

at n . Fourth, we draw nc , intersecting ij at p . Fifth, we draw fv and extend it to intersect mo at o . Sixth, we draw eo , intersecting kv at q . Now, $a_{22(1)} = ip$, $a_{23(1)} = kq$, and $\beta_{13.2} = kq/ip$.

We can evaluate $\beta_{13.2}$ graphically by drawing the heavy dotted lines below the segments representing the second equation. These lines are located on the 0-line by the intersection of line cn and a line parallel to the 0-line at a distance 1. The value of $\beta_{13.2}$ is measured by line st .

A "back solution" is shown also for $\beta_{12.3}$. First, draw line rd and extend it to meet the 0-line at u . Second, draw uu' horizontal. Third, draw line tf and extend it to intersect uu' at u' . Fourth, draw su' , intersecting ef at w . The value of $\beta_{12.3}$ is ew .

Whether the computations are carried out arithmetically or graphically, the number of steps in an n -variable problem increases rapidly as n increases about in proportion to n^3 . A diagram for a problem of four or five variables is much larger than Figure 2 and many more points must be located. But, as in the Doolittle method, the entire process is a repetition of the steps taken in Figure 1. And after a little practice it is unnecessary to draw all the lines indicated in Figures 1 and 2. All we need is to locate and mark points of intersection, using a transparent straightedge.

We have found that the reduction of each coefficient takes about five seconds if there are no complications. The complications that arise are caused by a lack of space if two of the initial coefficients are almost equal, causing near parallelism of two lines which must intersect. This can easily be rectified by first going through one of the equations involved and halving each of its coefficients and the constant term.

The scale should be chosen so that there is no length of as long as 8 inches. Generally the result will have accuracy of three significant figures, occasionally less. It should be noticed in the computation that negative coefficients are measured to the left of the 0-line, positive to the right. If more accuracy is needed this process gives an excellent first approximation to use in one of the iterative methods discussed by HOTELLING.¹

The equipment necessary includes a good drawing board, a T-square and a triangle, together with a rule, preferably transparent with several scales. If one uses colored pencils for consecutive reductions of the set of equations one can do at least a 5×5 set without tracing the partial solution to another sheet.

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¹ HAROLD HOTELLING, "Some new methods in matrix calculation," *Annals Math. Stat.*, v. 14, 1943, p. 1-34.

BIBLIOGRAPHY Z-XVII

12. H. T. AVERY, "The design of office machines—such as calculators," *Mech. Engineering*, v. 71, 1949, p. 719-724. Discussion by W. F. BERCK & J. S. BEGGS, v. 72, p. 344-346.

This paper by the chief engineer of the Marchant Calculating Machine Company discusses the engineering questions associated with the mass production of calculating machines and similar devices and presents an

example of the design of such a calculator. The design example includes a method for introducing an addend continuously by means of gears into the machine. The keyboard selects the proper gear ratio but the actual gear shifting is done automatically. There is also a discussion of a carry system based on differentials, the use of differentials to position the wheels of a register and a continuous method of comparing the remainder in division with the divisor in order to control automatic division.

F. J. M.

13. B. S. BENSON, "Description of an automatic data-analyzing machine," *Mech. Engineering*, v. 72, 1950, p. 285-288.

This device has as its input the film record telemetering data from flight tests of airplanes and missiles in bar graph form. In each frame the values of six out of twenty-eight variables are chosen, measured, and plotted. The selection is governed by a counting relay and integrating networks are provided for the measurement by photo cells and calibration of the quantities. The six quantities are distinguished in the graphical recorder by color. The time between frames is used as a reference for checking the counter relays.

F. J. M.

14. R. L. BISPLINGHOFF, T. H. H. PIAN & L. I. LEVY, "A mechanical analyzer for computing transient stresses in airplane structures," *Jn. Appl. Mech.*, v. 17, 1950, p. 310-314.

If the point of suspension of a torsion pendulum is rotated in accordance with a function $\psi(t)$, the rotation θ of the pendulum satisfies the differential equation $\ddot{\theta} + \omega^2(\theta - \psi) = 0$ or $\ddot{\theta} + \omega^2\theta = \omega^2\psi(t)$. Thus a torsion pendulum permits an easy realization of this last equation. In the application described here, three such pendulums are set up with the angular output θ measured by wire strain gauge elements which when used in bridge circuits will yield a.c. voltages proportional to the angle of twist. A linear combination of these a.c. voltages may be obtained by using transformers and potentiometers.

