On a Device for Computing the $e_m(S_n)$ Transformation

1. Introduction. The transformation

$$(1) \qquad e_m(S_n) = \frac{\begin{vmatrix} S_n & S_{n+1} & \cdots & S_{n+m} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+m-1} & & \cdots & \Delta S_{n+2m-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+m-1} & & \cdots & \Delta S_{n+2m-1} \end{vmatrix}}$$

has been developed by Schmidt [1] and Shanks [2]. It is shown by both authors that

$$e_m(S_n) = a$$

if

$$(2) \qquad S_n = a + \sum_{r=1}^{m} b_r \epsilon_r^n.$$

The former writer shows that iterates of the form (2) occur in the Gauss-Seidel method for the solution of linear equations, and derives the condition

$$(3) \qquad \begin{vmatrix} \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+m} & \Delta S_{n+m+1} & \cdots & \Delta S_{n+2m} \end{vmatrix} = 0$$

for the applicability of the transformation.

A consideration which militates against the use of the transformation is that the vast amount of labor spent in evaluating the determinants in (1) serves only to produce one transformed result. The computation of further results proceeds quite independently with a similar expenditure of effort, so that use of (1) in its present form as a sequence to sequence transformation in which members of the transformed sequences $e_m(S_n)(n = \bar{n}, \bar{n} + 1, \cdots, m = 1, 2, \cdots)$ may be examined for any tendency to approach a limit, involves a prohibitive expenditure of labor. It is proposed to show that the transformation (1) may be effected by a simple algorithm, in which transformed results for $n = \bar{n}, \bar{n} + 1, \cdots, m = 1, 2, \cdots$ are progressively available for comparison.

2. Theorem. *If*

$$\epsilon_{2m}(S_n) = e_m(S_n)$$

and

$$\epsilon_{2m+1}(S_n) = \frac{1}{e_m(\Delta S_n)}$$

then

$$(4) \qquad \epsilon_{s+1}(S_n) = \epsilon_{s-1}(S_{n+1}) + \frac{1}{\epsilon_s(S_{n+1}) - \epsilon_s(S_n)}, \quad s = 1, 2, \cdots,$$

provided that none of the quantities $\epsilon_{2m}(S_n)$ becomes infinite.

Proof. The above result is easily verified when $s = 1$, for:

$$\begin{aligned}\epsilon_0(S_{n+1}) + \frac{1}{\epsilon_1(S_{n+1}) - \epsilon_1(S_n)} &= S_{n+1} + \cfrac{1}{\cfrac{1}{\Delta S_{n+1}} - \cfrac{1}{\Delta S_n}} \\ &= \frac{S_{n+1}(\Delta S_n - \Delta S_{n+1}) + \Delta S_{n+1}\cdot\Delta S_n}{\Delta S_n - \Delta S_{n+1}} \\ &= \frac{S_n\Delta S_{n+1} - S_{n+1}\cdot\Delta S_n}{\Delta S_{n+1} - \Delta S_n} \\ &= \epsilon_2(S_n).\end{aligned}$$

For larger values of s, consider first the case $s = 2k$. It is then to be proved that

$$\frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^2 S_{n+k-1} & & \cdots & \Delta^2 S_{n+2k-1} \end{vmatrix}}{\begin{vmatrix} \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}} - \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^2 S_{n+k-1} & & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+2} & \Delta S_{n+3} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}} \tag{5}$$

$$= \cfrac{1}{\cfrac{\begin{vmatrix} S_{n+1} & S_{n+2} & \cdots & S_{n+k+1} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}} - \cfrac{\begin{vmatrix} S_n & S_{n+1} & \cdots & S_{n+k} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k-1} & & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}}.$$

Rearranging columns and rows, the left hand side of equation (5) may be written

$$\frac{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} & \Delta^2 S_{n+k-1} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} & \Delta^2 S_{n+k-1} \end{vmatrix}} - \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^2 S_{n+k-1} & & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^2 S_{n+k-1} & \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}},$$

and by the Schweinsian expansion of the quotient of two determinants [3] this expression is equivalent to

$$
(6)\qquad \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \Delta^2 S_{n+k-1} & \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k-2} & \Delta^2 S_{n+k-2} \end{vmatrix} \begin{vmatrix} \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \Delta^2 S_{n+2} & \Delta^2 S_{n+3} & \cdots & \Delta^2 S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} & \Delta^2 S_{n+k-1} \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta^2 S_{n+k-1} & \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}}
$$

$$
= \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}}{\begin{vmatrix} \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+2} & \Delta S_{n+3} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}.
$$

