A Modified Chebyshev-Everett Interpolation Formula

1. Introduction. It is a well-known property of the Chebyshev polynomials that, of all polynomials with leading coefficient unity, they possess the smallest absolute upper bound when the argument is allowed to vary between their limits of orthogonality. This property suggests the use of such polynomials as a means of interpolation. As is explained in Kopal's recent book on numerical analysis [1], a power series rearranged as a series of Chebyshev polynomials is reduced to an economic form, insofar as the “maximum” accuracy is attained with a very small number of terms. We use Kopal's notation [1] here.

The purpose of this note is to rearrange the Everett interpolation formula,

\[ f(m) = f(0) + m \Delta_1 + E_0 \Delta_0 + E_1 \Delta_1 + \cdots + E_0(2n) \Delta_0(2n) + E_1(2n) \Delta_1(2n) \]

as a series of suitable Chebyshev polynomials, thereby reducing considerably the number of terms depending on \( m \), required to give a prescribed accuracy, while retaining the use of even order differences only. Following Miller [2] the polynomials \( C_{2j+1}(2 - 2m) \) and \( C_{2j+1}(2m) \) could be used, where \( C_j(2m) = 2 \cos (j \cos^{-1} m) \). A disadvantage in doing so is the non-vanishing of such polynomials at tabular points. To overcome this difficulty Kopal [1] and Miller [7] have suggested an expansion in terms of the polynomials

\[ \left( 2 - 2m \right) \cos \frac{\pi}{4j + 2} \]

and

\[ \left( 2m \cos \frac{\pi}{4j + 2} \right) \]

Such an expansion Kopal calls the Modified Chebyshev-Everett Interpolation Formula and states that “...it represents potentially the most powerful and economic interpolation formula which can as yet be devised” [1]. See also Clenshaw and Olver [9].

Explicitly, following [1], p. 513, let

(1) \[ f(m) = f(0) + m \Delta_1 + \sum_{i=1}^{n} \left( \beta_0^i C_{2j+1} \left( 2 - 2m \right) \cos \frac{\pi}{4j + 2} \right)^i \]

\[ + \beta_1^i C_{2j+1} \left( 2m \cos \frac{\pi}{4j + 2} \right)^i \]

\[ = f(0) + m \Delta_1 + \sum_{i=1}^{n} \frac{\sec^2 \left( \frac{\pi}{4j + 2} \right)}{2^i + 1} \cdot \frac{1}{(2j + 1)!} \]

\[ \times \left[ N_0^{(2j)} C_{2j+1} \left( 2 - 2m \right) \cos \frac{\pi}{4j + 2} + N_1^{(2j)} C_{2j+1} \left( 2m \cos \frac{\pi}{4j + 2} \right) \right] \]

where the coefficients \( N_0^{(2j)} \) and \( N_1^{(2j)} \) will be designated modified differences of the \( 2j \)-th order. Owing to the symmetry of the Everett coefficients \( E^{(2n)} \) and to the
fact that \( C_{2j+1}(x) \) is an odd function of \( x \), the \( N_i^{(2j)} \)'s will be the same function of the \( \Delta_1 \)'s as the \( N_0^{(2j)} \)'s are of the \( \Delta_0 \)'s.

2. Calculation of the Modified Differences \( N_{0,1}^{(2j)} \). The coefficients \( N_{0,1}^{(2j)} \) are determined from the equivalence of the two expressions

\[
f(m) = f(0) + m\Delta_1 + \sum_{j=1}^{n} (E_0^{(2j)}\Delta_0^{(2j)} + E_1^{(2j)}\Delta_1^{(2j)})
\]

\[
= f(0) + m\Delta_1 + \sum_{j=1}^{n} \frac{\sec^{2j+1} \left( \frac{\pi}{4j + 2} \right)}{2^{2j+1}(2j + 1)!} 
\times \left[ N_0^{(2j)}C_{2j+1}\left(2 - 2m\cos \frac{\pi}{4j + 2}\right) + N_1^{(2j)}C_{2j+1}\left(2m\cos \frac{\pi}{4j + 2}\right) \right]
\]

