$$\frac{1}{t}\bar{J}_0(8t) = 7.99999,99990 - 42.66666,64204t^2 + 102.39999,00866t^4 - 130.03159,01993t^6 + 101.13454,38222t^8 - 52.95243,71745t^{10} + 19.89838,57672t^{12} - 5.60245,72363t^{14} + 1.20146,77449t^{16} - 0.18673,71001t^{18} + 0.01624,75465t^{20} \quad (10 \times 10^{-10})$$

$$\sqrt{\frac{2}{\pi}}\bar{P}_0\left(\frac{8}{t}\right) = 0.79788,45600 - 0.01256,42405t^2 + 0.00178,70944t^4 - 0.00067,40148t^6 + 0.00041,00676t^8 - 0.00025,43955t^{10} + 0.00011,07299t^{12} - 0.00002,26238t^{14} \quad (8 \times 10^{-10})$$

$$\frac{1}{t}\sqrt{\frac{2}{\pi}}\bar{Q}_0\left(\frac{8}{t}\right) = -0.06233,47304 + 0.00404,03539t^2 - 0.00100,89872t^4 + 0.00053,66169t^6 - 0.00039,92825t^8 + 0.00027,55037t^{10} - 0.00012,70039t^{12} + 0.00002,68482t^{14} \quad (8 \times 10^{-10})$$

$$\sqrt{\frac{\pi}{2}}\bar{G}_0\left(\frac{1}{t}\right) = 1.25331,39163 - 0.78323,44963t + 1.25733,12033t^2 - 3.09054,43850t^3 + 9.02560,45356t^4 - 25.43912,19592t^5 + 60.46288,82856t^6 - 112.80726,52384t^7 + 158.83274,70627t^8 - 163.74821,02377t^9 + 119.22659,27008t^{10} - 57.88900,96515t^{11} + 16.78876,58787t^{12} - 2.19748,24449t^{13} \quad (25 \times 10^{-10} \text{ in } \bar{K}_0).$$

An attempt to derive similar formulas for $I_0(x)$, $I_1(x)$, and $\bar{I}_0(x)$ by the same methods proved unsuccessful, since these functions cannot be defined by a differential equation and a boundary condition at infinity.

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An Iterative Method for the Solution of Linear Equations Based on the Power Method for Proper Vectors

Introduction. A computing machine program for obtaining the largest proper value and proper vector of an $(n + 1, n + 1)$ matrix A by the power method (i.e., by the iteration $X_{\nu+1} = AX_\nu$) may be used for solving a system of n linear equations in n unknowns. In the following note a method of setting up the iteration together with a simple criterion for convergence of the iteration are given.

Method. Let B be an $(n + 1, n + 1)$ matrix with real elements and real proper value k such that $|k|$ is greater than the moduli of all the remaining proper values of B. If X is a non-exceptional non-zero column vector, $X = \text{col}\,(\xi_1, \xi_2, \cdots, \xi_n, \xi_{n+1})$, with real components, it is well known that the iteration $X_0 = X$, $X_1 = BX_0$, $\cdots$, $X_{\nu+1} = BX_\nu$ is such that $X_{\nu+1} \doteq kX_\nu$. (There are exceptional choices of X for which the iteration will not converge to the largest proper value and vector; but round-off error during the iteration will usually make the X_ν non-exceptional

and convergence will actually take place, though possibly quite slowly.) To keep the components within size it is usual in a practical computation to modify X_r after each iteration by multiplying through by a factor which makes $\xi_{n+1} = 1$. If this last feature is incorporated into a program, the whole computation can be expressed as the following iterative algorithm:

