A Continued Fraction for $e^x$

Let $A_n/B_n$ denote the convergents to the continued fraction

$$b_0 + \frac{a_1}{b_1 + b_2 + \cdots + b_n + \cdots}.$$

It is well known [1] that the continued fraction

$$b_0 + \frac{a_1 b_2}{b_1 b_2 + a_2 - \frac{a_2 a_3 b_4}{(b_3 b_4 + a_4) b_4 + b_5 a_6 - \frac{a_4 a_5 b_6}{(b_6 b_7 + a_7) b_7 + \cdots}}}$$

has convergents $A_{2n}/B_{2n}$.

From the continued fraction of Gauss, one can obtain [2]

$$e^x = 1 + \frac{x}{1} - \frac{x/1 \cdot 2}{1} + \frac{x/2 \cdot 3}{1} - \frac{x/2 \cdot 5}{1} + \cdots.$$

For this expansion, we have $b_0 = 1$, $a_1 = x$, $a_{2n} = -x[2(2n - 1)]$, $a_{2n+1} = x/[2(2n + 1)]$, and $b_n = 1$ ($n = 1, 2, 3, \cdots$). Thus, we obtain

$$-a_{2n} a_{2n+1} b_{2n-1} b_{2n+2} = x^2/[4(4n^2 - 1)],$$

and

$$(b_{2n} b_{2n+1} + a_{2n+1}) b_{2n+2} + b_{2n} a_{2n+2} = 1, \quad (n = 2, 3, \cdots);$$
and so
\begin{align*}
e^x &= 1 + \frac{x}{1 - x/2} + \frac{x^2/(4\cdot3)}{1} + \frac{x^2/(4\cdot15)}{1} + \ldots \\
&\quad+ \frac{x^2/[4(\nu - 1)^2 - 1]}{1} + \ldots
\end{align*}

A convenient equivalent form is
\begin{align*}
e^x &= 1 + \frac{x}{1 - x/2} + \frac{x^2/4}{3} + \frac{x^2/4}{5} + \ldots + \frac{x^2/4}{2\nu - 1} + \ldots
\end{align*}

These expansions converge quite rapidly for all \(x\). For example, if \(-1 \leq x \leq 1\), we have \(e^x - A_4/B_4 < .000084\), \(e^x - A_6/B_6 < .000000033\), and \(e^x - A_6/B_6 < .000000000081\).