The right hand side of equation (5) may be written

$$
(7)\qquad \frac{(-1)^k \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \\ S_{n+1} & S_{n+2} & \cdots & S_{n+k+1} \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \cdot & \cdots & \cdot & \cdot \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & \Delta S_{n+k-1} \\ 1 & \cdots & 1 & 1 \end{vmatrix} - \begin{vmatrix} \Delta S_{n+1} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \cdot & \cdots & \cdot & \cdot \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & \Delta S_{n+k-1} \\ S_{n+1} & \cdots & S_{n+k} & S_n \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \\ 1 & 1 & \cdots & 1 \end{vmatrix}}
$$

and by using an extensional identity derived from

$$
\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} |a_1| & |b_2| - |b_1| & |a_2| \end{vmatrix}
$$

(7) is equivalent to

$$\frac{(-1)^k \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k+1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_{n+k+1} & \Delta S_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} & \Delta S_{n+2k} & \Delta S_{n+k-1} \\ S_{n+1} & S_{n+2} & \cdots & S_{n+k} & S_{n+k+1} & S_n \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+2} & \Delta S_{n+3} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}$$

which is equivalent to expression (6). Thus the left and right hand sides of (4) are equal when $s = 2k$. That they are also equal when $s = 2k+1$ is demonstrated in a similar fashion.

3. Algorithm. It can now be seen that the quantities S_n, $\epsilon_s(S_n)$; $n = \bar{n}, \bar{n} + 1, \cdots$; $s = 1, 2, \cdots$; may be arranged as in a table of S_n and its differences, in which $\epsilon_s(S_n)$ takes the place of $\Delta^s S_n$. This table is composed of groups of four quantities $\epsilon_s(S_n)$, $\epsilon_s(S_{n+1})$, $\epsilon_{s-1}(S_{n+1})$, $\epsilon_{s+1}(S_n)$, arranged in the form

$$\begin{array}{ccc} & \epsilon_s(S_n) & \\ \epsilon_{s-1}(S_{n+1}) & & \epsilon_{s+1}(S_n). \\ & \epsilon_s(S_{n+1}) & \end{array}$$

The quantity $\epsilon_{s+1}(S_n)$ is formed by the relation

$$\epsilon_{s+1}(S_n) = \epsilon_{s-1}(S_{n+1}) + \frac{1}{\epsilon_s(S_{n+1}) - \epsilon_s(S_n)},$$

and it may be remarked that this simple calculation is easily programmed for a digital computer, all the entries in the second and subsequent columns being obtained from this programme by trivial alterations in storage parameters.

In this table the even order columns $\epsilon_{2r}(S_n)$ display the transformed results $e_r(S_n)$. There is no need for the explicit evaluation of the determinant in equation (3), as it occurs in the table as the denominator of $\epsilon_{2m+1}(S_n)$; if equation (3) holds when $n = \bar{n}, \bar{n} + 1, \cdots$, then $\epsilon_{2m}(S_{\bar{n}}) = \epsilon_{2m}(S_{\bar{n}+1}) = \cdots$.

4. A numerical illustration. A numerical illustration is provided by the transformation of the sequence produced by the linearly convergent iterative scheme

$$S_{n+1} = \tfrac{1}{4}(S_n^2 + 2)$$

which may be used to derive the smaller zero of the Laguerre polynomial $L_2(x) = x^2 - 4x + 2$, i.e., $x = 0.58578\ 64375$.

It will be noted from Table I (and this is a consistent feature of the transformation when meeting with numerical success), that

$$\left|\frac{1}{\epsilon_{2s}(S_{m+1}) - \epsilon_{2s}(S_m)}\right| \gg |\epsilon_{2s-1}(S_{m+1})|$$

and hence

$$\epsilon_{2s+2}(S_m) \doteq \cfrac{1}{\cfrac{1}{\epsilon_{2s}(S_{m+2}) - \epsilon_{2s}(S_{m+1})} - \cfrac{1}{\epsilon_{2s}(S_{m+1}) - \epsilon_{2s}(S_m)}} + \epsilon_{2s}(S_{m+1})$$

$$= \epsilon_{2s}(S_{m+1}) - \frac{[\epsilon_{2s}(S_{m+2}) - \epsilon_{2s}(S_{m+1})][\epsilon_{2s}(S_{m+1}) - \epsilon_{2s}(S_m)]}{[\epsilon_{2s}(S_{m+2}) - 2\epsilon_{2s}(S_{m+1}) + \epsilon_{2s}(S_m)]}.$$

It is thus an approximate rule that there are as many significant figures in the difference $\epsilon_{2s+2}(S_m) - \epsilon_{2s}(S_{m+1})$ as there are in the second difference $\epsilon_{2s}(S_{m+2}) - 2\epsilon_{2s}(S_{m+1}) + \epsilon_{2s}(S_m)$. From this it is seen that loss of figures due to cancellation may occur in the transformation. This possible loss of figures in the Aitken transformation $e_1(S_n)$ has been remarked upon by Olver [4] and Shanks [5].