or, in effect, from

\[
(3) \quad \sum_{j=1}^{n} E_1^{(2j)}\Delta_1^{(2j)} = \sum_{j=1}^{n} \frac{\sec^{2j+1} \left( \frac{\pi}{4j + 2} \right)}{2^{2j+1}(2j + 1)!} \left[ N_0^{(2j)}C_{2j+1}\left(2 - 2m\cos \frac{\pi}{4j + 2}\right) + N_1^{(2j)}C_{2j+1}\left(2m\cos \frac{\pi}{4j + 2}\right) \right].
\]

Both sides of (3) are expanded in powers of \( m \) and the coefficients of equal powers equated. The procedure is quite straightforward, but the algebra is rather unwieldy, and is not reproduced here.

The following general relations can be obtained by equating the coefficients of \( m^{2n-3} \) and \( m \):

\[
N_0^{(2n-4)} = \Delta_0^{(2n-4)} - \frac{1}{4} \left( \frac{n}{3} - \frac{1}{2(n - 1)} \sec^2 \frac{\pi}{4n - 2} \right) \Delta_0^{(2n-2)}
\]

\[
+ \left( \frac{n + 1)(5n + 6)}{1440} - \frac{\sec^4 \frac{\pi}{4n + 2}}{64n(2n - 1)} - \frac{\sec^2 \frac{\pi}{4n - 2}}{32(n - 1)} \right)
\times \left( \frac{n + 1}{3} - \frac{1}{2n} \sec^2 \frac{\pi}{4n + 2} \right) \Delta_0^{(2n)} + \ldots
\]

and

\[
\sum_{j=1}^{n} (-1)^j \frac{(j!)^2}{(2j + 1)!} \Delta_1^{(2j)} = \sum_{j=1}^{n} \left( \frac{\sec^{2j} \frac{\pi}{4j + 2}}{2^{2j}(2j)!} \right) N_1^{(2j)}.
\]

These relations are of assistance in computing the modified differences.
If differences up to the tenth are retained the following closed forms for the 
coefficients $N^x$, $N^{iv}$, $\cdots$, $N^{x}$ are obtained:

\[
N^x = \Delta^x,
\]

\[
N^{iv} = \Delta^{iv} + a_4 \Delta^x,
\]

\[
N^{vi} = \Delta^v + a_4 \Delta^{iv} + b_4 \Delta^x,
\]

\[
N^{iv} = \Delta^v + a_4 \Delta^{iv} + b_4 \Delta^{vii} + c_6 \Delta^x,
\]

\[
N^{ii} = \Delta^v + a_4 \Delta^{iv} + b_4 \Delta^{vii} + c_6 \Delta^{viii} + d_6 \Delta^x,
\]

where

\[
a_j = -\frac{1}{4} \left( \frac{j + 1}{3} - \frac{1}{2j} \sec^2 \frac{\pi}{4j + 2} \right),
\]

\[
b_j = \left( \frac{j + 1}{1440} \right) \sec^4 \frac{\pi}{4j + 2} - \frac{1}{64j(2j - 1)} \sec^2 \frac{\pi}{4j + 2} + a_j \left( \frac{1}{8j - 1} \right),
\]

\[
c_6 = -\frac{139}{6048} + \frac{1}{5 \cdot 3^2 \cdot 2^3} \sec^6 \frac{\pi}{22} - \frac{a_5}{7 \cdot 2^8} \sec^4 \frac{\pi}{18} + \frac{b_5}{24} \sec^2 \frac{\pi}{10},
\]

\[
c_4 = -\frac{41}{3024} + \frac{5}{16 \cdot 8!} \sec^6 \frac{\pi}{18} - \frac{a_4}{960} \sec^4 \frac{\pi}{14} + \frac{b_4}{16} \sec^2 \frac{\pi}{10},
\]

\[
d_6 = \frac{479}{151200} - \frac{15}{27 \cdot 10!} \sec^6 \frac{\pi}{22} + \frac{5a_5}{16 \cdot 8!} \sec^6 \frac{\pi}{18} - \frac{b_6}{8 \cdot 5!} \sec^4 \frac{\pi}{14} + \frac{c_5}{16} \sec^2 \frac{\pi}{10},
\]

The numerical values of the coefficients $a$, $b$, $c$, $d$ are given in Table I.

