HARVARD UNIVERSITY, "Proceedings of a symposium on large-scale digital calculating machinery," Cambridge, Mass., 1948.

HARVARD UNIVERSITY, "Proceedings of a second symposium on large-scale digital calculating machinery," Cambridge, Mass., 1951.

M. A. HYMAN, "On the numerical solution of partial differential equations," thesis, Delft, 1953, 108 p.

N. METROPOLIS & S. ULAM, "The Monte Carlo method," Am. Stat. Assn., *Jn.*, v. 44, 1949, p. 335–341.

J. VON NEUMANN & H. H. GOLDSTINE, "Numerical inverting of matrices of high order," Am. Math. Soc., *Bull.*, v. 53, 1947, p. 1021–1099, and Am. Math. Soc., *Proc.*, v. 2, 1951, p. 188–202.

F. W. J. OLVER, "The evaluation of zeros of high-degree polynomials," R. Soc. London, *Phil. Trans.*, v. 244, 1952, p. 385–415.

E. STIEFEL, "Relaxationsmethoden bester Strategie zur Lösung linearer Gleichungssysteme," *Commentarii Mathematici Helvetici*, v. 29, 1955, p. 157–179.

JOHN TODD, "Motivation for working in numerical analysis," *Comm. on Pure and Applied Math.*, v. 8, 1955, p. 97–116.

DAVID YOUNG, "Iterative methods for solving partial difference equations of elliptic type," Am. Math. Soc., *Trans.*, v. 76, 1954, p. 92–111.

GEORGE E. FORSYTHE

University of California
Los Angeles, California

This note was sponsored jointly by the Office of Naval Research and the Office of Ordnance Research.

## Modified Quotients of Cylinder Functions

The name in the title of this note is applied to the function, $\mathfrak{C}_\nu(z)$, defined by the following equation,

$$\mathfrak{C}_\nu(z) = \frac{zC_{\nu-1}(z)}{C_\nu(z)} \tag{1}$$

where $C_\nu(z)$ is a cylinder function [1] which satisfies the pair of recurrence formulae,

$$\begin{aligned} \frac{2\nu}{z} C_\nu(z) &= C_{\nu-1}(z) + C_{\nu+1}(z) \\ 2C_\nu'(z) &= C_{\nu-1}(z) - C_{\nu+1}(z) \end{aligned} \tag{2}$$

$\mathfrak{C}_\nu(z)$ has not repeated zeros and poles with possible exception of the origin and satisfies the following RICCATI's equation,

$$\frac{dy}{dz} + \frac{1}{z}(y^2 - 2\nu y) + z = 0. \tag{3}$$

Its derivative,

$$\frac{\partial \mathfrak{C}_\nu(z)}{\partial z} = \frac{z}{C_\nu^2(z)} \{C_{\nu-1}(z)C_{\nu+1}(z) - C_\nu^2(z)\} \tag{4}$$

has a close relation to the famous integral due to LOMMEL,

$$\int zC_\nu^2(z)dz = \left[\frac{z^2}{2}\{C_\nu^2(z) - C_{\nu-1}(z)C_{\nu+1}(z)\}\right]. \tag{5}$$

Introducing each of the three kinds of Bessel functions, $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, into the equation (1) in place of $C_\nu(z)$, we obtain corresponding kinds of modified quotients, $\mathfrak{J}_\nu(z)$, $\mathfrak{Y}_\nu(z)$, $\mathfrak{H}_\nu^{(1)}(z)$ and $\mathfrak{H}_\nu^{(2)}(z)$, respectively. In various boundary value problems of mathematical physics, we encounter quite often the Bessel functions in quotient forms [2]. It is obvious that the modified quotients defined here give a convenient approach to mathematical analysis and numerical estimates of these problems. Moreover there is a remarkable parallelism between the modified quotients and the trigonometric cotangent and tangent, as there is between the Bessel functions and the sine and cosine. Therefore the modified quotients should have the same *raison d'être* in cylinder functions as the cotangent and tangent in trigonometric functions.

From the above consideration it seems highly desirable to give permanent symbols to these modified quotients, to collect their formulae and to construct their tables. To this end an attempt was made [3], but further cooperation and criticism of interested workers are necessary.

MORIO ONOE

Institute of Industrial Science
University of Tokyo
Chiba City, Japan

1. G. N. WATSON, *Theory of Bessel Functions*, Cambridge Univ. Press, 1922, p. 82 ff.
2. For example:
DENNISON BANCROFT, "The velocity of longitudinal waves in cylindrical bars," *Phys. Rev.*, 59, 1941, p. 588.
G. E. HUDSON, "Dispersion of elastic waves in solid circular cylinders," *Phys. Rev.*, 63, 1943, p. 46.
H. SUHL & L. R. WALKER, "Topics in guided-wave propagation through gyromagnetic media," *Bell System Tech. J.*, 33, 1954, p. 579.
3. M. ONOE, "Formulae and tables, modified quotients of cylinder functions," Report of the Institute of Industrial Science, University of Tokyo, No. 32, 1955.

## Flip-flop as Generator of Random Binary Digits

The aim of the present note is to show that a well known electronic element of digital computers, the flip-flop, may be used for generating a series of random binary digits with equal probabilities.

Let us consider a flip-flop as shown on fig. 1 and let A and B denote two possible stable states of the flip-flop. If we switch on the contact $S$, the flip-flop will be randomly set in one of its states A or B. We may obtain by the aid of the flip-flop a sequence of $2k$ random elements $X_1, X_2, \cdots, X_{2k}$, (abbreviated $\{X_{2k}\}$), where

$$X_j = \begin{cases} \text{A, if } j\text{-th switching on the contact } S \text{, set flip-flop in state A} \\ \text{B, if } j\text{-th switching on the contact } S \text{, set flip-flop in state B} \end{cases}$$

and $1 \leq j \leq 2k$.