Coupon Collector's Test for Random Digits

1. Introduction. Increasing use of random numbers, especially in Monte Carlo procedures and in large computing installations, has served to focus attention on the various tests for randomness. Kendall and Babington-Smith\(^1\) list four tests for so-called local randomness. While not giving the coupon collector's test (to be described below) a place in their now classical list of four tests, they did use a modified coupon collector's test in some of their investigations.

In an ordered set of digits, say, one may count the length of a sequence, beginning at a specified position, necessary to give or include the complete set of all ten digits. Or one may count the length required to give a set of \(k\), \(k < 10\), different digits. The distribution of these observed lengths for different initial positions can then be compared with a theoretically computed distribution. Such a test will be called the coupon collector's test from an analogy with certain sales promotion schemes.

The theoretical distribution may be computed from formulas given by H. von Schelling,\(^2\) which formulas hold for the case where the individual category probabilities might be unequal. For the random digital case with category probabilities equal to 1/10, von Schelling's formulas simplify readily and may be conveniently related to the "differences of zero." These latter quantities are tabulated by Fisher and Yates\(^3\) up to sequences of length 26. But in using the coupon collector's test for a complete set of all 10 digits it has been found that the mean of the length distribution is slightly greater than 29, and a table of probabilities associated with the sequence lengths 10 to 26 inclusive would hardly give a realistic picture of the entire distribution.

The present author has therefore extended this tabulation, and exact probabilities are given for sequence lengths 10 \(\leq n \leq 35\) and approximate probabilities for sequence lengths 36 \(\leq n \leq 75\). The probabilities were computed from the relation

\[
p_n = \frac{1}{10^{n-1}} \sum_{j=0}^{9} (-1)^j \binom{q}{j} (q - j)^{n-1},
\]

and are listed in Table 3.

If one were interested in sequence lengths necessary to obtain five different digits, the mean of this distribution is approximately 6.46. The range 5 to 26 inclusive available from Fisher and Yates\(^3\) might be sufficient here.

2. Sequence Lengths for Decimal Expansions of \(\pi\) and \(e\). It would be a simple matter to program a large digital computing machine so that it would tabulate the distribution of the sequence lengths needed for complete sets for a given ordered digital collection. However, the author did not have such a digital computing machine available, and he made a tabulation by hand for the decimal expansion of \(\pi\). The 2035 decimal approximation to \(\pi\) given by George W. Reitwiesner\(^4\) was used as the raw material for this count. Beginning with the initial position \(3\) in \(\pi \simeq 3.14159\cdots\), it was recorded that a sequence length of 33
positions was needed to get all the ten digits. Beginning anew with the thirty-fourth position digit (which is a 2), it was recorded that a sequence of 18 positions was needed to get a complete set of all the 10 digits. Continuing this procedure, 67 sequences of complete sets were obtained, plus an incomplete sequence (at the end of the decimal expansion) of length 15. It was considered advisable to make the sequences non-overlapping as described above since there is considerable dependence among the set of sequence lengths if every position in the decimal expansion of π is regarded as a new starting point.

The sequence lengths for π are also included in Table 3. This tabulation for π was checked by Mr. Wayne Jones of the Department of Defense, Washington, D. C.

Mr. Jones also made a tabulation based on the decimal expansion of e. Reitwiesner\textsuperscript{4} gave a 2010 decimal approximation to e. An additional 490 places was given by Metropolis, Reitwiesner and von Neumann.\textsuperscript{5} Mr. Jones found 82 complete sequences using 2486 digits in the expansion of e. This tabulation is also given in Table 3. The author desires to thank Mr. Jones for this count.

3. Statistical Tests. The mean and the standard deviation of the theoretical distribution may be computed from results given by von Schelling\textsuperscript{2} or Feller.\textsuperscript{6} These theoretical values and the corresponding observed values for π and e are given below.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Observed</th>
<th>Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>29.29</td>
<td>30.16</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>11.21</td>
<td>11.83</td>
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<tr>
<td></td>
<td></td>
<td>10.64</td>
</tr>
</tbody>
</table>

To use a chi-square test, it is desirable that the expected values all exceed 10 in size. Since the sample size for π is small (67) some grouping of the sequence lengths is necessary to meet this desired minimum. The following results were obtained for a convenient grouping.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>π</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed</td>
<td>Expected</td>
<td>Observed</td>
</tr>
<tr>
<td>Sequence lengths, n</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20–23</td>
<td>13</td>
<td>11.720</td>
</tr>
<tr>
<td>24–27</td>
<td>9</td>
<td>11.491</td>
</tr>
<tr>
<td>28–32</td>
<td>5</td>
<td>11.480</td>
</tr>
<tr>
<td>33–39</td>
<td>13</td>
<td>10.195</td>
</tr>
<tr>
<td>40 and over</td>
<td>14</td>
<td>10.510</td>
</tr>
<tr>
<td>Totals</td>
<td>67</td>
<td>67.000</td>
</tr>
<tr>
<td>Chi-squared test values</td>
<td>6.436</td>
<td>2.826</td>
</tr>
</tbody>
</table>

Neither of these chi-square test values is unusually out of line. It has been previously reported\textsuperscript{4,7} that (using a sample of 2000 digits for e) excessive flatness in the single frequencies was noted, and an indication was obtained that the single digits in e are "non-random."\textsuperscript{5} Apparently, this phenomenon did not reflect itself
Table 3

