A proof depends on the fact that a non-zero quaternion $\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ effects the rotation $\mathbf{v} \to \mathbf{q}\mathbf{v}\mathbf{q}^{-1}$ of vectors $\mathbf{v}$, and that conversely every rotation can be so produced by some quaternion.

Select $\mathbf{q}$ from a uniform distribution on the set $|\mathbf{q}| < 1$ as above. Then $\mathbf{q}\mathbf{k}\mathbf{q}^{-1}$ will have on the surface of the unit sphere a distribution invariant under rotations because, if $\mathbf{p}$ is any other non-zero quaternion, then

$$\mathbf{p}(\mathbf{q}\mathbf{k}\mathbf{q}^{-1})\mathbf{p}^{-1} = \left(\frac{\mathbf{p}}{|\mathbf{p}|}\mathbf{q}\right)\mathbf{k}\left(\frac{\mathbf{p}}{|\mathbf{p}|}\mathbf{q}\right)^{-1},$$

but multiplication of $\mathbf{q}$ by a quaternion of unit norm is a rotation of quaternion space and hence preserves the distribution from which we selected $\mathbf{q}$.

But there exists on the surface of a sphere only one distribution invariant under all rotations, namely, the uniform distribution.

Now let $\mathbf{q}\mathbf{k}\mathbf{q}^{-1} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and we have the above formulae.

The method extends to $n$ dimensions and to more general spaces acted upon by a compact group of transformations.

J. M. Cook

Argonne National Laboratory
Lemont, Illinois

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## On the Numerical Solution of Integral Equations

For the numerical solution of linear integral equations we must of course replace all our variables, both dependent and independent, by discrete variables. In choosing the grid for the independent variables we should naturally be influenced by the storage facilities of our calculating machines. One purpose of this note is to emphasize the obvious fact that we should not be too strongly influenced by the storage facilities since we may then take longer than necessary for the job. The following technique is formulated in terms of finding the inverse of a kernel, $K(x, y)$, but a similar technique could be applied for solving a single integral equation. The idea seems too simple to be new but I have not met it before, and it may have some applications.

First choose a wide grid. This will convert our kernel into a small matrix, $M_1$. Invert it. Next choose a less wide grid, and hence a larger matrix, $M_2$. Use the result of stage 1 to construct an approximate inverse of $M_2$ (by smoothing the elements of the inverse of $M_1$ so as to obtain a matrix of the right size). This step will usually be justifiable if $K(x, y)$ is a smooth function. Starting with our approximate inverse of $M_2$ we may obtain a more exact inverse by means of a well known iterative procedure known to be rapidly convergent when we start with a good approximation (I. J. Good [1] and H. Hotelling [2]). We can next choose a still narrower grid and proceed as before. The whole process can be stopped when adjacent entries in our matrix are numerically close.

In one-dimensional problems of mathematical physics the word "adjacent" can here be usually interpreted positionally on the matrix itself, if the rows and columns are in an appropriate order. Otherwise the word should be interpreted in terms of the physical application. For example, in the inverse of the matrix $L_n$ treated by W. L. Wilson [3], which arises from a two-dimensional physical problem, the elements at the positions (4, 14) and (7, 19) may be regarded as adjacent because they correspond to *pairs* of points that are close together in the physical problem. This question of the definition of "adjacent" is related to that of how to extrapolate from a matrix to a larger one. It seems difficult to formulate any general rule for extrapolation. For the present each case would have to be treated on its own merits by the exercise of judgment, as in the reference cited.

I. J. Good

25 Scott House
Princess Elizabeth Way
Cheltenham, England

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2. Harold Hotelling, "Some new methods in matrix calculation," *Annals Math. Stat.*, v. 14, 1943, p. 1–34.
3. W. L. Wilson, Jr., *Tables of Inverses to Laplacian Operators over Triangular Grids*, UMT File [*MTAC*, this issue, Rev. 53, p. 108].

## A Rotation Method for Computing Canonical Correlations

Given two sets of variates, $x_i (i = 1, 2 \cdots p)$, $y_j (j = 1, 2 \cdots q)$ with $p \geqslant q$, Hotelling [1] has shown that it is possible to find linear transforms $u_i$, $v_j$ of the $x$'s and $y$'s respectively with the properties

$$\begin{aligned} &1. \quad \operatorname{var}(u_i) = \operatorname{var}(v_j) = 1 \\ &2. \quad \operatorname{cov}(u_i, u_k) = 0, \quad i \neq k \\ &\qquad \operatorname{cov}(v_j, v_l) = 0, \quad j \neq l \\ &3. \quad \operatorname{cov}(u_i, v_j) = 0, \quad i \neq j. \end{aligned}$$

The variates $u$ and $v$ are called canonical variates, and the correlations $\rho_i (i = 1, 2 \cdots q)$ between corresponding variates $u_i$, $v_i$ are called canonical correlations.

The same problem may be stated in terms of matrix algebra. Suppose that the dispersion matrix of the $x$'s and $y$'s, considered as a single vector variate, is

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}' & \mathbf{B} \end{pmatrix}$$

where **A** has $p$ rows and $p$ columns, **B** has $q$ rows and $q$ columns, and **C** has $p$ rows and $q$ columns. The whole matrix is symmetric and positive definite. We