TECHNICAL NOTES AND SHORT PAPERS

A Method for the Numerical Evaluation of Certain Infinite Integrals

The solution of many physical problems often necessitates the numerical evaluation of infinite real integrals, a common example being that of solutions obtained with the aid of integral transforms. The evaluation of such integrals is often a laborious task, particularly if the integrand is oscillatory, so that it is usual to resort to special methods which give information for certain ranges of values of the variables; methods of this type are those involving asymptotic expansions or the related techniques of steepest descent and of stationary phase. The purpose of the present note is to outline a method in which the value of such integrals is expressed in terms of a convergent series obtained by a modification of the corresponding asymptotic expansion. The development is given below for a special case only, namely one which might arise in conjunction with the use of sine transforms; it will be clear however that these results can be readily generalized to other types of integrals which are usually reduced to an asymptotic representation. Examples may be found in Erdélyi [1]. The method is thus valid whether the integrand is oscillatory or not; in fact, though the special integrand considered in detail below does oscillate, inspection of the convergence proofs shows that this fact is of little importance to the developments presented. A method which holds in the case of oscillatory integrands has been described by I. M. Longman [2].

Basic expansions. Consider a convergent integral \( I(a) \) of the form

\[
I(a) = \int_{a}^{\infty} f(x) \sin x \, dx; \quad f(x) \to 0 \text{ steadily as } x \to \infty.
\]

By \( f(x) \to 0 \) steadily, we mean that \( f(x_1) \geq f(x_2) > 0 \) if \( x_1 < x_2 \) and \( \lim f(x) = 0 \); see Whittaker and Watson [3]. \( N \) successive integrations by parts may be shown to give the following result

\[
I(a) = \sum_{i=0}^{N} f^{(i)}(a) \cos \left[ a + i(\pi/2) \right] + \int_{a}^{\infty} f^{(N)}(x) \sin \left[ x + N(\pi/2) \right] dx
\]

where \( f^{(i)} = (d/dx)^i f \), provided that \( f(x) \) is differentiable the required number of times, and that

\[
f^{(i)}(x) \to 0 \text{ steadily as } x \to \infty; \quad i = 0, 1, 2, \ldots.
\]
for \( I(a) \) may now be derived in the following manner. Integration by parts gives

\[
(3a) \quad I(a) = \int_a^{\infty} f(x) \sin x \, dx + f(a_1) \cos a_1 + \int_{a_1}^{\infty} f^{(1)}(x) \cos x \, dx
\]

and further

\[
(3b) \quad I(a) = \int_a^{\infty} f(x) \sin x \, dx + \int_{a_1}^{\infty} f^{(1)}(x) \cos x \, dx + f(a_1) \cos a_1 - f^{(1)}(a_2) \sin a_2 - \int_{a_2}^{\infty} f^{(2)}(x) \sin x \, dx.
\]

Repetition of this process finally gives

\[
(4) \quad I(a) = \sum_{i=0}^{\infty} f^{(i)}(a_{i+1}) \cos \left[ a_{i+1} + i(\pi/2) \right] + \sum_{i=0}^{\infty} \int_{a_i}^{a_i+1} f^{(i)}(x) \sin \left[ x + i(\pi/2) \right] \, dx
\]

where one may set

\[
(4a) \quad a_{i+1} \geq a_i; \quad a_0 = a.
\]

It will now be shown that the quantities \( a_i \) may be chosen in such a manner that the two series on the right-hand side of equation (4) converge.

**Convergence of series expansion.** The first series on the right-hand side of equation (4) will certainly converge if the \( a_i \)'s are chosen so that the series

\[
(5) \quad S_1 = \sum_{i=0}^{\infty} f^{(i)}(a_{i+1})
\]

converges; and this series will converge (absolutely) if a positive number \( \rho \) independent of \( i \) exists such that

\[
(5a) \quad 1 > \rho > \left| f^{(i)}(a_{i+1}) / f^{(i-1)}(a_i) \right|
\]

for all \( i \geq 1 \). It will now be shown that such a choice of \( a_i \)'s is always possible. (The author is indebted to Dr. C. C. Chao for his valuable suggestions concerning this proof.)

Choose the quantity \( a_i \geq a_0 \) arbitrarily; then the value of \( f^{(0)}(a_1) \) is known and \( a_2 \) must be selected so that

\[
(5b) \quad \left| f^{(1)}(a_2) \right| < \rho \left| f^{(0)}(a_1) \right|
\]

as may always be done because of relation (2a). Now however the value of \( f^{(2)}(a_2) \) is known, and so \( a_3 \) can be chosen by a similar procedure. Repetition of this process yields values of all \( a_i \)'s in such a manner that relation (5a) is satisfied for all \( i \geq 1 \) and therefore series \( S_1 \) converges absolutely. It should be noted that the choice of \( a_i \)'s is not unique, and that in fact if such a choice has been made \( (a_i = a_i', \text{ say}) \) then the values \( a_i = a_i'' \) will also insure convergence of \( S_1 \) provided only that

\[
(6) \quad a_i'' \geq a_i'
\]

in view of the steadiness requirement of equation (2a).
It will now be shown that the \( a_i's \) may be taken in conformity with requirement (6) and, in addition, so that the second series of equation (4), namely

\[
S_2 = \sum_{i=0}^{\infty} I_i; \quad I_i = \int_{a_i}^{a_{i+1}} f^{(i)}(x) \sin \left[ x + i(\pi/2) \right] dx
\]

also converges. Note first that it follows from equation (2a) that, for any \( i \), a number \( A_i \) exists such that

\[
|f^{(i-1)}(x_i)| < |f^{(i-1)}(x)| \quad \text{for all } x_1 > x > A_i.
\]

Let now the quantities \( a_i \) be selected (consistently with inequality (6)), so that

\[
a_i > A_i.
\]

Because of equation (4a) then the relation

\[
|f^{(i-1)}(a_{i+1})/f^{(i-1)}(a_i)| < 1
\]

holds for all \( i \).