If the $N^{(2i)}$ are multiplied by $\frac{\sec^{2i+1} \left( \frac{\pi}{4j + 2} \right)}{2^{2i+1}(2j + 1)!}$, the actual coefficients of the 
Chebyshev polynomials $\beta_{a,1}^{(i)}$ in equation (1) are obtained. These are given in 
Table II ($i = 0, 1$).

As will be seen, the coefficients $\beta^{(i)}$ diminish very rapidly and in practice it 
will only be necessary to use about two terms of the series (1). Should it be found 
unnecessary to retain the higher differences, they should simply be omitted from 
the $\beta^{(i)}$ or $N^{(2i)}$.

3. Besselian and Comrie-type Modified Differences. Clearly, the well-known 
Bessel interpolation formula can also be regrouped in terms of the Chebyshev 
polynomials, and the modified differences corresponding to it obtained in a similar 
maner. Modified differences arising in connection with Bessel's formula were 
first introduced into computational practice by E. W. Brown [3], and subse-
sequently (but independently) by Camp [4] and Comrie [5]. Comrie, in particular, 
has done so much to propagate their use that they are usually associated with his 
name. It must be emphasized that the Comrie type of (Besselian) modified 
differences is not identical with what may be called the Chebyshev-Bessel modi-
fied differences, which would be obtained by an analogous procedure to that 
developed in the preceding sections of this note.
### Table I

<table>
<thead>
<tr>
<th>( N_i^{ii} )</th>
<th>( \Delta_i^{ii} )</th>
<th>( \Delta_i^{iv} )</th>
<th>( \Delta_i^{vi} )</th>
<th>( \Delta_i^{viii} )</th>
<th>( \Delta_i^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{i^{iv}} )</td>
<td>1</td>
<td>(-0.18090) 16994 37</td>
<td>(0.03717) 66239 55</td>
<td>(-0.00807) 38992 79</td>
<td>(0.00180) 85304 04</td>
</tr>
<tr>
<td>( N_{i^{vi}} )</td>
<td>1</td>
<td>(-0.28949) 60381 83</td>
<td>(0.07283) 14707 77</td>
<td>(-0.01772) 12103 93</td>
<td>(0.11351) 63325 91</td>
</tr>
<tr>
<td>( N_{i^{viii}} )</td>
<td>1</td>
<td>(-0.38444) 50665 04</td>
<td>(0.11351) 63325 91</td>
<td>(-0.47448) 31950 54</td>
<td>(1)</td>
</tr>
</tbody>
</table>