F. J. M.

15. M. DOPORTO, "Theory and description of a gradient wind computer," Department of Industry and Commerce, Meteorological Service Geophysical Publications, v. 3, No. 1, 1950, 8 p., Stationery Office, Dublin, Eire.

The author devises a graphical computer based on a rather evident geometrical construction which solves for the real positive roots of certain quadratic equations expressing the meteorological gradient wind velocity. Many meteorologists are of the opinion that other approximations (such as $x = c$ for large $|b|$), or nomograms based on a different gradient wind equation, give the gradient velocity with sufficient accuracy for most purposes, considering the approximations inherent in the wind equations used and the errors existing in the forecasted values of the parameters involved.

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16. S. FIFER, A. S. MENGEL, G. W. EVANS, W. A. MERSMAN, L. BAUER, L. C. MERRILL, W. S. MELAHN, J. L. BURNSIDE, R. N. BRETOI, D. L. MARKUSEN, C. F. COOK, D. S. TEAGUE, R. D. GILPIN, H. H. GOODE, W. A. WHEATLEY, G. G. DEN BROEDER, R. R. BENNETT, A. S. FULTON, H. EHLERS, E. VOGEL, W. A. MCCOOL, P. BROCK, S. SHERMAN, A. H. MILLER, H. MEISSINGER, C. BECK, W. F. RICHMOND, *Symposium on REAC techniques*, Reeves Instrument Corporation, 215 E. 91 Street, New York 28, N. Y., 1951, 148 p.

Since the war, the use of REAC equipment has become widespread in certain fields. A REAC is an electromechanical differential analyzer, manufactured in "packages" consisting of seven feedback amplifiers and associated equipment to permit their use as integrators or summers, potentiometers and four servo units which are used as multipliers, dividers or resolvers. The purpose of "Symposium I" was to coordinate the experience of the many users of REAC equipment for their mutual benefit. (The symposium refers to "40 installations" many of which have more than one package.) The symposium took place on March 15, 16, 1951.

Session I. The first paper given here was by A. S. Mengel and concerns the application of the Rand REAC installation to an unusual type of problem in the calculus of variations concerning the motion of a dynamic system in which one of the end values is to be maximized. This problem has been studied mathematically in previous Rand reports by M. R. HESTENES who showed that its solution depended on the simultaneous solution of the equations of motion of the dynamic system and a set of Euler-Lagrange equations. The REAC equipment can be effectively utilized both to obtain solutions and also to determine the sensitivity of the solutions to various parameters which appear. In practical applications, this last is frequently a very important question.

The second paper of this session by G. W. EVANS is entitled "Some partial differential equations solvable by the REAC." A single first order partial differential equation can be solved by the method of characteristics, which is also applicable to certain linear partial differential equations of the second order in the hyperbolic and parabolic case. Two examples of these last with constant coefficients are given with detailed REAC circuits.

The third paper by W. A. MERSMAN is concerned with the solution of an equation in the form $L(x) = M(y)$ where L and M are linear differential operators with constant coefficients and y has a discontinuity at $t = 0$.

The fourth paper by L. BAUER & S. FIFER discusses the realizations of a polynomial in a complex variable z by either resolvers which generate sine and cosine or by obtaining these functions as the solution of differential equations. Either method is applicable to the REAC and such a realization of the polynomial can be used to locate the roots.

The concluding paper of this session is by L. C. MERRILL and is concerned with the evaluation of multidimensional integrals. These can be evaluated as a properly averaged one-dimensional integral on an ergodic path through the region of integration. The path is generated by sawtooth voltages of incompatible frequencies, and irregular regions of integration in the multidimensional space are handled by suitable on-off circuits. The range of the integration is extended by a direction reversing method.

Session II. W. S. MELAHN describes in the first paper of this session certain additions to the REAC equipment at Rand including an IBM plugboard method of set up, which permits one to retain the set up of an individual problem while the machine is used on a different problem, an operator's console, a digital read out and certain stabilized power supplies. By additional wiring to the parts of components, the Rand installation has attained a much greater flexibility and by additional checking and adjusting techniques a greater accuracy than the usual REAC installation.

The next paper by J. L. BURNSIDE advocates a number of changes in REAC design. One of these is the separation of heat producing non-manipulated components such as power supplies from the rest, a second is the substitution of a switching system for the patch bay set up. He also recommends the addition of certain equipment like high speed multipliers and printing readers for the servo units.

