3. A Higher Order Formula. The formulas corresponding to (1) and (2) for an error of order h^5 are as follows:

$$(16)\quad y_1 = y_0 + \frac{h}{24}[9f(x_0, y_0) + 19f(x_1, y_1) - 5f(x_2, y_2^*) + f(x_3, y_3^*)] + 0(h^5)$$

$$(17)\quad y_2^* = y_0 + \frac{h}{3}[f(x_0, y_0) + 4f(x_1, y_1) + f(x_2, y_2^*)]$$

$$(18)\quad y_3^* = 9y_1 - 8y_0 - 3h[f(x_0, y_0) + 2f(x_1, y_1) - f(x_2, y_2^*)].$$

These formulas are used to find y_1 as follows:

a) Guess $y_1 = y_2^*(x - h)$, $y_2^* = y_3^*(x - h)$.
b) Calculate improved values of y_3^*, y_2^*, y_1 in that order.
c) Repeat from b) until y_1 has converged before proceeding to the next point.

Note that no starter formulas are required since initially we may guess $y_0 = y_1 = y_2^* = y_3^*$ and proceed from b) above.

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TECHNICAL NOTES AND SHORT PAPERS

Note on the Computation of the Zeros of Functions Satisfying a Second Order Differential Equation

By D. J. Hofsommer

It has been pointed out by P. Wynn [1] that, if a function satisfies a second order differential equation, this fact may be used with advantage in the computation of its zeros. In his note he only pays attention to Richmonds formula which, incidentally, was already known to Schröder [2]. We will elaborate his idea to construct another iteration formula.

Let $f(x)$ be the function, the roots of which are to be computed. Let α be such a root and let x be a first approximation. If the approximation is sufficiently close,

$$(1)\quad \alpha = x - f/f' - \tfrac{1}{2}(f''/f')(f/f')^2 - \tfrac{1}{6}[3(f''/f')^2 - f'''/f'](f/f')^3 + 0[(f/f')^4].$$

This series may be used either for direct computation in taking enough terms, or for obtaining an iterative process if only few terms are retained. If $f(x)$ satisfies the homogeneous differential equation

$$(2)\qquad f'' = 2Pf' + Qf + 2S,$$

substitution in the series (1) yields

$$(3)\quad \alpha = x - f/f' + (P + S/f')(f/f')^2 - \tfrac{1}{3}(4P^2 - P' + Q + 10PS/f' + S'/f' + 6S^2/f'^2)(f/f')^3 + 0[(f/f')^4]$$

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or,

$$\alpha = x - f/f' - (P + S/f')(f/f')^2 + 0[(f/f')^3] \tag{4}$$

$$= x - \frac{1}{f'/f - P - S/f'} + 0\,[(f/f')^3] \tag{5}$$

From (4) and (5) it follows that the iterative processes

$$x_{k+1} = x_k - f(x_k)/f'(x_k) - [P(x_k) + S(x_k)/f'(x_k)][f(x_k)/f'(x_k)]^2 \tag{6}$$

or

$$x_{k+1} = x_k - \frac{1}{f'(x_k)/f(x_k) - P(x_k) - S(x_k)/f'(x_k)} \tag{7}$$

are cubically convergent.

Wynn's process gives

$$x_{k+1} = x_k - \frac{1}{f'(x_k)/f(x_k) - P(x_k) - S(x_k)/f'(x_k) - \frac{1}{2}Q(x_k)f(x_k)/f'(x_k)}$$

if $f(x)$ satisfies (2).

From equality (7) it is seen that the last term in the denominator may be left out without affecting the order of the iteration process.

For comparison we have calculated by aid of (7) the first three zeros of J_0 with one iteration step. The results are

x_0	x_1	x_1 (Wynn)
2.405	2.404 825 557 694	2.404 825 557 697
5.520	5.520 078 110 286	5.520 078 110 286
8.654	8.653 727 912 904	8.653 727 912 914

Of course one iteration step amounts to using the development (4) or (5). If in such a case the required accuracy is almost, but not quite, obtained, application of (3) is preferable to taking a second iteration step.