### Table II

<table>
<thead>
<tr>
<th>( \beta_i )</th>
<th>( \Delta_i^{ii} )</th>
<th>( \Delta_i^{iv} )</th>
<th>( \Delta_i^{vi} )</th>
<th>( \Delta_i^{viii} )</th>
<th>( \Delta_i^x )</th>
<th>( j )</th>
<th>( \cos \frac{\pi}{4j+2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 )</td>
<td>0.03207 50149 55</td>
<td>(-0.00580) 24247 15</td>
<td>(0.00119) 24407 69</td>
<td>(-0.00025) 89704 40</td>
<td>(0.00005) 80086 40</td>
<td>1</td>
<td>0.86602 54037 84</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.00033 46856 93</td>
<td>(-0.00009) 68901 81</td>
<td>(0.00002) 43756 51</td>
<td>(-0.00000) 59310 36</td>
<td>(0.95105) 65162 95</td>
<td>2</td>
<td>0.97492 79121 82</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>0.00000 18516 20</td>
<td>(-0.00000) 07118 46</td>
<td>(0.00000) 02101 89</td>
<td>(0.98480) 77530 12</td>
<td>(0.98982) 14418 81</td>
<td>3</td>
<td>(0.98480) 77530 12</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>0.00000 00061 77</td>
<td>(-0.00000) 0029 31</td>
<td>(0.00000) 00000 14</td>
<td>(0.98982) 14418 81</td>
<td>(0.98480) 77530 12</td>
<td>4</td>
<td>(0.98982) 14418 81</td>
</tr>
</tbody>
</table>
Comrie used modified differences $M^{(i)}$ whose coefficients were determined to minimize the error of the Bessel interpolation formula truncated after a given number of terms. Comrie never published details of the process by which these constants were determined; his argument has been reconstructed in [1], Chapter II. Although these constants were determined by Comrie with specific reference to Bessel's formula, it has become customary to use them in Everett's formula also. In this case an expansion

$$f(m) = f(0) + m\Delta_1 + E_0^{i}\Delta_0^{i} + E_1^{i}\Delta_1^{i} + \cdots$$

$$+ E_0^{(2)}(\Delta_0^{(2)}) + A_1\Delta_1^{(2j+2)} + B_1\Delta_0^{(2j+4)} + \cdots$$

$$+ E_1^{(2)}(\Delta_1^{(2)}) + A_1\Delta_1^{(2j+4)} + B_1\Delta_1^{(2j+4)} + \cdots$$

is used, where the coefficients $A_j, B_j$ are those determined by Comrie for the Bessel formula.

For convenience, write

$$r_{2j+1} = \frac{\sec^{2j+1} \left( \frac{\pi}{4j+2} \right)}{2^{2j+1}(2j+1)!} C_{2j+1} \left( 2m \cos \frac{\pi}{4j+2} \right).$$

Consider a specific example where differences up to the eighth are retained. Then $f(m)$ may be expanded:

$$(4) \quad f(m) = f(0) + m\Delta_1 + E_1^{i}\Delta_1^{i} + E_1^{iv}\Delta_1^{iv} + E_1^{vii}\Delta_1^{vii} + \cdots$$

$$+ \text{similar terms in } E_0\Delta_0,$$

$$(5) \quad f(m) = f(0) + m\Delta_1 + r_3(\Delta_1^{ii} + a\Delta_1^{iv} + b\Delta_1^{vi} + c\Delta_1^{viii})$$

$$+ r_6(\Delta_1^{iv} + d\Delta_1^{vi} + e\Delta_1^{viii})$$

$$+ r_7(\Delta_1^{vi} + f\Delta_1^{viii})$$

$$+ r_9\Delta_1^{viii} + \cdots.$$

A comparison of these formulae shows that

$$r_3 = E_1^{ii}$$

$$ar_3 + r_6 = E_1^{iv}.$$ 

Now if we regard the last three terms of (5) as error terms then

$$f(m) \sim f(0) + m\Delta_1 + r_3(\Delta_1^{ii} + a\Delta_1^{iv} + b\Delta_1^{vi} + c\Delta_1^{viii}) + \cdots$$

$$= f(0) + m\Delta_1 + E_1^{ii}N_1^{ii} + \cdots.$$ 

Reference to Comrie's paper [5] shows that $N_1^{ii}$ is in fact identical with the modified second difference used by Comrie.

However, there seems to be no simple relation between $N_1^{iv}$ and higher order modified differences, and the higher Comrie modified differences.
4. Example. As an example, an interpolation has been made in a table of Fermi-Dirac functions at present being prepared by the author.

A portion of this table, together with 1st, 2nd, 4th, 6th, 8th, and 10th differences, is reproduced below. It has been chosen because the differences do not fall off rapidly, as can be seen.

