Simplified Calculation of the Exponential Integral

By James Miller and R. P. Hurst

Calculations of the negative exponential integral function [1]

\[
- \text{Ei} (- x) = \int_{x}^{\infty} \frac{e^{-t}}{t} \, dt
\]

are usually made using one method for \( x < 3 \), another for \( 3 < x < 20 \), and a third for \( x > 20 \). The intermediate region is of considerable interest for many applications, and it is here that the function is most difficult to compute. The Ruedenberg [2] and Kotani [3] methods are valid for any argument, but difficulties in application limit their usefulness to the intermediate region where less complicated methods fail.

The authors wish to present a simple method which is useful over the entire range of arguments. The method is based on a Taylor's expansion of the slowly varying exponential product function, \(- e^x \text{Ei} (- x)\). An interpolation formula is obtained by which this function may be computed from a table value, \(- e^{x_0} \text{Ei} (- x_0)\); then \(- \text{Ei} (- x)\) is obtained upon dividing by the exponential. The accuracy of the result is limited only by the accuracy of the table value and that of the computed exponential. The positive exponential integral

\[
\text{Ei} (x) = \int_{-\infty}^{x} \frac{e^t}{t} \, dt
\]

may be computed by analogous means.

The formulae for the negative exponential product function are developed as follows:

Let

\[
g(x) = - e^x \text{Ei} (- x),
\]

then

\[
g'(x) = g(x) - x^{-1}
\]

\[
g''(x) = g'(x) + x^{-2}
\]

\[
g'''(x) = g''(x) - 2! x^{-3}
\]

\[
g^{iv}(x) = g'''(x) + 3! x^{-4}.
\]

One may show by induction that

\[
g^{(n)}(x) = (-1)^n (n-1)! x^{-n}.
\]

The Taylor's expansion, then, is

\[
- e^x \text{Ei} (- x) = g(x_0) + g'(x_0) (x - x_0) + \frac{1}{2!} g''(x_0) (x - x_0)^2
\]

\[
+ \frac{1}{3!} g'''(x_0) (x - x_0)^3 + \cdots + \frac{1}{n!} g^{(n)}(x_0) (x - x_0)^n + R_{n+1}
\]

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where

\[ R_{n+1} = \frac{1}{(n+1)!} g^{(n+1)}(x_1)(x - x_0)^{n+1} \]

with \( x < x_1 < x_0 \).

The preceding equations may be combined to produce the simple formula

(5) \[ -e^x Ei(-x) = T_0 + \sum_{n=1}^{\infty} T_n \]

where

(6) \[ T_0 = g(x_0) = -e^{x_0} Ei(-x_0) \]

and the \( T \)'s are related by the recursion relation

(7) \[ T_n = \frac{x - x_0}{n} \left[ T_{n-1} + \frac{(x - x_0)^{n-1}}{(-x_0)^n} \right] \quad (n \geq 1). \]

For the positive exponential product function we have

(8) \[ e^{-x} Ei(x) = U_0 + \sum_{n=1}^{\infty} U_n \]

where

(9) \[ U_0 = e^{-x_0} Ei(x_0) \]

and

(10) \[ U_n = -\frac{x - x_0}{n} \left[ U_{n-1} + \frac{(x - x_0)^{n-1}}{(-x_0)^n} \right] \quad (n \geq 1). \]

Equations (5) through (10) are very convenient for computing the negative or positive exponential product functions where a table value is known. For example, with \( \left| \frac{x - x_0}{x_0} \right| \leq \frac{1}{50} \), convergence to 16 figures is obtained with 12 or fewer terms; convergence to 8 figures requires 6 or fewer terms. The spacing of the arguments in the accompanying table is such that this condition is fulfilled over most of the range \( 2.5 \leq x \leq 80 \).

To prevent loss of figures, \( x - x_0 \) should be negative for computing \(-e^x Ei(-x)\) and positive for computing \(e^{-x} Ei(x)\). The accompanying table of \(-e^x Ei(-x)\) was computed starting with the value for \( x = 80 \), where \(-e^x Ei(-x)\) was computed using the asymptotic expansion; each computed value became the \( T_0 \) for the next lower argument. The table of \( e^{-x} Ei(x) \) was computed going up from the value of the function for \( x = 0.2 \), where \( Ei(x) \) was computed using the usual MacLaurins series, then multiplied by \( e^{-x} \) to obtain the product function. All calculations were made on an IBM 650 computer using double precision, floating point arithmetic.
Simplified calculation of the exponential integral

Exponential integrals and exponential product functions

\begin{array}{ll}
\hline
x & e^{-x}E_1(x) \\
\hline
0.20 & \cdots \\
0.25 & \cdots \\
0.30 & \cdots \\
0.35 & \cdots \\
0.40 & \cdots \\
0.45 & \cdots \\
0.50 & \cdots \\
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0.60 & \cdots \\
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0.70 & \cdots \\
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In using this table each tabulated entry is to be multiplied by the power of 10
given in the parentheses located just to the right of the last significant digit listed.

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agement.

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1. See Frank E. Harris, “Tables of the exponential integral Ei(x),” MTAC, v. 11, 1957, p. 9–16 for a summary of these methods. The authors are grateful to Dr. Harris for the use of his materials prior to publication.