Consider now the integrals \( I_i \); because of the steadiness requirement in equation (2a) the quantity \( f^{(i)}(x) \) does not change sign within \( a_i \leq x \leq a_{i+1} \) and

\[
|I_i| < \left| \int_{a_i}^{a_{i+1}} f^{(i)}(x) dx \right| = |f^{(i-1)}(a_{i+1}) - f^{(i-1)}(a_i)| = |f^{(i-1)}(a_i) - 1 - \left[ f^{(i-1)}(a_i + 1)/f^{(i-1)}(a_i) \right]| < 2 |f^{(i-1)}(a_i)| \quad i \neq 0
\]

in view of relation (7c). Series \( S_2 \) (with the possible omission of the first term) is then term-by-term less than the series

\[
2 \sum_{i=0}^{\infty} |f^{(i-1)}(a_i)| = 2 \sum_{i=0}^{\infty} |f^{(i)}(a_{i+1})|
\]

which has been shown to converge. Hence \( S_2 \) also converges.

**Example.** As an illustration of the procedure indicated above, the special case of \( f(x) = x^{-k} \) will be considered; thus

\[
I(a) = \int_{a}^{\infty} x^{-k} \sin xdx; \quad k > 0.
\]

Here one may take (as will be shown)

\[
a_i = a + i\alpha
\]

where \( \alpha \) is a constant; equation (4) then reduces to

\[
I(a) = S_1(a) + S_2(a)
\]

where

\[
S_1(a) = \sum_{i=1}^{\infty} \frac{(1)(k)(k+1) \cdots (k+i-2)}{(a+i\alpha)(k+i-1)} \sin \left[ i(\pi/2 - \alpha) - a \right]
\]

\[
S_2(a) = \sum_{i=1}^{\infty} \frac{(1)(k)(k+1) \cdots (k+i-2)}{\int_{a+i-1}^{a+i} x^{-(k+i-1)} \cos \left[ i(\pi/2 - x) \right] dx}.
\]
Series $S_1$ converges if
\begin{equation}
1 > \lim_{i \to \infty} \left( \frac{(k + i - 1)(a + i\alpha)}{[a + (i + 1)\alpha]^{(k + 1)}} \right) = \lim_{i \to \infty} \left( \frac{(k + i - 1)}{a + i\alpha} \right) \lim_{i \to \infty} \left( \frac{a + i\alpha}{a + (i + 1)\alpha} \right) \lim_{i \to \infty} \left( \frac{a + i\alpha}{a + (i + 1)\alpha} \right) = \frac{1}{(a/\alpha)} \lim_{i \to \infty} \left( 1 - \frac{1}{1 + \left( a/\alpha \right) + i} \right) = \frac{1}{(ae)}
\end{equation}
or in other words if
\begin{equation}
\alpha > \left( \frac{1}{e} \right).
\end{equation}

Series $S_2$ may now be considered by expanding the integrals it contains in a manner entirely analogous to that of equations (8a) and (8b), and it can thus be easily shown that this series also converges if $\alpha$ is chosen as specified in equation (10d); the latter condition then represents in the present case the only requirement for convergence of expansion (10a).

An advantageous choice of $\alpha$, consistent with requirement (10d), is $\alpha = \pi/2$, since in this case the sine-term in $S_1$ is constant; in particular, if $a = 0$ (or $a = n\pi$), note that $S_1 = 0$. No choice of $\alpha$ is of course possible which will make $S_2 = 0$, so that numerical evaluations of integrals are still necessary. Series $S_2$ converges quite rapidly, however; as an example consider in fact the case of $a = 0$, $k = 1$ and $\alpha = \pi/2$ for which the value of the integral in question is well known. The result may be written as

\begin{equation}
(11) \quad \frac{2}{\pi} \int_0^\infty \left( \frac{1}{x} \right) \sin x \, dx = 1 = \frac{2}{\pi} \sum_{i=1}^{\infty} (i - 1)! \int_{i-1}^{i+1} \frac{1}{\sqrt{x-i}} x^{-i} \cos [i(\pi/2) - x] \, dx.
\end{equation}

The integrals in this summation were evaluated by Simpson's rule with the relatively coarse interval of $(\pi/8)$. The value of the summation itself may be expressed as the limit of the sequence of the partial sums $S_i$ of the first $i$ terms of the series; the first few terms of this sequence were found to be as follows (to four significant figures):

\begin{equation}
(11a) \quad S_1 = .8727; \quad S_2 = .9762; \quad S_3 = .9951; \quad S_4 = .9988; \quad S_5 = .9996
\end{equation}

and may therefore be said to converge fairly rapidly. Almost the same results were obtained when the coarser interval of $(\pi/4)$ was used in Simpson's rule.

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