very briefly the computer used. No reference is made to the large amount of literature on the paper.

As far as computing technique goes the main point of interest is an electronic circuit used to represent the boundary conductance which is not constant but rather a function of temperature. This nonlinear condition has heretofore on other computers been represented in discrete finite steps. The author shows in figure 8 a circuit for representation of such boundary conductance. The check of the computations with actual experiments is rather unsatisfactory possibly because of poor assumption of physical constants of the system; the correction of assumptions determines in analog computers the validity of the result. The author, in designing the computing circuit, disregards a number of influences (for example the thermal resistance of the ice).

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The paper deals with the deicing of a propeller of ovoid cross section. The butt end of this ovoid will be called A, the extension, B. Heat is generated by electric heaters only in the region A and is conducted to the region B. A cross section through the propeller in the region A shows (proceeding inwards) the steel blade (extending all the way to B), a layer of nylon, the heater, another layer of nylon, and finally sponge rubber. Disregarding the heat flow along the propeller length, the problem is two dimensional.

The greatest problem in solving two dimensional problems on analog computers, one on which little information is available, is that of how to section the body in which heat flow occurs. The authors disregard the thermal resistance across the thickness of the steel, and along the length of the nylon and heater layers. The sponge rubber is represented by a number of parallel sections, ending in a fictitious center with zero volume (Fig. 3 of the paper). This design is not discussed or analyzed; thus, the most crucial problem, from a computational view-point, is not dealt with in the paper. Regarding deicing, the paper shows the desirability of using high rates of energy production, which results in a lower total heat consumption than heating at a lower rate.

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NOTES

143.—Analytical Approximations. [Editorial Note: In Note 139, MTAC, v. 6, p. 251–253, Cecil Hastings has described the Rand Collection of Illustrative Approximations. The interest that these approximations have aroused during the past year is considerable. It is hoped to publish as Notes
from time to time additional examples of such approximations contributed by our readers. To encourage this hope, Mr. Hastings is submitting a dozen new examples, prepared with the assistance of Mr. James P. Wong and Mrs. David K. Hayward. These differ from the RAND Collection in form, especially since they do not give illustrative error curves. For convenience in future references we are numbering these approximations consecutively.]

(1) Square Root: To better than 1 part in 12 over .1 ≤ x ≤ 10,
    \[ x^\frac{1}{2} = (1 + 4x)/(4 + x). \]

(2) Pearson Cosine Transformation: To .003 over 0 ≤ x ≤ 1,
    \[ r(x) = \cos \left( \pi / (1 + x^4) \right) = (-1 - 4x + 5x^2)/(1 + 8x + 6x^2). \]
    \[ r(x^{-1}) = -r(x) \] can be used to obtain function values over 1 ≤ x ≤ ∞.

(3) Common Logarithmic Function: To better than .005 over .1 ≤ x ≤ 1,
    \[ \log x = - .076 + .281 x - .238/(x + .15). \]
    This approximation is the result of a request for a very simple formula to use in the reduction of certain data.

(4) Common Logarithmic Function: To better than .000,004 over 1 ≤ x ≤ 10,
    \[ \log x = \frac{1}{2} + .86857y + .29059y^3 + .15783y^5 + .20269y^7, \]
    where \( y = (x - \sqrt{10})/(x + \sqrt{10}). \)

(5) Common Logarithmic Function: To better than .000,000,015 over 1 ≤ x ≤ 10,
    \[ \log x = \frac{1}{2} + .8685888y + .2895497y^3 + .1731159y^5 + .1314381y^7 + .0547562y^9 + .1832415y^{11}, \]
    where \( y = (x - \sqrt{10})/(x + \sqrt{10}). \)

(6) Inverse Tangent: To better than .005 over -1 ≤ x ≤ 1,
    \[ \arctan x = x/(1 + .28x^2). \]
    This approximation is the result of a request for a very simple formula to use in the reduction of certain data.

(7) Descending Exponential Function: To better than .000,000,11 over 0 ≤ x ≤ ∞,
    \[ e^{-x} = (1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5)^{-8}, \]
    where \( a_1 = .125,000,204, \ a_2 = .007,811,604, \ a_3 = .000,326,627, \ a_4 = .000,009,652 \) and \( a_5 = .000,000,351. \)

(8) Incomplete Gamma Function Type Integral: To better than .000,000,1 over 0 ≤ x ≤ 1,
    \[ F(x) = \int_1^\infty e^{-t}t^{-x-1}dt \]
    \[ \approx [.219384 + x(0.024717 + .000803x)]/[1 + x(.558651 + .090584x)]. \]
    \( F(x) + (x + 1)F(x + 1) = e^{-1} \) can be used to obtain function values outside the range indicated.

(9) Exponential Integral of Negative Argument: To better than .000,000,1 over 10 ≤ x ≤ ∞,
    \[ xe^x \int_x^\infty t^{-1}e^{-t}dt \]
    \[ \approx [1.15198 + x(4.03640 + x)]/[4.19160 + x(5.03637 + x)]. \]
(10) Segmental Area Function: To better than .0012 over \(-1 \leq x \leq 1,
A(x) = \int_{-x}^{x} (1 - t^2)^{\frac{1}{2}} dt \approx 2.0083x - 0.4160x^3 + 0.1604x^5 - 0.1808x^7.

In terms of elementary functions, \(A(x) = \arcsin x + x(1 - x^2)^{\frac{1}{2}}\).

(11) Segmental Area Function: To better than .00016 over \(-1 < x < 1,
A(x) = \int_{-x}^{x} (1 - t^2)^{\frac{1}{2}} dt \\
\approx (1.99916x - 2.39484x^3 + 0.58673x^5)/(1 - 1.03472x^2 + 0.15634x^4).