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p_n )</th>
<th>( \pi )</th>
<th>( e )</th>
</tr>
</thead>
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<tr>
<td>10</td>
<td>.0003 6288</td>
<td>0</td>
<td>0</td>
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<tr>
<td>11</td>
<td>.0016 3296</td>
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<td>0</td>
</tr>
<tr>
<td>12</td>
<td>.0041 9126 4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>.0080 9315 2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>.0130 4560 8576</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>.0186 3435 9744</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>.0243 5958 6451 2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>.0297 8461 8864</td>
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<td>2</td>
</tr>
<tr>
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<td>2</td>
</tr>
<tr>
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</tr>
<tr>
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<td>.0415 3577 5577 4998 4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>21</td>
<td>.0435 8654 2461 1780 8</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>22</td>
<td>.0447 3311 6259 6932 2752</td>
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<td>3</td>
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<td>.0450 6836 4358 6388 8896</td>
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<td>24</td>
<td>.0447 0706 5704 2485 9072</td>
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<td>1</td>
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</tr>
<tr>
<td>27</td>
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<td>3</td>
<td>5</td>
</tr>
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<tr>
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</tr>
<tr>
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<td>2</td>
</tr>
<tr>
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<td>2</td>
</tr>
<tr>
<td>35</td>
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<td>3</td>
</tr>
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<td>3</td>
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<td>1</td>
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<td>1</td>
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<td>43</td>
<td>.0112 1807 0507 6953 1223</td>
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</table>
Table 3—Continued

<table>
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<th>n</th>
<th>p_n</th>
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<th>( e )</th>
</tr>
</thead>
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</tr>
<tr>
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<td>.0045</td>
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<td>0</td>
</tr>
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<td>.0040</td>
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</tr>
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<td>.0036</td>
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</tr>
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<td>.0033</td>
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<td>0</td>
</tr>
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</tr>
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<td>0</td>
</tr>
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<td>0</td>
</tr>
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</tr>
<tr>
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<td>.0019</td>
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</tr>
<tr>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
</tr>
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<td>.0010</td>
<td>5660 0172 2241 9129</td>
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</tr>
<tr>
<td>69</td>
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<td>7124 1073 6049 1625</td>
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</tr>
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<td>.0006</td>
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</tr>
<tr>
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</tr>
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<td>72</td>
<td>.0005</td>
<td>6273 6471 7289 2795</td>
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<td>1</td>
</tr>
<tr>
<td>73</td>
<td>.0005</td>
<td>0658 1228 5628 1531</td>
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<td>0</td>
</tr>
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<td>74</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>.0036</td>
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<td>(77) 0</td>
</tr>
<tr>
<td>and over</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Total 67 82

in materially changing the characteristics of the sequence length distribution for the coupon collector’s test. Some question arises as to whether the single frequency test and the coupon collector’s test are independent, and also which test has the greater power.

The chi-square test values in Table 2 were calculated by assuming that the sequence lengths for complete sets of digits are independent draws from a known (infinite) multinomial probability distribution. (Null hypothesis.) The alternatives would include unspecified sorts of dependency and other underlying probabilities different from those given in Table 3.

ROBERT E. GREENWOOD

The University of Texas
Austin, Texas


A Method for the Evaluation of a System of Boolean Algebraic Equations

With the advent of large scale electronic devices whose logical design is described by a system of Boolean algebraic equations, a method to mechanize the evaluation of such a system and shorten this evaluation with respect to time will be increasingly useful. Such a method will be described in this paper.

The problem may be described as follows: Given a set of n variables, Q_k, (k = 1, 2, ..., n) each of which may take on the value 1 (true) or 0 (false) at any time t; then the value of any Q_k at time t + 1 may be defined by the system of Boolean equations

\[
\begin{align*}
\text{(1)} & \quad R_t^k = f_k(Q_t^k) \\
\text{(2)} & \quad S_t^k = g_k(Q_t^k) \\
\text{(3)} & \quad Q_{t+1}^k = \phi(Q_t^k, R_t^k, S_t^k)
\end{align*}
\]

where 1 ≤ q ≤ n. For example, the recirculation loop of a dynamic flip-flop may be defined simply by

\[Q_{t+1}^k = \phi(Q_t^k, R_t^k, S_t^k) = R_t^k.\]

In another system, a more complex definition

\[Q_{t+1}^k = \phi(Q_t^k, R_t^k, S_t^k) = Q_t^k \cdot \overline{R_t^k} \cdot \overline{S_t^k} + R_t^k \cdot \overline{S_t^k} + \overline{Q_t^k} \cdot R_t^k \cdot S_t^k\]

may be taken, where \(R_t^k\) and \(S_t^k\) are the two inputs to flip-flop \(Q_t^k\).

We shall use the symbols for conjunction, disjunction, and negation

\[
\begin{align*}
Q \cdot Q & = \text{"}Q\text{" and } Q\text{"} \\
Q + Q & = \text{"}Q\text{" or } Q\text{"} \\
\overline{Q} & = \text{"}Not Q\text{"}
\end{align*}
\]

which are defined by the truth tables\(^1\)

<table>
<thead>
<tr>
<th>Q</th>
<th>(\overline{Q})</th>
<th>(Q \cdot Q)</th>
<th>(Q + Q)</th>
<th>(\overline{Q})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
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<tr>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

A "term" is defined as one or more variables conjoined together, e.g., \(Q_1 \cdot Q_2 \cdot \overline{Q_3}\) and an "equation" as \(M\) terms, \(T_m\), (\(m = 1, 2, \ldots, M\)) disjoined together, e.g., \(Q_1 \cdot Q_2 \cdot \overline{Q_3} + Q_4 \cdot Q_5\). Now note that the value of a term is zero if any variable in