3. A Higher Order Formula. The formulas corresponding to (1) and (2) for an error of order $h^{5}$ are as follows:

$$
\begin{align*}
y_{1} & =y_{0}+\frac{h}{24}\left[9 f\left(x_{0}, y_{0}\right)+19 f\left(x_{1}, y_{1}\right)-5 f\left(x_{2}, y_{2}^{*}\right)+f\left(x_{3}, y_{3}^{*}\right)\right]+0\left(h^{5}\right)  \tag{16}\\
y_{2}^{*} & =y_{0}+\frac{h}{3}\left[f\left(x_{0}, y_{0}\right)+4 f\left(x_{1}, y_{1}\right)+f\left(x_{2}, y_{2}^{*}\right)\right] \\
y_{3}^{*} & =9 y_{1}-8 y_{0}-3 h\left[f\left(x_{0}, y_{0}\right)+2 f\left(x_{1}, y_{1}\right)-f\left(x_{2}, y_{2}^{*}\right)\right] .
\end{align*}
$$

These formulas are used to find $y_{1}$ as follows:
a) Guess $y_{1}=y_{2}{ }^{*}(x-h), y_{2}{ }^{*}=y_{3}{ }^{*}(x-h)$.
b) Calculate improved values of $y_{3}{ }^{*}, y_{2}{ }^{*}, y_{1}$ in that order.
c) Repeat from b) until $y_{1}$ has converged before proceeding to the next point.
Note that no starter formulas are required since initially we may guess $y_{0}=$ $y_{1}=y_{2}{ }^{*}=y_{3}{ }^{*}$ and proceed from b) above.
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1. Herbert S. Wilf, "An open formula for the numerical integration of first order differential equations," MTAC, v. 11, 1957, p. 201-203.

## TECHNICAL NOTES AND SHORT PAPERS

## Note on the Computation of the Zeros of Functions Satisfying a Second Order Differential Equation

By D. J. Hofsommer

It has been pointed out by P. Wynn [1] that, if a function satisfies a second order differential equation, this fact may be used with advantage in the computation of its zeros. In his note he only pays attention to Richmonds formula which, incidentally, was already known to Schröder [2]. We will elaborate his idea to construct another iteration formula.

Let $f(x)$ be the function, the roots of which are to be computed. Let $\alpha$ be such a root and let $x$ be a first approximation. If the approximation is sufficiently close,
(1) $\quad \alpha=x-f / f^{\prime}-\frac{1}{2}\left(f^{\prime \prime} / f^{\prime}\right)\left(f / f^{\prime}\right)^{2}-\frac{1}{6}\left[3\left(f^{\prime \prime} / f^{\prime}\right)^{2}-f^{\prime \prime \prime} / f^{\prime}\right]\left(f / f^{\prime}\right)^{3}+0\left[\left(f / f^{\prime}\right)^{4}\right]$.

This series may be used either for direct computation in taking enough terms, or for obtaining an iterative process if only few terms are retained. If $f(x)$ satisfies the homogeneous differential equation

$$
\begin{equation*}
f^{\prime \prime}=2 P f^{\prime}+Q f+2 S \tag{2}
\end{equation*}
$$

substitution in the series (1) yields

$$
\begin{align*}
\alpha= & x-f / f^{\prime}+\left(P+S / f^{\prime}\right)\left(f / f^{\prime}\right)^{2}  \tag{3}\\
& -\frac{1}{3}\left(4 P^{2}-P^{\prime}+Q+10 P S / f^{\prime}+S^{\prime} / f^{\prime}+6 S^{2} / f^{\prime 2}\right)\left(f / f^{\prime}\right)^{3}+0\left[\left(f / f^{\prime}\right)^{4}\right]
\end{align*}
$$

[^0]or,
\[

$$
\begin{align*}
\alpha & =x-f / f^{\prime}-\left(P+S / f^{\prime}\right)\left(f / f^{\prime}\right)^{2}+0\left[\left(f / f^{\prime}\right)^{3}\right]  \tag{4}\\
& =x-\frac{1}{f^{\prime} / f-P-S / f^{\prime}}+0\left[\left(f / f^{\prime}\right)^{3}\right] \tag{5}
\end{align*}
$$
\]