The third paper by R. N. BRETOI & D. L. MARKUSEN is concerned with the use of the REAC in the design of the Autopilot for an airplane. The design of such a device involves a number of stability considerations which are normally considered first in a simplified form by direct calculation and then all factors are studied in full generality on the REAC.

In testing airplane equipment and instruments, it is necessary to consider their performance under conditions simulating flight. This is usually done by mounting these on a rotating mount. C. F. COOK discusses the control of one such mount by REAC equipment and finds a region of stable control.

The next paper was by D. S. TEAGUE & R. D. GILPIN, "Preliminary investigation of the suitability of REAC for experimental curve fitting." The objective here is the determination of certain unknown coefficients in a differential equation, when a solution is given.

The concluding paper of this session by H. H. GOODE, W. A. WHEATLEY and G. G. DEN BROEDER is concerned with the problem of simulating N chance variables with a joint N dimensional distribution with given means and given covariance matrix, by analog equipment. This is accomplished by means of certain transformations on N variables which are the output of N noise generators, whose specifications are given.

F. J. M.

Session III. This session consisted of three papers on noise and frequency analysis and one on the solution of integral equations.

A. S. FULTON and R. R. BENNETT, "The generation and measurement of ultra low frequency random noise." In this article an electronic noise generator suitable for use with an electronic analog computer is described. The type of noise produced has a normal amplitude distribution. Its frequency spectrum is essentially flat at low frequencies with a half power point at 25 c.p.s. The mean value is practically zero and is obtained by passing the noise through a high pass filter. The frequency spectrum can be altered by passing the noise through various types of filters and the amplitude distribution can even be changed by suitable combinations of linear and non-linear filters. The (residual) mean value of the noise is obtained by integrating the voltage for a sufficiently long length of time. The mean square value can be determined essentially by half-wave recti-

fyng the noise and then integrating. The spectrum of a particular noise can, of course, be computed from its autocorrelation function. In simulator work it is often simpler to pass the noise through various sharp band pass filters and measure the transmitted power.

H. EHLERS & E. VOGEL, "Simulation of noise in missile homing problems." (Only an abstract of this paper appears.) It is said to contain a description of the noise generator and the shaping of the output to simulate the noise as recorded in flight.

W. A. McCOOL, "Frequency analysis by electronic analog methods." In this paper an electronic analog computer technique for evaluating Fourier transforms is described. The method is to consider the complex Fourier series expansion of period T of an arbitrary function $g(t)$, $g(t) = \sum_{-\infty}^{\infty} y_n(t) = \sum_{-\infty}^{\infty} F(\omega_n) \exp(j\omega_n t)$. For large values of T it is reasonable to assume $F(\omega_n)$ is a good approximation to the Fourier transform of $f(t)$. If, in the above equation, $F(\omega_n)$ is replaced by the Fourier transform $F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) dt$, there results, on dropping subscripts, $y(t) = \int_0^t f(\tau) \exp(j\omega(t - \tau)) d\tau$. From this result, a linear differential equation on $y(t)$ (which can be solved by analog techniques) is derived. The method also applies to the evaluation of the inverse Fourier transform.

P. BROCK and S. SHERMAN, "Applications of REAC equipment to the solution of problems involving integral operators." In this paper practical methods (successfully used on REAC equipment) are given for the solution of typical problems involving integral operators. The method is by iteration using the "point-plot method." For example, the following problem was solved: $\varphi(s) = s + 2 \int_0^1 st\varphi(t) dt$ where $\varphi_0(s) = s$ and (*) $\varphi_n(s) = s + 2 \int_0^1 st\varphi_{n-1}(t) dt$. $\varphi_n(s)$ is determined from (*) using the previously determined φ_{n-1} curve as a function of t . To solve (*), $\varphi_n(s)$ is determined as a point function $[\varphi_n(s_i)]$ for a sequence of constant s values $[s_i]$. The REAC solves $\varphi_n(s_i, t) = s_i + 2 \int_0^1 s_i t \varphi_{n-1}(t) dt$ for each s_i . Note that $\varphi_n(s_i, 1) = \varphi_n(s_i)$. This function $\varphi_n(s_i)$ is plotted for each value of s_i on an output table.

Session IV. This session on numerical checks was devoted to a discussion by many of the members of the Symposium (under the chairmanship of P. Brock) of the general problem of determining the accuracy of REAC solutions. No papers were presented. Most of the remarks were of a qualitative nature.