Table I also displays terms in the transformed sequence $e_1{}^2(S_n)$, and it will

TABLE I

n	S_n	ϵ_1	ϵ_2	ϵ_3	ϵ_4	$e_1{}^2$
0	0.0					
		2.0				
1	0.5		0.57142 85714			
		16.0		89.11111 08		
2	0.5625		0.58510 63830		0.58574 34871	0.58576 19630
		60.2352 941		1658.713		
3	0.57910 15625		0.58573 19781		0.58578 57313	0.58578 61706
		211.05540		20262.3		
4	0.58383 96549		0.58578 18504		0.58578 64208	0.58578 64316
		725.9366		239060.		
5	0.58521 71856		0.58578 60462			
		2483.836				
6	0.58561 97886					

be noted that comparison of the results given by the transformations $e_2(S_n)$ and $e_1{}^2(S_n)$ in this case reacts slightly in favor of the latter. The difference between the two results however is not greatly significant, and there seems to be some point in using the transformation $e_m(S_n)$ consistently (in place of m successive applications of Aitken's transformation) just in case S_n might very accurately be represented by some equation of the form (2). There is no reason to believe that this is true for the example chosen, but results for a favorable numerical example are given in Schmidt's paper. Further examples in which success for $e_m(S_n)$ might be anticipated from analytical knowledge of S_n are given by Shanks.

5. An application. Having shown that the various transformations $e_m(S_n)$ may be effected without much effort, the primary purpose of this paper has been achieved. Functions of the form (2) are of such widespread occurrence in the theory of Mathematical Physics, that an exhaustive account of the problems in which the transformation $e_m(S_n)$ might prove useful (besides being irrelevant to the main thesis of this paper which is primarily one of computational expediency) would be excessively discursive and, in view of possible developments, necessarily incomplete. By way of a footnote however it may be pointed out that the transformation may be applied to establish a criterion for the fitting of certain types of statistical data. The determination of the constants α_i, b_i when fitting tabular data, which is given at regularly spaced intervals in t of magnitude w, to a function of the form

$$\phi(t) = \sum_{i=1}^{h} b_i e^{-\alpha_i t}, \tag{8}$$

when m is given, is already well known [6]. To establish the suitability of (8) as an adequate representation of the data, and further to determine h, it is merely necessary to apply the transformation $e_m(S_n)$ to the data, writing $S_{n+r} = \phi(t + rw)$, when $e_h(S_n) = 0$. (Various refinements, such as partitioning the data in groups of p values, and so on, are obvious and need not be discussed here.) This was one of the first problems upon which the transformation $e_m(S_n)$ [5] was used.

It is the author's hope that by demonstrating the ease with which the various transformations may be effected, their field of application might be widened, and deeper insight thereby obtained into the problems for whose solution the transformations have been used.

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1. J. R. Schmidt, "On the numerical solution of linear simultaneous equations by an iterative method," *Phil. Mag.*, Ser. 7, v. 32, 1941, p. 369–383.
2. D. Shanks, "Non-linear transformations of divergent and slowly convergent sequences," *J. Math. and Physics*, v. 34, 1955, p. 1–42.
3. A. C. Aitken, *Determinants and Matrices*, Oliver and Boyd, 6th Ed., 1949, Ch. V, p. 108, formula (2) and p. 49, formula (9).
4. F. W. J. Olver, "The evaluation of zeros of high degree polynomials," Roy. Soc. Phil., *Trans.*, v. 244, 1952, p. 385–415.
5. D. Shanks, "An analogy between transients and mathematical sequences and some nonlinear sequence-to-sequence transforms suggested by it," Part I. Naval Ordnance Laboratory Memorandum 9994, White Oak, Md., 1949.
6. E. T. Whittaker & G. Robinson, *The Calculus of Observations*, Blackie and Son Ltd., 4th Ed., Ch. XV, 1944, p. 369.

TECHNICAL NOTES AND SHORT PAPERS

Note on the Computation of Certain Highly Oscillatory Integrals

The purpose of this note is to draw attention to the possible use of the Faltung theorem for Fourier transforms as an aid to the computation of highly oscillatory integrals.

If F is the Fourier cosine transform of f, defined by

$$F(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos ut \, dt$$