From (4) and (5) it follows that the iterative processes

$$
\begin{equation*}
x_{k+1}=x_{k}-f\left(x_{k}\right) / f^{\prime}\left(x_{k}\right)-\left[P\left(x_{k}\right)+S\left(x_{k}\right) / f^{\prime}\left(x_{k}\right)\right]\left[f\left(x_{k}\right) / f^{\prime}\left(x_{k}\right)\right]^{2} \tag{6}
\end{equation*}
$$

or

$$
\begin{equation*}
x_{k+1}=x_{k}-\frac{1}{f^{\prime}\left(x_{k}\right) / f\left(x_{k}\right)-P\left(x_{k}\right)-S\left(x_{k}\right) / f^{\prime}\left(x_{k}\right)} \tag{7}
\end{equation*}
$$

are cubically convergent.
Wynn's process gives

$$
x_{k+1}=x_{k}-\frac{1}{f^{\prime}\left(x_{k}\right) / f\left(x_{k}\right)-P\left(x_{k}\right)-S\left(x_{k}\right) / f^{\prime}\left(x_{k}\right)-\frac{1}{2} Q\left(x_{k}\right) f\left(x_{k}\right) / f^{\prime}\left(x_{k}\right)}
$$

if $f(x)$ satisfies (2).
From equality (7) it is seen that the last term in the denominator may be left out without affecting the order of the iteration process.

For comparison we have calculated by aid of (7) the first three zeros of $J_{0}$ with one iteration step. The results are
$x_{0} \quad x_{1} \quad x_{1}$ (Wynn)

| 2.405 | 2.404825 | 557 | 694 | 2.404825 | 557 | 697 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 5.520 | 5.520 | 078 | 110 | 286 | 5.520 | 078 |
| 8.654 | 8.653 | 727 | 912904 | 8.653 | 727 | 912 |
| 8.614 |  |  |  |  |  |  |

Of course one iteration step amounts to using the development (4) or (5). If in such a case the required accuracy is almost, but not quite, obtained, application of (3) is preferable to taking a second iteration step.

In such a way we have calculated the zeros of $P_{37}(\cos \varphi)$. Using Smith's table [3] the zeros $\varphi_{m}$ were found in hundredths of a degree. Then new values for $\varphi_{m}$ were calculated by aid of (3) or (4) using the NBS tables [4]. In this way interpolation was avoided. We found

| $m$ | $\varphi_{m}$ (radians) | $m$ | $\varphi_{m}$ (radians) |
| :---: | :---: | :---: | :---: |
|  |  | 9 | 0.8168974876 |
| 0 | 1.5707963268 | 10 | 0.7331369028 |
| 1 | 1.4870279834 | 11 | 0.6493794383 |
| 2 | 1.4032597455 | 12 | 0.5656265169 |
| 3 | 1.3194917255 | 13 | 0.4818805368 |
| 4 | 1.2357240479 | 14 | 0.3981458831 |
| 5 | 1.1519568593 | 15 | 0.3144315401 |
| 6 | 1.0681903387 | 16 | 0.2307593047 |
| 7 | 0.9844247150 | 17 | 0.1471977156 |
| 8 | 0.9006602917 | 18 | 0.0641267810 |

The approximation in hundredths of a degree could have been obtained also by
starting with the approximation for the zeros of $P_{n}(\cos \varphi)$

$$
\begin{array}{ll}
\varphi_{m} \doteqdot\left(\frac{1}{2}-\frac{2 m}{2 n+1}\right) \pi, & n \text { odd } \\
\varphi_{m} \doteqdot\left(\frac{1}{2}-\frac{2 m+1}{2 n+1}\right) \pi, & n \text { even }
\end{array}
$$

