Table of Integers Not Exceeding 10 00000 That Are Not Expressible as the Sum of Four Tetrahedral Numbers

By Herbert E. Salzer and Norman Levine

For the past 21 years one of the authors has been concerned with empirical theorems expressing certain classes of positive integers as the sum of four tetrahedral numbers, or "tetrahedrals" \( T_n = \frac{n(n + 1)(n + 2)}{6}, n \geq 0 \), the results being contained in short notes and abstracts, as well as unpublished tables [1]–[9]. We use \( \Sigma_p \) to denote a sum of \( p \) \( T_n \)'s. Among the more interesting past findings were: every square \( \leq 10 \ 00000 \) a \( 24 \) [7], [8], every \( T_n \leq 10 \ 00000 \) a \( 4 \), other than the trivial decomposition \( T_n = T_n + 0 + 0 + 0 \) [3], [4], every multiple of 5 \( \leq 1 \ 00000 \) a \( 4 \) [9] (see also [11]), and verification of Pollock's conjecture that every integer is a \( 4 \) for the first 20000 integers [5], [10]. Incidentally all investigations in [1]–[9] were done by hand, employing at the most a desk calculator.

The exceptional numbers, which by definition are those not expressible as \( 4 \), were tabulated previously only up to 2000. But even that far interesting features turned up, such as 1314 being the only exceptional number ending in 4, and the very few ending in 1 or 9 [4]. Then for numbers ending in 6 and \( \leq 20000 \), 6186 turned out to be the only exceptional one [6]. Also there was no striking difference between the density of exceptional numbers in the first thousand and in the second thousand brackets, decreasing from around 4\% to 3\%, so that it was interesting to speculate upon the approximate density in the neighborhood of say 10 00000. Then finally, the verification of the conjectures that every integer is a \( 4 \) and that every integer \( m = 5r \) is a \( 4 \) for the first 20000 cases, while every \( 10r + 6, r = 0, 1, 2, \ldots, 617 \), is a \( 4 \) until 6186, posed the question as to whether an exception might occur to either of the former two empirical theorems even after verification in those first 20000 cases. Thus it was felt that tabulation of the exceptional numbers \( \leq 10 \ 00000 \) would afford a much clearer picture as to their distribution and density, as well as stronger evidence for the truth of Pollock’s conjecture and the author’s (and Richmond’s [11]) conjecture that every \( m = 5r \) is a \( 4 \).

This table of exceptional numbers \( \leq 10 \ 43999 \) presents a great surprise in its picture of their distribution which is entirely different from that envisioned from those \( \leq 2000 \) [4]. Most strikingly unexpected is the decrease in the density of exceptional numbers from several percent in the neighborhood of 2000 to what appears to be practically zero near 10 00000. The scarcity of exceptional numbers in the higher ranges is in accordance with Hua’s result that "almost all" positive integers are expressible as a \( 4 \) [15], [16]. But even more, this table shows that the likelihood of a given number \( m \) being exceptional falls off so rapidly with increasing \( m \) that it appears to be a plausible conjecture that there might be some \( m_0 \) sufficiently large such that every \( m > m_0 \) is a \( 4 \) (a conjecture unlikely to suggest itself from the exceptional numbers among only the first few thousand).

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A quick glance over this table is sufficient to verify Pollock’s conjecture for each of the first million integers. Furthermore, from the absence of any exceptional numbers in the range 3 43867 to 10 00000 and from an upper bound to the magnitude of the first differences of tetrahedral numbers \( \leq 2500 00000 \), it is easily shown that for any number \( m \) between 10 00000 and 2500 00000, it is always possible to find a \( T_n \) such that \( 3 43867 < m - T_n < 10 00000 \). Thus every \( m \leq 2500 00000 \) is a \( \Sigma_b \). (This fact and the extreme rarity of large exceptional numbers makes the empirical theorem of Pollock that every integer is an \( \Sigma_b \) as overwhelmingly certain as one can be short of actual proof.) This table verifies that every \( m = 5r \) is a \( \Sigma_i \) for the first 2 00000 values of \( r \), making it also a very plausible empirical theorem. In the class of the first 1 00000 values of both \( m = 10r + 4 \) and \( m = 10r + 6 \), 1314 and 6186 respectively are the sole exceptional numbers. Among the first 1 00000 values of \( m = 10r + 1 \), only 271, 1731 and 2251 are exceptional. Among the first 1 00000 values of \( m = 10r + 9 \), only 219 and 709 are exceptional. All exceptional numbers \( m \), where 6186 < \( m < 10 00000 \), are here seen to be of the form 10r + 2, 10r + 3, 10r + 7 or 10r + 8, and that appears to be another plausible conjecture for every exceptional \( m > 6186 \).