$$
\begin{aligned}
\lambda &= b_{n+1,1}\xi_1 + b_{n+1,2}\xi_2 + \cdots + b_{n+1,n}\xi_n + b_{n+1,n+1} \\
\xi_1' &= \frac{1}{\lambda}\{b_{11}\xi_1 + b_{12}\xi_2 + \cdots + b_{1n}\xi_n + b_{1,n+1}\} \\
\xi_2' &= \frac{1}{\lambda}\{b_{21}\xi_1 + b_{22}\xi_2 + \cdots + b_{2n}\xi_n + b_{2,n+1}\} \qquad (1) \\
&\cdot \\
&\cdot \\
&\cdot \\
\xi_n' &= \frac{1}{\lambda}\{b_{n1}\xi_1 + b_{n2}\xi_2 + \cdots + b_{nn}\xi_n + b_{n,n+1}\}.
\end{aligned}
$$

The algorithm is repeated until each ξ_i' agrees with ξ_i to the required degree of accuracy. The last λ and col $(\xi_1, \xi_2, \cdots \xi_n, 1)$ are the required proper value and vector respectively.

This algorithm has appeared in numerous places, e.g., von Mises and Pollaczek-Geiringer [1].

The program thus set up can be directly applied to a system of a linear equations in n unknowns as follows. The system of equations $AX = b$, where A is a non-singular (n, n) matrix with real entries and X and b are the real column vectors col $(X_1, X_2, \cdots, X_n)$, col $(b_1, b_2, \cdots, b_n)$, respectively, is equivalent to the system

$$\begin{pmatrix} A + kI & -b \\ 0 & k \end{pmatrix} y = ky$$

where y is the vector col $(X_1, X_2, \cdots, X_n, 1)$ and k is any real constant. This shows that y is a proper vector of the matrix

$$B = \begin{pmatrix} A + kI & -b \\ 0 & k \end{pmatrix}$$

corresponding to the proper value k.

The iteration corresponding to the equations (1) will converge to y provided $|k|$ is larger than the modulus of any other proper value of B. The existence or non-existence of an appropriate k is covered by the following theorem.

THEOREM 1. *Let A be non-singular. A necessary and sufficient condition that there exist a real number k such that each eigenvalue distinct from k of the matrix*

$$B = \begin{pmatrix} A + kI & -b \\ 0 & k \end{pmatrix}$$

be of modulus less than $|k|$ is that all eigenvalues of the matrix A have real parts different from zero of the same sign.

Proof: Necessity: Let A have proper values $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then B has proper values $k, k + \lambda_1, k + \lambda_2, \cdots, k + \lambda_n$. If two of the λ_i, e.g., λ_1, λ_2 have real parts of opposite sign, then $|k + \lambda_1| < |k|$ implies that the real part of λ has sign opposite to that of k. Hence λ_2 has a real part of the same sign as that of k so that $|k + \lambda_2| > |k|$. Also, if any λ_i has real part 0, then $|k + \lambda_i| > |k|$ unless $|A| = 0$, for all real k.

Sufficiency: If $\lambda_1, \lambda_2, \cdots, \lambda_n$ have real parts all of the same sign, first suppose this sign to be negative. In the complex plane (see figure 1) each λ_i is to the left of the y-axis. Let μ_i be the point at which the line joining the origin to λ_i meets the circle $|z + 1| = 1$. Let k be any real number such that $k > |\lambda_i|/|\mu_i|$ for $i = 1, 2, \cdots, n$. Then λ_i/k lies inside the circle $|z + 1| = 1$. Hence $|1 + \lambda_i/k| < 1$, i.e., $|k + \lambda_i| < k$ for all i. In the case where the real parts of λ_i are all positive

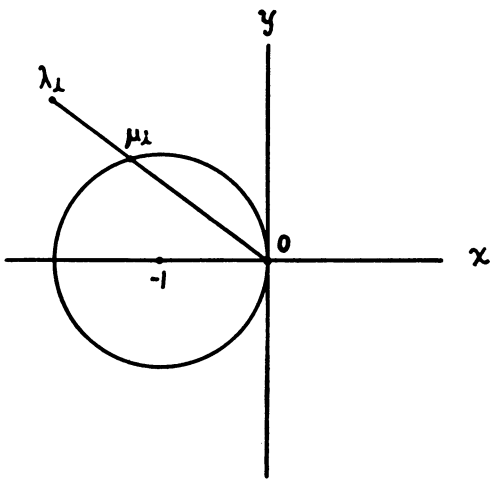

FIG. 1

it is only necessary to replace the circle $|z + 1| = 1$ by the circle $|z - 1| = 1$ and then choose k in such a way that $-k > |\lambda_i|/|\mu_i|$ for all i.