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( F(\eta) )</th>
<th>( \Delta_1 )</th>
<th>( \Delta_{11} )</th>
<th>( \Delta_{1v} )</th>
<th>( \Delta_{1v} )</th>
<th>( \Delta_{1v} )</th>
<th>( \Delta_{x} )</th>
</tr>
</thead>
<tbody>
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<td>-2.0</td>
<td>0.1145 8783</td>
<td>6921 403</td>
<td>3748 501</td>
<td>297 363</td>
<td>-135 029</td>
<td>65 578</td>
<td>-20 496</td>
</tr>
<tr>
<td>-1.5</td>
<td>1838 0186</td>
<td>10669 904</td>
<td>5259 311</td>
<td>130 749</td>
<td>-125 114</td>
<td>110 942</td>
<td>-119 432</td>
</tr>
<tr>
<td>-1.0</td>
<td>2905 0090</td>
<td>15929 215</td>
<td>6900 870</td>
<td>-160 979</td>
<td>37 361</td>
<td>36 873</td>
<td>-68 774</td>
</tr>
<tr>
<td>-0.5</td>
<td>4497 9305</td>
<td>22830 085</td>
<td>8381 450</td>
<td>-456 963</td>
<td>153 475</td>
<td>-105 970</td>
<td>113 458</td>
</tr>
<tr>
<td>0.0</td>
<td>6780 9389</td>
<td>31211 535</td>
<td>9405 068</td>
<td>-599 471</td>
<td>205 237</td>
<td>-135 355</td>
<td>124 858</td>
</tr>
<tr>
<td>0.5</td>
<td>9902 0925</td>
<td>40616 603</td>
<td>19829 215</td>
<td>-536 743</td>
<td>211 643</td>
<td>-32 156</td>
<td>-17 467</td>
</tr>
<tr>
<td>1.0</td>
<td>1.3963 7528</td>
<td>50445 818</td>
<td>9716 618</td>
<td>-352 371</td>
<td>5 895</td>
<td>53 577</td>
<td>-82 922</td>
</tr>
<tr>
<td>1.5</td>
<td>1.9008 3346</td>
<td>60162 437</td>
<td>9716 618</td>
<td>-352 371</td>
<td>5 895</td>
<td>53 577</td>
<td>-82 922</td>
</tr>
</tbody>
</table>

From this table we find, e.g., the value of \( F(-0.3) \) from the formula: \( m = 0.4 \),

\[
F(-0.3) = F(-0.5) + 0.4\Delta_1 + E_{0}^{11}(0.4)N_{0}^{11} + E_{1}^{11}(0.4)N_{1}^{11}
\]

\[
+ \beta_0^2 C_5 \left( 1.2 \cos \frac{\pi}{10} \right) + \beta_1^2 C_5 \left( 0.8 \cos \frac{\pi}{10} \right)
\]

where

\[
N_{0}^{11} = 0.06929 411 \quad \beta_0^2 = -0.00000 052
\]

\[
N_{1}^{11} = 0.08470 882 \quad \beta_1^2 = -0.00000 171.
\]

The values of \( C_5 \left( 1.2 \cos \frac{\pi}{10} \right) \) and \( C_5 \left( 0.8 \cos \frac{\pi}{10} \right) \) are obtained by linear interpolation in the table of Chebyshev polynomials [6].

For comparison, columns (1) and (2) below give respectively the value of \( F(-0.3) \), as calculated with the same number of terms in the interpolation formula, by the straight Everett formula,

\[
f(m) = f(0) + m\Delta_1 + E_0^{11}\Delta_0^{11} + E_1^{11}\Delta_1^{11} + E_0^{1v}\Delta_0^{1v} + E_1^{1v}\Delta_1^{1v}
\]

and the modified Everett formula using Comrie's coefficients in the modified fourth difference,

\[
f(m) = f(0) + m\Delta_1 + E_0^{11}\Delta_0^{11} + E_1^{11}\Delta_1^{11} + E_0^{1v}M_0^{1v} + E_1^{1v}M_1^{1v}
\]

where \( M_0^{1v} = \Delta_0^{1v} - 0.207\Delta_0^{vi} + 0.045\Delta_0^{vii} \). Column (3) gives the value computed by the present method. The value obtained by direct calculation is 0.53193 157. It will be seen that the present method gives substantially better results for this case.
A MODIFIED CHEBYSHEV-EVERETT INTERPOLATION FORMULA

\[ F(-0.3) = 0.44979305 + 0.44979305 - 9132034441656 - 4693611875 - 49130.53193534 \]

