

unit errors occur in 121 of the entries of the 1624 volume. In neither table did we make any checking of difference entries.

The idea of constructing a table in which the logarithm of unity was zero originated with Napier. Napier and Briggs never thought of logarithms as exponents of a base. An excellent exposition of their ideas is given in G. A. GIBSON, "Napier's logarithms and the change to Briggs's logarithms," p. 111-137 of C. G. KNOTT, *Napier Tercentenary Memorial Volume*, London, 1915; see also, H. S. CARSLAW, "The discovery of logarithms by Napier," *Mathematical Gazette*, v. 8, 1915, p. 76-84, 115-119. It was not till considerably later that our modern definition of a logarithm as an exponent was put forward by such mathematicians as DAVID GREGORY, 1684; WM. GARDINER, 1742; LEONARD EULER, 1748, 1770.

R. C. ARCHIBALD

Brown University
Providence, R. I.

**On the Numerical Integration of Functions Tabulated
in Logarithmic Form***

In a number of physical problems which can be described by differential equations, it occurs that one or more of the variables show a range of several orders of magnitude. In such cases, it is convenient to tabulate the variables in logarithmic form. The purpose of this note is to show that the usual finite-difference formulae for numerical integration can be easily adapted to integrate a function when only its logarithm is given.

We shall limit our attention to the Newton-Gregory (backward-difference) formula, which is the most commonly used in the hand-integration of differential equations. Let y' be the function to be integrated with respect to the independent variable x between the limits $x_0 - h$ and x_0 , where h is the interval of tabulation. Writing $x = x_0 + hm$, we can approximate $\ln y'$ by the "Newton-backward" interpolating polynomial $f(m)$ as follows:

$$(1) \ln y' = f(m) = f_0 + \binom{m}{1} \Delta'_{-1} + \binom{m+1}{2} \Delta''_{-1} + \dots$$

$$= f_0 + \sum_{j=1}^n \binom{m+j-1}{j} \Delta^{(j)}_{-1/2} + \text{truncation error.}$$

$(j = 1, 2, 3, \dots, n).$

We now want to obtain an integration formula of the type

$$(2) \ln \left(\frac{1}{h} \int_{x_0-h}^{x_0} y' dx \right) = \ln \int_{-1}^0 \exp f(m) dm$$

$$= f_0 + \ln \int_{-1}^0 \left\{ 1 + \sum_{j=1}^n \binom{m+j-1}{j} \Delta^{(j)}_{-1/2} \right.$$

$$\left. + \frac{1}{2!} \left[\sum_{j=1}^n \binom{m+j-1}{j} \Delta^{(j)}_{-1/2} \right]^2 + \dots \right\} dm$$

$$= f_0 + N' \Delta'_{-1} + N'' \Delta''_{-1} + \dots.$$

Dropping the subscripts under the $\Delta^{(j)}$'s, which are always $-\frac{1}{2}j$, and making n successively equal to 1, 2, etc., we have

$$\exp(N'\Delta') = \int_{-1}^0 \exp(m\Delta') dm = \frac{1}{\Delta'} (1 - e^{-\Delta'}) = F', \quad (3)$$

$$\exp(N'\Delta' + N''\Delta'') = \int_{-1}^0 \exp[m\Delta' + \frac{1}{2}(m^2 + m)\Delta''] dm = F'', \text{ etc.}$$

The individual coefficients $N^{(j)}$ can be computed from

$$N^{(j)}\Delta^{(j)} = \ln F^{(j)} - \ln F^{(j-1)}. \quad (4)$$

The evaluation of the coefficients $N^{(j)}$ in a power series of $\Delta^{(j)}$ is quite laborious for $j > 2$; it can be easily shown, however, that the first term of the series is always the corresponding Newton-Gregory coefficient:

$$\begin{aligned} N' &= -\frac{1}{2} + \frac{1}{24}\Delta' - \frac{1}{2880}\Delta'^3 + \dots \\ (5) \quad N'' &= -\frac{1}{12} + \frac{1}{1440}\Delta'' + \frac{1}{720}\Delta'^2\Delta'' + \frac{1}{181440}\Delta''^2 + \dots \\ N''' &= -\frac{1}{24} + \dots \end{aligned}$$