In general, while criteria exist for deciding whether the real parts of the proper values of a matrix A are all of the same sign, such criteria are not computationally of much use. If it is known that the proper values of A are all real (e.g., in the case where A is symmetric), then if a value of k exists for which the iteration converges, the value $k = -\operatorname{tr} A$ will be adequate. In any case the sign of a suitable k will always be opposite to that of $\operatorname{tr} A$.

From a practical point of view for any A a simple procedure is to assume a value of k exists and choose a large multiple of $-\operatorname{tr} A$ to be this value. If the iteration does not appear to converge, it is suggested that the equations be modified by the usual type of linear transformation. If one cannot find an obvious row transformation which makes the system converge it may be noted that, if A is non-singular, A^TA is positive definite so that the system $AX = b$ may be replaced by $A^TAX = A^Tb$. For this system $k = -\max \alpha_{ii}$, where α_{ij} are the

elements of A^TA, is always adequate. While as Taussky shows [2], the system $A^TAX = A^Tb$ is more ill-conditioned than $AX = b$, this ill condition reflects itself in a larger computing time but does not affect accuracy.

The system has been applied with success at the Canadian Armament Research and Development Establishment on the ALWAC III machine. The circumstance was such that the routine for proper vectors was programmed prior to that for matrix inversion. It was originally meant as a stop-gap measure but has turned out to be a practical method of much merit.

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The Computation of Complex Proper Values and Vectors of a Real Matrix with Application to Polynomials

Introduction. The power method for computing the largest real proper value and corresponding proper vector of a real matrix A can with very minor adjustments be applied to the complex proper values and vectors. A good exposition of the power method may be found in von Mises and Pollaczek-Geiringer [4]. When applied to the companion matrix of a real polynomial the method yields a real root or pair of complex roots of the polynomial. It turns out that in this application the method is equivalent to Bernoulli's method, as was shown by Aitken [1]. Having obtained the root or roots of maximum modulus other roots are usually obtainable by the elementary devices indicated below. The method admits of easy programming on a digital computer. The program has been carried out on the ALWAC III computer at the Canadian Armament Research and Development Establishment and has turned out to be very practical indeed.

Method and Theory. Let A be an (r, r) real matrix with proper values $\lambda_1, \lambda_2, \cdots, \lambda_r$ and corresponding column proper vectors $x_1, x_2, \cdots, x_r$. It is assumed that $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_r|$. If any one of the λ_i is not real its conjugate appears among the remaining λ_i and the corresponding proper vectors are conjugate. If η is an arbitrary non-zero column vector with r components, η can be expressed in the form $\eta = c_1x_1 + c_2x_2 + \cdots + c_rx_r$. It follows that $\eta_n = A^n\eta = c_1\lambda^nx_1 + c_2\lambda_2{}^nx_2 + \cdots + c_r\lambda_r{}^nx_r$.

If $|\lambda_1| > |\lambda_2|$, then λ_1 is real and

$$\eta_n = \lambda_1{}^n \left\{ c_1x_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^n x_2 + \cdots + c_r\left(\frac{\lambda_r}{\lambda_1}\right)^n x_r \right\}.$$

If $c_1 \neq 0$, the dominant term in η_n is $\lambda_1{}^nc_1x_1$. Hence, for sufficiently large n, $\eta_{n+1} \doteq \lambda_1\eta_n$. It is ordinarily convenient at each stage to normalize η_n so that the