Session V. This session consisted of five papers devoted to particular techniques used on the REAC for solving various special problems.

A. H. MILLER, "A REAC solution of a linear proton accelerator design problem." The mathematical problem considered in this paper is that of solving a number of systems of equations. Each system is composed of one linear and one non-linear differential equation. The initial conditions for the n th system are determined by the solutions of the $(n - 1)$ st system. The problem that arises in solving these equations on the REAC is that seven servos (necessary to generate $1/n$, cotangents, tangents, etc.) are needed while only four are available. By considering the numerical range of the variables involved, certain approximations and other circuit arrangements can be used. For example, $1/n$ can be generated by solving the differential equation, $dy/dn = -y^2/100$.

H. MEISSINGER, "Representation of functions of several variables on REAC equipment." One can represent a function of two variables by using

a three dimensional cam or a multi-wire pick-off device. Neither of these methods is particularly feasible. A method is described where certain functions can be broken up into sums and products of functions of a single variable. For example, it is shown that a certain function $z(x, y)$ which occurred in a recent problem on guided missile trajectory work could be approximated by $z(x, y) = f_3(x) - f_2(x)f_1(y)$. Approximate analytic expressions were determined for f_1 , f_2 and f_3 and the result set up on the REAC.

C. BECK, "A method for solving problems on the REAC by use of transfer functions without passive networks." Some problems are stated directly in terms of the transfer function $H(s)$ of a linear system, that is, (*) $x_0(s) = H(s)x_i(s)$ where s is the Laplace transform variable and $H(s)$ is a rational function of s . A method is described whereby (*) can be solved without resorting to the original differential equation. The case where the system is not originally in the relaxed state (zero initial conditions) is also considered.

W. F. RICHMOND, "REAC techniques." Various techniques for treating continuous linear functions, continuous non-linear functions, discontinuous linear functions or linear functions with discontinuous derivatives, discontinuous non-linear functions or non-linear functions with discontinuous derivatives are discussed. Numerous circuit diagrams are included.

R. R. BENNETT, "The generation of straight line transfer relationships." The use of relay (sampling, clamping) amplifiers in generating straight line segments is described. Such techniques are necessary in the solution of non-linear problems on electronic analog computers.

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17. G. D. McCANN & R. H. MACNEAL, "Beam vibration analysis with electric-analog computer," *Jn. Appl. Mech.*, v. 17, 1950, p. 13-26.

Four systems of partial differential equations in two independent variables x and t are solved by an analog method in which x is permitted to have only a small number of values so that the partial differential equations become an ordinary system on the variable t , which can be realized by an electric analog [*MTAC*, v. 3, p. 501-513]. The systems treated are linear of the second and fourth order on one, two or three unknown functions and the coefficients may depend on the variable x . The validity of the results also depends on an analogy: results from the analog are compared with analytic solutions in certain simple cases.

F. J. M.

18. A. D. MOORE, "Soap film and sandbed mapper techniques," *Jn. Appl. Mech.*, v. 17, 1950, p. 291-298.

19. MULTIX *Multiplying Sticks*. Larchmont, N. Y. \$1.00.

This set consists of ten cardboard strips and two mimeographed pages of instruction. The strips are similar to the well known "Napier bones" for multiplying a number of any size by a one digit number. For each possible digit of the multiplicand a strip is given. When these are suitably arranged the product can be read off. However in these "Multix," the same strip can

be used for either 5 in the multiplicand or 55 or 555 and, consequently, only ten strips are needed. The price seems high and many people will find multiplication tables more useful, even if less intriguing.

F. J. M.

20. D. W. PENDERY, "604 electronic calculator control panels for multiply, divide, and square root on the card-programmed electronic calculator," *IBM Tech. Newsletter*, No. 1, 1950, p. 4-8.

W. BELL, "The 'DUZ' general-purpose control panel for the IBM type 604, electronic calculator," *IBM Tech. Newsletter*, No. 1, 1950, p. 9-11.

D. B. MACMILLAN & R. H. STARK, "A general-purpose 604 electronic calculator control panel for the card-programmed electronic calculator," *IBM Tech. Newsletter*, No. 1, 1950, p. 12-18.

B. OLDFIELD, "604 electronic calculator diagrams for the calculation of $\sin x$, $\cos x$, e^x , e^{-x^2} , $\sinh x$, and $\cosh x$ on the card-programmed electronic calculator," *IBM Tech. Newsletter*, No. 1, 1950, p. 19-25.