The method of computation was to obtain \( \binom{i+2}{3} + \binom{j+2}{3} + \binom{k+2}{3} + \binom{l+2}{3} \) for every \( i, j, k, l \geq 0 \) until one found for every \( m \leq 10 43999 \) either a representation as a \( \Sigma_i \) or that that \( m \) was exceptional. The calculation was begun upon the Univac Scientific Computer (ERA 1103) at the Convair Digital Computing Laboratory, the initial part re-run, and then those results were checked and continued upon the IBM 704 Digital Computer. At the start upon the ERA 1103, 500 words of high speed storage with 36 binary bits in each word permitted the investigation of 18000 numbers at a time. Then the IBM 704 had at its disposal 3000 words of high speed memory, each of the 36 binary bits in a word representing a number, so that 1 08000 numbers could be investigated at one time. The 1 08000 binary bits were filled with 1’s at the start, and a 0 introduced into the binary position of the word which represented a non-exceptional \( m \). After all possible combinations of \( i, j, k, l \) had been exhausted, each of the 3000 words was searched for binary bits that remained 1 and the exceptional numbers \( m \) corresponding to those bits were printed out. In choosing combinations of \( i, j, k, l \), repetitions due to symmetry were avoided, as well as combinations yielding an \( m \) that was either too large or too small for the group of 1 08000 numbers under consideration.

Those interested in actual mathematical proofs (which appear to be rather involved) may consult Dickson \[12\], \[13\] for earlier work, and Watson \[14\] for the sharpest results to date. It is rather amazing that the proved \( p \) in \( m = \Sigma_p \) for every \( m \) is \( p = 8 \), and no better than \( p = 8 \) for arbitrarily large \( m \), while the actual \( p \) (according to the evidence in this table) may be only 5 for every \( m \), and 4 for sufficiently large \( m \), less by 3 and 4 respectively. Considering the difficulty of the existing proof for \( p = 8 \) \[14\], one may well wonder, should \( p = 5 \) or \( p = 4 \) be the truly minimum values, for every \( m \), and \( m \) sufficiently large, respectively, how long the world must wait and how difficult and sharp the mathematical
tools must be, until the desired proofs would be found.

References [15] and [16] were called to the author's attention by K. A. Hirsch.

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7. —, "Table expressing every square up to one million as a sum of four non-negative tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 54, 1948, p. 830.

Empirical Theorems of Other Authors:

Mathematical Papers:
13. L. E. Dickson, *Modern Elementary Theory of Numbers*, University of Chicago Press, Chicago, 1939, Chap. VII, "Sums of nine values of a cubic function," p. 130-146. This contains the proof of the theorem that every integer is the sum of nine tetrahedrals, with further references to the most advanced work up to that time, that of R. D. James and L. K. Hua on the theorem that every sufficiently large integer is the sum of eight tetrahedrals.

GROUPS OF PRIMES HAVING MAXIMUM DENSITY

By John Leech

The following lists give groups of six or more primes which minimize the difference between first and last, the lists being complete for the range 50 to 100 00000. Four numbers out of nine can be prime, such as 191, 193, 197, 199. There are 897 such groups of four in the range. Five numbers out of thirteen can be prime; there are 318 such groups in the range. Six numbers out of seventeen can be prime, such as 97, 101, 103, 107, 109, 113; there are seventeen such groups in the range, centered on:

Received February 7, 1955. Due to misfiling in the *MTAC* office, this paper is appearing later than was scheduled; see *MTAC*, Review 110, v. 11, 1957, p. 274. Some of the results have meanwhile appeared in "On a generalization of the prime pair problem," by Herschel F. Smith, *MTAC*, v. 11, 1957, p. 274.