In such a way we have calculated the zeros of P_{37} (cos φ). Using Smith's table [3] the zeros φ_m were found in hundredths of a degree. Then new values for φ_m were calculated by aid of (3) or (4) using the NBS tables [4]. In this way interpolation was avoided. We found

m	φ_m(radians)	m	φ_m(radians)
		9	0.816 897 487 6
0	1.570 796 326 8	10	0.733 136 902 8
1	1.487 027 983 4	11	0.649 379 438 3
2	1.403 259 745 5	12	0.565 626 516 9
3	1.319 491 725 5	13	0.481 880 536 8
4	1.235 724 047 9	14	0.398 145 883 1
5	1.151 956 859 3	15	0.314 431 540 1
6	1.068 190 338 7	16	0.230 759 304 7
7	0.984 424 715 0	17	0.147 197 715 6
8	0.900 660 291 7	18	0.064 126 781 0

The approximation in hundredths of a degree could have been obtained also by

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starting with the approximation for the zeros of $P_n(\cos \varphi)$

$$\varphi_m \doteq \left(\tfrac{1}{2} - \frac{2m}{2n+1}\right)\pi, \qquad n \text{ odd}$$

$$\varphi_m \doteq \left(\tfrac{1}{2} - \frac{2m+1}{2n+1}\right)\pi, \qquad n \text{ even}$$

and applying one Newton step $x_{k+1} = x_k - f(x_k)/f'(x_k)$.

Finally we remark that if $f(x)$ is a member of a set $f_n(x)$, it often happens that $f_n'(x)$ satisfies a recurrence relation

$$f_n'(x) = A(x)f_{n-1}(x) + B(x)f_n(x).$$

It then may be useful to transform

$$f_n'(x_k)/f_n(x_k) = A(x_k)f_{n-1}(x_k)/f_n(x_k) + B(x_k),$$

especially if $f_n(x)$ is tabulated and $f_n'(x)$ not or if the tables of $f_n(x)$ are more extensive than those of $f_n'(x)$.

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A New Mersenne Prime

by Hans Riesel

On September 8, 1957, the Swedish electronic digital computer BESK established that the Mersenne number $M_{3217} = 2^{3217} - 1$ is a prime. The result was recalculated on September 12. The method used for testing this number was a Lucas' test, and the running time was $5^h\ 30^m$. The new prime has 969 decimal digits and on the BESK was found to be

$$2^{3217} - 1$$

2	591 17086	013 20262	777 62467	679 22441	530 94181	888 75531	254 27303
974 92316	187 40192	665 86362	086 20120	951 68004	834 06550	695 24173	319 41774
416 89509	238 80701	741 03777	095 97512	042 31306	662 40829	163 53517	952 31118
615 48622	656 04547	691 12759	584 87756	105 68757	931 19101	771 14088	262 52153
849 03583	040 11850	721 16424	747 46182	303 14713	983 40229	288 07454	567 79079
410 37288	235 82070	589 23510	684 33882	986 88861	665 86502	809 27692	080 33960
586 93087	905 00409	503 70987	590 21190	183 71991	620 99400	256 89351	131 36548
829 73911	265 67973	032 41986	517 25011	641 27035	097 05427	773 47797	234 98216
764 43446	668 38311	932 25400	996 48994	051 79024	162 40565	190 54483	690 80961
606 16257	430 42361	721 86333	941 58524	264 31208	737 26659	196 20617	535 35748
892 89459	962 91951	830 82621	860 85340	093 79328	394 20261	866 58614	250 32514
507 73096	274 23537	682 29386	494 07127	700 84607	712 42118	230 80804	139 29808
705 75047	138 25264	571 44837	937 11250	320 81826	126 56664	908 42516	994 53951
887 78961	365 02484	057 39378	594 59944	433 52311	882 80123	660 40626	246 86092
121 50349	937 58478	229 22371	443 39628	858 48593	821 57388	212 32393	687 04616
067 73629	093 15071						

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