While the presence of the large term $+\frac{1}{24}\Delta'$ in the expression for N' makes it necessary to tabulate this coefficient (or, rather, the product $N'\Delta'$) for practical work, it appears that the first term (i.e., the Newton-Gregory coefficient) is a sufficient approximation of $N^{(j)}$ for $j \geq 2$. The series for N'' shows that even in the extremely unlikely case of $\Delta' = \Delta'' = 1$, we have $N''' = -1/12.3$, and numerical integration of Equation (3) shows that when Δ' , Δ'' , and Δ''' vary between 0 and 1, N''' varies between $-1/24.0$ and $-1/24.5$. In practice, the interval of tabulation is always chosen such that $|\Delta'| \leq 1$ ($\Delta' = 1$ corresponds to an increment by a factor of e in the variable from one tabular value to the next!), $|\Delta''| \leq 1/10$, $|\Delta'''| < |\Delta''|$, etc. Under such circumstances, the coefficients N'' and N''' do not differ from the Newton-Gregory coefficients $-1/12$ and $-1/24$ by more than 1% and it stands to reason (although convergence proofs have not been undertaken) that the same will be true of the coefficients of the higher differences.

The method here described has been used with great success in the integration of the differential equations governing the motion and the loss of mass of meteors. Decimal, rather than natural, logarithms have been used throughout, and a small table of $N'\Delta'$ to four places, computed for the range from -0.40 to $+0.40$ in Δ' at 0.01-intervals has proved more than adequate. The table is not reproduced here, since it can be duplicated by any computer with very little effort.

The advantage of using the above formula should be quite obvious in the case of near-exponential functions, for which the step of integration can be made

much larger, without any increase in the truncation error, when the logarithm rather than the natural value of the function is tabulated.

LUIGI JACCHIA

Harvard College Observatory
Cambridge, Massachusetts

* Work performed under Contract DA 19-020-ORD-2556 with the Office of Ordnance Research, U. S. Army.

Note on a Logarithm Algorithm

In a recent note,¹ D. SHANKS developed a well-known algorithm for the computation of logarithms,² in a way particularly suitable for use on automatic computing machines. In what follows I should like to point out that, if we use the value of $\mu = \ln a_0$ (the natural logarithm of a_0), the number of operations necessary for this computation can be cut down considerably in replacing about a third of the single steps indicated by Shanks by one division and one addition.

We assume as in the paper quoted that $a_0 > a_1 > 1$. To compute $\lambda = \log_{a_0} a_1$, we determine a sequence of numbers a_2, a_3, a_4, \dots , ($a_i > 1$) and a sequence of positive integers n_1, n_2, \dots , by the relations

$$a_i^{n_i} \leq a_{i-1} < a_i^{n_i+1}, \quad a_{i+1} = a_{i-1}/a_i^{n_i} \quad (i = 1, 2, \dots).$$

We then have

$$\lambda = \frac{1}{n_1 +} \frac{1}{n_2 +} \dots;$$

if we stop at the calculation of n_i we have an approximate value of λ by taking the i th convergent:

$$\mu_i = \frac{P_i}{Q_i} = \frac{1}{n_1 +} \frac{1}{n_2 +} \dots \frac{1}{n_i} = \lambda - \eta_i.$$

We will show that we have for η_i the formula

$$(I) \quad \eta_i = \eta_i^* + \rho_i, \quad \eta_i^* = (-1)^i \frac{a_{i+1} - 1}{\mu Q_i},$$

where the *error term* ρ_i can be estimated by

$$(II) \quad |\rho_i| \leq \mu Q_i \eta_i^2 \leq \mu |\eta_i|^{3/2} \leq \mu/Q_i^3$$

as soon as we have

$$(III) \quad \mu/Q_i \leq 1.7933 \dots$$

For $a_0 = 10$ we have $\mu = \mu_{10} \doteq 2.3026$ and (III) is certainly satisfied from $i = 2$ on. For $a_0 = 2$ we have

$$\mu = \mu_2 \doteq 0.6931.$$