(h^6) \\
U_4^3 &= U_4^3 + O(h^6).
\end{align*}
\]

Therefore, by (9),

\[
(10) \quad k_3 = k_1 + T^2h^2 + k_2h^3 + k_3h^4 + k_4h^5 + O(h^6),
\]

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with
\[
\begin{align*}
k_{33} &= \frac{(2Tf_v + T^2)\cdot 8}{8} \\
k_{34} &= \frac{(3f_v T^2 + 6ST + T^3)\cdot 48}{48} \\
k_{35} &= \frac{(4f_v T^3 + 12(ST^2 + Pf_{vv} + TS^3) + T^4)\cdot 384}{384}.
\end{align*}
\]

Finally we compute
\[
\begin{align*}
k_4 &= hf(\tau_0 + h, y_0 + k_3) \\
&= h[f + h f_x + k_3 f_v + \frac{1}{2}(h^2 f_{xx} + 2h k_3 f_{uv} + k_3^2 f_{vv}) + \cdots ] \\
&= h[f + U_3 + (U_4^2/2) + (U_4^3/6) + (U_4^4/24) + \cdots ].
\end{align*}
\]

Since, by (10),
\[
\begin{align*}
&k_3^2 = (1 + (T/2)h^2 + k_3 h^3 + \cdots)^2 \\
&k_4^2 = k_3^2 + fT h^3 + (1/4)(P + f k_3 h^4) + O(h^5),
\end{align*}
\]

there result the following expressions:
\[
\begin{align*}
U_3 &= hf_x + k_3 f_v \\
&= U_3 + (1/4)T f_v h^3 + (1/16)f_v(f_v T^2 + 2ST)h^4 \\
U_4^2 &= U_4^2 + (1/2)ST f_v h^4 \\
U_4^3 &= U_4^3, \quad U_4^4 = U_4^4.
\end{align*}
\]

Thus it follows from (11) that
\[
\begin{align*}
k_4 &= k_1 + Th^2 + k_43 h^3 + k_44 h^4 + k_45 h^5 + O(h^6),
\end{align*}
\]

with
\[
\begin{align*}
k_{43} &= \frac{(1/2)(Tf_v + T^2)}{2} \\
k_{44} &= \frac{(1/6)(6k_{33} f_v + 3ST + T^2)}{3} \\
k_{45} &= \frac{(1/24)(24(k_{34} f_v + Sk_{33}) + 3Pf_{vv} + 6TS^2 + T^4)}{12}.
\end{align*}
\]

In the Runge-Kutta expression (3) the coefficient \(C_6\) of \(h^6\) is thus found to be
\[
\begin{align*}
C_6 &= \frac{(k_{16} + 2k_{26} + 2k_{36} + k_{46})}{6} \\
&= \frac{(36Pf_{vv} + 60TS^2 + 72f_v ST + 36ST^2 + 12f_v T^2 + 8f_v T^3 + 10T^4)}{1152}.
\end{align*}
\]

It follows that
\[
\begin{align*}
\epsilon &= C_6 - \frac{(y_0 V_120)}{120} \\
&= \frac{(- 127f_v T + 9Pf_{vv} + 3TS^2 + 6ST f_v - 3ST^2 + 3f_v T^2 - 2f_v T^3 + (T^4/2))}{1440},
\end{align*}
\]

or, more explicitly,
\[
\begin{align*}
\epsilon &= \left[ -127f_v T + 3(f_v^2 - S)f_{xx} + 3(2Tf_v + 2f f_v^2 - 2S - f f_{vv})f_{vv} + 3(3P + 2T f f_v + f^2 f_v^2 - f^2 f_{vv})f_{vv} - 2f f_v f_{xxx} + 3(f_v - f f_v)f_{xxx} + 6f f_v f_{vv} + (T + 2f_v^2) f f_{vv} + (T^4/2)\right]\]/1440.
\]

4. **Estimate of the Error.** Let it be assumed, now, that in a certain
region $B(x, y)$ containing $(x_0, y_0)$

$$|f(x, y)| \leq M,$$

$$|f_{x+y}| \leq L^{i+i}/M^{i-1},$$

where $M, L$ are positive constants independent of $x, y$. In that case clearly

$$|T^n| \leq \sum_{j=0}^{n} (\frac{\gamma}{j}) M^j(L^n/M^{i-1}) = M(2L)^n$$

$$|S^n| \leq \sum_{j=0}^{n} (\frac{\gamma}{j}) M^i(L^{n+i}/M^i) = L(2L)^n, \quad |P| \leq (2ML)^2.$$