B. G. OLDFIELD, "The calculation of variance, covariance, and averages," *IBM Tech. Newsletter*, No. 2, 1951, p. 2-6.

B. G. OLDFIELD, "Numerical integration by Simpson's rule," *IBM Tech. Newsletter*, No. 2, 1951, p. 7-12.

W. D. BELL & AMY MCAVORY, "Graphical aids for determining shift codes," *IBM Tech. Newsletter*, No. 2, 1951, p. 13-15.

D. B. MACMILLAN & R. H. STARK, "'Floating decimal' calculations on the IBM card-programmed electronic calculator," *IBM Tech. Newsletter*, No. 2, 1951, p. 16-26.

The introduction to *Technical Newsletter*, No. 1, reads as follows:

"Technical Newsletter Number 1 is the first of a series to be published by the Applied Science Department of the International Business Machines Corporation. The purpose of these Newsletters is to allow rapid interchange of information between computers concerning computational methods on IBM punched-card equipment.

Technical Newsletter Number 1 describes general purpose control panel diagrams which have been used successfully. Three of these diagrams are for the IBM Card-Programmed Electronic Calculator. The fourth diagram is for the IBM Type 604 Electronic Calculator."

In both numbers, the solution to the stated problems is given without mathematical discussion in the technical language of the IBM manuals. The "shift codes" referred to in the title of the BELL & MCAVORY paper deal with the position of the decimal point.

F. J. M.

21. H. PORITSKY, B. E. SELLS & C. E. DANFORTH, "Graphical, mechanical and electrical aids for compressible fluid flow," *Jn. Appl. Mech.*, v. 17, 1950, p. 37-46.

The flows studied in this paper are two dimensional with no vorticity and no viscosity. Consequently, they can be described by two potential functions, φ and ψ , by plotting the curves $\varphi = C_1$ and $\psi = C_2$ for equally

spaced values of C_1 and C_2 . φ is the velocity potential and ψ is the streamline function and thus the two sets of curves intersect at right angles. The plot divides the given area into curvilinear rectangles, which in the case of incompressible flows are equisided. For the compressible case, Bernoulli's equation gives a relation between the sides of these curvilinear rectangles. The authors give a historical discussion of graphical methods in this problem and describe, with examples, a process of starting with the incompressible case and arriving at the desired result. For this purpose a number of mechanical aids are described. A four legged caliper is described which will give the proper size for the other side of the rectangle, when one side is given and a cam device for the same purpose is described. To permit a ready modification of the plot, the latter is realized by wires and pegs which have two holes bored at right angles through their sides. Lord Rayleigh has pointed out that the incompressible case can be plotted very effectively by using a double mesh of wires. For this purpose each peg has an additional pair of perpendicular holes making an angle of 45° with the first pair and the double mesh involves both the sides and the diagonals of the smaller squares. The authors also discuss the representation of the equations of fluid flow by means of a resistance network. Here the resistances are to be adjusted until the proper relation between potential and current flow is obtained.

F. J. M.

22. W. W. SOROKA, "Experimental aids in engineering design analysis," *Mech. Engineering*, v. 71, 1949, p. 831-837.

This is a survey article with a bibliography, covering certain analogue techniques for the solutions of partial and ordinary differential equations and algebraic equations.

23. M. A. WOODBURY, "Inverting modified matrices," Princeton Univ., Statistical Research Group, *Memorandum Report 42*, Princeton, New Jersey. 4 hectographed leaves.

This is a resumé, including the work of SHERMAN and MORRISON, of formulas for inverting matrices which have been obtained from a matrix with a known inverse by bordering or by modifying either an individual element or the elements in one row or column or both or in a number of rows and columns. "None of the proofs are difficult but considerable exploration was necessary to discover useful forms."

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NOTES

128. GAUSS TO GERLING ON RELAXATION.—In his recent book⁷ on matrices ZURMÜHL traces the relaxation method² of solving linear equations back to DEDEKIND's report of GAUSS's lectures.⁷ It is believed by some computers, however, that Gauss's method is a different one—namely, the related method given by SEIDEL.³ In the interest of giving Gauss his proper credit as a proponent of relaxation, the following translation of a letter by Gauss⁴ is offered. Moreover, in this same letter Gauss introduces a useful trick,⁷ now generally forgotten, which seems to improve the convergence