In most applications high order differences will fall off rapidly and probably only the modified second difference will be required. If this is calculated and printed with a table, interpolation becomes a very simple procedure.

5. Use of the Expansion. An interesting discussion about the merits of a Chebyshev-Everett interpolation formula is recorded in Appendix 11 of reference [7] to which the reader is referred. (This report was received after the major part of this note had been written.)

It should be emphasized that the chief advantage of this type of expansion is that the number of terms in the interpolating expansion of a function \( f(m) \), dependent on \( m \), is reduced considerably. Clearly, the expansion will have its greatest use when several interpolations within a given interval are required. The complicated modified differences, once calculated, are available for all interpolations within the relevant interval, the number of terms dependent on \( m \) being usually only two, \textit{viz.},

\[ C_3 \left\{ 2m \cos \frac{\pi}{6} \right\} \quad \text{and} \quad C_3 \left\{ (2 - 2m) \cos \frac{\pi}{6} \right\}. \]

A table of the above functions would be advantageous, especially when the expansion is being used for sub-tabulation. For purposes other than sub-tabulation, values of the Chebyshev functions can be obtained by simple interpolation in the existing tables [6] and [8]. For convenience, values of \( \cos \frac{\pi}{4j + 2} \) are given to twelve decimal places in Table II.

G. A. CHISNALL

Department of Astronomy
The University
Manchester 13, England

I am indebted to Professor Kopal for allowing me to read the relevant sections of his book prior to publication and for many constructive criticisms. This note has been written during the tenure of a scholarship from the N.R.D.C., to whom I am also indebted.

Iterative Procedure for Evaluating a Transient Response Through its Power Series

Introduction. We discuss here a particular method of evaluating a time function, such as a transient response, from its Laplace transform. We shall assume the Laplace transform of $F(t)$,

$$F(S) = \int_0^\infty e^{-St} F(t) \, dt$$

is given, and is of the form

$$F(S) = \frac{A_0 S^{n-1} + A_1 S^{n-2} + \cdots + A_{n-2} S + A_{n-1}}{S^n + b_1 S^{n-1} + b_2 S^{n-2} + \cdots + b_{n-1} S + b_n}.$$  

(The superscript 0 of the $A$'s does not indicate a power, but is used as a superscript for reasons which will appear presently.)

We require in (2) that the denominator be of higher degree than the numerator, which is necessary if $F(t)$ is a Laplace transform. We shall always assume also that we have $\bar{F}(S)$ written with the coefficient of $S^n$ in the denominator equal to unity, as in equation (2).

The method described below is an iterative method for obtaining the quantities $a_j$ in MacLaurin series for $F(t)$, which we write in the form

$$F(t) = \sum_{j=0}^\infty a_j \frac{t^j}{j!}.$$  

The method has the advantage that it is not necessary to know the roots of the numerator or denominator of $\bar{F}(S)$. The series (3) will converge for all positive values of $t$, although for larger values of $t$ a great many terms may be needed. This, however, is not too great a disadvantage for digital purposes, since the method for computing the $a_j$'s is an iterative one. It is necessary, however, to have some idea of how many terms will be needed, and a criterion for this is also given below and discussed. The theoretical derivations are given in the Appendix.

Method for Computing the $a_j$'s. The method for computing the $a_j$'s, the theory of which is discussed in the Appendix, is as follows: We start with the coefficients

$$A_0, A_1, \ldots, A_{n-1}, b_1, b_2, \ldots, b_n$$  

of (2).