By (13), then

$$|E| = |eh^k| \leq (73/720)ML^4h^5.$$  

To calculate a suitable value of $L$ one may first compute bounds $L_{i,k}$ for $|f_{x+y}|$, then define quantities

$$L_{i+j} = \max \left\{ \frac{(L_{i+j,0}/M)^{i+j}(i+j)}{i+j}, \frac{(ML_{i+j-2,1})^{i+j}(i+j)}{i+j}, \cdots, \frac{(M^{i+j}L_{0,i+j})^{i+j}}{i+j} \right\},$$

for $i + j < 4$. Then one may put

$$L = \max (L_{1,1}, L_{2,2}, L_{3,3}, L_{4,4}).$$

5. Example. Let us consider the differential equation

$$y' = x + y, \quad y(0) = 0,$$

and restrict $x, y$ to $0 \leq x \leq 0.2, \ 0 \leq y \leq 0.2$. The exact solution is $y = e^x - x - 1$, so that $y(0.2) = 0.02140276$. For $h_1 = 0.2$ Runge-Kutta's method gives $y_1 = 0.02140276$. For $h_2 = 0.1$ it gives $y_2 = 0.02140257$. Now for the above example we may take $M = 1, L = 1$, so that for $h = 0.2$, by formula (15),

$$|E| \approx (1/10)(2/10)^5 = 0.0000032;$$

actually the error is less than 0.000003. According to the expression exhibited by BIEBERBACH

$$|E| < 6MNh^5|N^n - 1|/(N - 1),$$

where $|f| < M, |f_{x+y}| < N/M^{i-1}$, and, further, $hN < 1, aM < b$, where $|x - x_0| < a, |y - y_0| < b$. With $N = 1$ formula (17) leads to

$$|E| < 0.0096,$$

which is considerably larger than the error estimate obtained by (15).

6. Systems of Equations. For a system of differential equations

$$y_i' = f_i(x, y_1, y_2), \quad y_i(x_0) = y_{i0}, \quad i = 1, 2,$$

the Runge-Kutta formulae become:

$$k_{1i} = hf_{i0}, \quad k_{0i} = f_i(x_0, y_{10}, y_{20})$$

$$k_{i2} = hf_{i1}(x_0 + h/2, y_{10} + k_{11}/2, y_{20} + k_{21}/2)$$

$$k_{i3} = hf_{i1}(x_0 + h/2, y_{10} + k_{12}/2, y_{20} + k_{22}/2)$$

$$k_{i4} = hf_{i1}(x_0 + h, y_{10} + k_{13}, y_{20} + k_{23})$$

$$y_i(x_1) = y_{i0} + (1/6)(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4}).$$
Proceeding in the manner described in the foregoing sections for the case of a single differential equation, and making use of the abbreviations

\[ R^n = D^n f_z, \quad \tau^n = D^n g_z, \quad \sigma^n = D^n g_y, \quad \rho^n = D^n g_x, \quad \pi = [\tau]^z \]

one obtains for \( \epsilon_1 = C_{18} = y^*(x_0, y_{10}, y_{20})/120 \) the expression

\[
1440\epsilon_1 = \left( \frac{T^4}{2} - 2(f_x T^3 + f_y T^2) + 3[T^2(f_y^2 + f_z g_y) + \tau^2(f_x f_z + f_y g_y)]
\right.
\]
\[
- 3(ST^3 + R T^2) + 12[S(T f_y + \tau f_x) + R(T g_y + \tau g_x)]
\]
\[
- 6[T(S f_y + R f_x) + \tau(R f_y + \rho f_x)] + 3(TS^2 + \tau R^2)
\]
\[
+ 9(P f_{xy} + 2T \tau f_{xz} + \pi f_{x})
\]
\[
- 12T(f_y^2 + f_z g_y(2f_y + g_y)) + \tau f_y(f_y^2 + f_z g_y + f_y g_y)
\]
\[
- 15f_{yz}(T g + \tau f_y).
\]

Assumptions similar to (14) now permit one to get a bound for \( \epsilon_1 h^k \).

If, namely, near \( (x_0, y_{10}, y_{20}) \),

\[
|f_1| \leq M, \quad |\partial^{p+q+r}f_i/\partial x^p y_1^q y_2^r| \leq L^{p+q+r}/M^{p+q+r-1}
\]

for \( 0 \leq p + q + r \leq 4 \), it is found that

(19) \[ |E_1| \leq (973/720)ML^4 h^6. \]

For the differential equation \( y''' - y = 1, y(0) = 0, y'(0) = 1 \), which is equivalent to the system \( y_1' = y_2, y_2' = 1 + y_1, y_3(0) = 0, y_4(0) = 1 \), the solution is \( y = e^x - 1 \). Thus \( y(0.1) = .1051709 \). With \( h = 0.1 \), R.K.'s method gives \( y(0.1) = .1051707 \), whence \( E = 2 \cdot 10^{-7} \). In the region \( 0 \leq x \leq 0.1, 0 \leq y_1 \leq 0.10, 0 \leq y_2 \leq 1.11 \); above estimate (19) asserts that \( |E_1| \leq 1.5 \cdot 10^{-7} \).

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RECENT MATHEMATICAL TABLES


The tables in this paper are rather special and are used to prove a result of Inkeri\(^1\) that the field \( k(m^4) \) is not Euclidean for

\[ m = 193,241,313,337,457,601. \]

There are six tables corresponding to these values of \( m \). Each gives a complete period of a periodic algorithm showing certain inequalities which establish the non-existence of Euclid’s algorithm in each field considered.