and applying one Newton step $x_{k+1}=x_{k}-f\left(x_{k}\right) / f^{\prime}\left(x_{k}\right)$.
Finally we remark that if $f(x)$ is a member of a set $f_{n}(x)$, it often happens that $f_{n}{ }^{\prime}(x)$ satisfies a recurrence relation

$$
f_{n}^{\prime}(x)=A(x) f_{n-1}(x)+B(x) f_{n}(x)
$$

It then may be useful to transform

$$
f_{n}^{\prime}\left(x_{k}\right) / f_{n}\left(x_{k}\right)=A\left(x_{k}\right) f_{n-1}\left(x_{k}\right) / f_{n}\left(x_{k}\right)+B\left(x_{k}\right),
$$

especially if $f_{n}(x)$ is tabulated and $f_{n}{ }^{\prime}(x)$ not or if the tables of $f_{n}(x)$ are more extensive than those of $f_{n}{ }^{\prime}(x)$.
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1. P. Wynn, "On a cubically convergent process for determining the zeros of certain functions," MTAC, v. 10, 1956, p. 97-100.
2. E. Schröder, "Ueber unendlich viel Algorithmen zur Auflösung der Gleichungen," Math. Ann., v. 2, 1870, p. 317-363.
3. E. R. Smith \& Archie Higdon, "Zeros of the Legendre polynomials," Jn. Science, Iowa State College, v. 12, 1938, p. 263-274." (MTAC, Rev. 132, v. 1, 1944, p. 153-514.)
4. NBSCL, Table of Sines and Cosines to Fifteen Decimal Places at Hundredths of a Degree ( NBS Applied Math. Series, no. 5), Govt. Printing Office, Washington, 1949. (MTAC, Rev. 662, v. 3, 1949, p. 515-516.)

## A New Mersenne Prime

## by Hans Riesel

On September 8, 1957, the Swedish electronic digital computer BESK established that the Mersenne number $M_{3217}=2^{3217}-1$ is a prime. The result was recalculated on September 12. The method used for testing this number was a Lucas' test, and the running time was $5^{h} 30^{m}$. The new prime has 969 decimal digits and on the BESK was found to be

| $2^{3217}-1$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 59117086 | 01320262 | 77762467 | 67922441 | 53094181 | 88875531 | 25427303 |
| 97492316 | 18740192 | 66586362 | 08620120 | 95168004 | 83406550 | 69524173 | 31941774 |
| 41689509 | 23880701 | 74103777 | 09597512 | 04231306 | 66240829 | 16353517 | 95231118 |
| 61548622 | 65604547 | 69112759 | 58487756 | 10568757 | 93119101 | 77114088 | 26252153 |
| 84903583 | 04011850 | 72116424 | 74746182 | 30314713 | 98340229 | 28807454 | 56779079 |
| 41037288 | 23582070 | 58923510 | 68433882 | 98688861 | 66586502 | 80927692 | 08033960 |
| 58693087 | 90500409 | 50370987 | 59021190 | 18371991 | 62099400 | 25689351 | 13136548 |
| 82973911 | 26567973 | 03241986 | 51725011 | 64127035 | 09705427 | 77347797 | 23498216 |
| 76443446 | 66838311 | 93225400 | 99648994 | 05179024 | 16240565 | 19054483 | 69080961 |
| 60616257 | 43042361 | 72186333 | 94158524 | 26431208 | 73726659 | 19620617 | 53535748 |
| 89289459 | 96291951 | 83082621 | 86085340 | 09379328 | 39420261 | 86658614 | 25032514 |
| 50773096 | 27423537 | 68229386 | 49407127 | 70084607 | 71242118 | 23080804 | 13929808 |
| 70575047 | 13825264 | 57144837 | 93711250 | 32081826 | 12656664 | 90842516 | 99453951 |
| 88778961 | 36502484 | 05739378 | 59459944 | 43352311 | 88280123 | 66040626 | 24686092 |
| 12150349 | 93758478 | 22922371 | 44339628 | 85848593 | 82157388 | 21232393 | 68704616 |
| 06773629 | 09315071 |  |  |  |  |  |  |

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[^0]:    Received 3 May 1957.