(12) Segmental Area Function: To better than .000,016 over \(-1 \leq x \leq 1,
A(x) = \int_{-x}^{x} (1 - t^2)^{\frac{1}{2}} dt \\
\approx x(1.999872 + 4.143151\eta - 3.153670\eta^2 - 1.430807\eta^3)\\
(1 + 2.901498\eta - 1.811287\eta^2 - 1.098016\eta^3),

where \(\eta = x/(5 - 4x^2)\).

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144.—Zeros of the Derivative of Bessel Functions of Fractional Order. The NBS Computation Laboratory\(^1\) has published extensive tables of Bessel functions of fractional order, \(J_\nu(x)\), \(\pm \nu = \frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}\), and zeros of \(J_\nu(x)\) have been tabulated by Abramowitz\(^2\) and by the Computation Laboratory when the latter was known as the Mathematical Tables Project.\(^3\) (The zeros in the last two sources are also given in the first, p. 384–385.) This present note gives the first seven or eight positive zeros of the first derivative of those Bessel functions. Following standard notation, \(j'_\nu, s\) will be used to denote the \(s\)-th positive root of \(J'_\nu(x) = 0\). The zeros \(j'_\nu, s\) were obtained from the published values\(^1\) of \(J'_\nu(x)\), and this table includes all such zeros within the range of tabulation of \(J'_\nu(x)\) itself (i.e., not exceeding 25).

These zeros were computed to the maximum accuracy obtainable from the NBS tables. All entries are given here to seven decimal places; although the seventh decimal place is not absolutely guaranteed, it has a high probability of being correct. The zeros were computed using the 5-point case of two different formulas for inverse interpolation for the derivative, given in Salzer\(^4\) on p. 214 and p. 215. They were also checked by (a) calculating an approximate expression for the error in \(j'_\nu, s\), (b) recomputing the last \(j'_\nu, s\) given here by the asymptotic formulas for each \(\nu\) (see below), and (c) computing the seventh divided difference of \(j'_\nu, s\) as a function of \(\nu\), using a formula in Salzer.\(^6\) (This divided difference check was not fully applicable to the first few zeros.)

For zeros > 25, the following asymptotic formulas, whose coefficients were calculated from the general expression for \(j'_\nu, s\) in Watson,\(^7\) will give at
least seven decimal accuracy:

\[ \pm \nu = \frac{3}{4}, \quad j_{n,s} = y - .65625000y^{-1} - .54931641y^{-3} \]

where \( y = \pi s - \left\{ \begin{array}{l}
.39269908 \quad \nu = -\frac{3}{4} \\
1.17809724 \quad \nu = \frac{3}{4}
\end{array} \right. \)

\[ \pm \nu = \frac{5}{6}, \quad j_{n,s} = y - .59722222y^{-1} - .41380530y^{-3} \]

where \( y = \pi s - \left\{ \begin{array}{l}
.26179939 \quad \nu = -\frac{5}{6} \\
1.30899694 \quad \nu = \frac{5}{6}
\end{array} \right. \)

\[ \pm \nu = \frac{7}{8}, \quad j_{n,s} = y - .43055556y^{-1} - .07507073y^{-3} \]

where \( y = \pi s + \left\{ \begin{array}{l}
.39269908 \quad \nu = -\frac{7}{8} \\
( -1.96349540) \quad \nu = \frac{7}{8}
\end{array} \right. \)

To obtain \( j_{n,s} \) for any other \( \nu \) that is less than one in absolute value, the present table may be used in conjunction with a special table of interpolation coefficients,\(^1\) p. 393–413. The user is cautioned that for interpolation as well as for forming divided differences, the values of \( j_{n,s} \) for \( \nu < 0 \) are not continued into \( j_{n,s} \) for \( \nu > 0 \), but into \( j_{n,s+1} \).

Mrs. Ruth E. Capuano and Miss Mary M. Dunlap assisted in the computations.

Herbert E. Salzer

NBSCL

Table of \( j_{n,s} \)

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145.—AN EXAMPLE IN THE USE OF THE DIFFERENTIAL ANALYSER. In a recent article Sprague\(^1\) has discussed, as an example, a differential analyser setup for solving the equation

\[ w'' - ww' - wt = 0, \]

Figure 1

[Diagram showing a differential analyser setup]

primes denoting differentiation with respect to \( t \). There are several simpler ways of setting up this equation. One is to take the once-integrated form

\[ w' = \int_0^t wd(w + \frac{1}{2}t^2) + w'(0) \]

and so place it on the analyser in accordance with Figure 1. This uses three integrators in lieu of the six shown in Figure 3 of Sprague's paper.

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CORRIGENDA

[Editorial note: Mr. Sprague informs us that an additional integrator will be required in case the differential analyzer is of digital type, thus the above method would require four integrators.]


146.—Two New Mersenne Primes. The program described in Notes 131(c) and 138 [MTAC, v. 6, p. 61, 204] has been continued. Two more Mersenne primes, $2^{2203} - 1$ and $2^{2281} - 1$, were discovered by the SWAC on October 7 and 9, 1952. The time required for either of the tests is one hour. This makes 17 Mersenne primes, and a corresponding number of perfect numbers, now known. They are $2^n - 1$ for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203,$ and $2281$.

D. H. L.

CORRIGENDA

v. 6, p. 129, l. 22 for $r_i(1 - \rho_{i-1})$ read $r_i(1 + \rho_{i+1})$

v. 6, p. 132, l. — 6 and — 18 for Thompson read Thomson

v. 6, p. 152, l. — 5 for 8 read 9

v. 6, p. 187, l. 12 for QVAC read QUAC

v. 6, p. 189, l. — 8 for Connoly read Connolly