Two Theorems on Inverses of Finite Segments of the Generalized Hilbert Matrix

By Richard B. Smith

1. Introduction. A quick check on the inverse $S_n$ of finite segments of the generalized Hilbert matrix $H_n = (h_{ij})$, $h_{ij} = 1/(p + i + j - 1)$, $i, j = 1, 2, \cdots, n, p \neq -1, -2, \cdots, -(2n - 1)$, may be made by summation of the $n^2$ elements of the inverse. The summation requires complete accuracy.

2. Inversion of the Matrix. The inverse of this matrix has been derived by Savage and Lukacs [1] for $p = 0$ and by Collar [2] for nonnegative integer values of $p$. By using the method employed by Savage and Lukacs which is based on a formula in [4], the element in the $i$th row and $j$th column of $S_n$ is

$$S_{n}^{ij} = \frac{(-1)^{i+j}}{p + i + j - 1} \left[ \prod_{k=0}^{n-1} \frac{(p + i + k)(p + j + k)}{(i - 1)!(n - i)!(j - 1)!(n - j)!} \right]$$

where $p \neq -1, -2, \cdots, -(2n - 1)$.

3. Summation Identity. Let binomial coefficients of the form $C_{s}^{r} = 0$ for $s < 0$ and $s > r$. Then

$$\sum_{j=0}^{n} (-1)^{i} C_{j}^{i} C_{j+i}^{i+n} = (-1)^{n} C_{n+i}^{n}$$

follows from formula (27) in [5].

4. Theorem I. Let $S_{n}^{ij}$ be defined by (1), then

$$\sum_{i,j=1}^{n} S_{n}^{ij} = n(p + n)$$

where $p \neq -1, -2, \cdots, -(2n - 1)$.

When $n = 1$ or 2 equation (3) is easily verified. By assuming (3) for $n$ it remains to be shown that

$$\sum_{i,j=1}^{n+1} S_{n+1}^{ij} = (n + 1)(p + n + 1)$$

or

$$\sum_{i,j=1}^{n+1} S_{n+1}^{ij} - \sum_{i,j=1}^{n} S_{n}^{ij} = p + 2n + 1$$

where $p \neq -1, -2, \cdots, -[2(n + 1) - 1]$. By substitution of (1) for $S_{n+1}^{ij}$ and $S_{n}^{ij}$ in (4) we have

Received March 4, 1957; revised April 7, 1958.

* The author is indebted to the referee for this reference.
\[
\sum_{i, j=1}^{n} \frac{(-1)^{i+j}}{p + i + j - 1} \left( \prod_{k=0}^{i+j-1} (p + i + k)(p + j + k) \right) \\
\cdot \left[ \frac{(p + n + i)(p + n + j)}{(n + 1 - i)(n + 1 - j)} - 1 \right]
\]

\[
+ 2(-1)^{n+1} \frac{\prod_{k=0}^{n+1} (p + n + k)}{n!} \sum_{i=1}^{n} (-1)^{i} \frac{\prod_{k=0}^{i+j-1} (p + i + k)}{(i - 1)(n + 1 - i)!} 
\]

\[
+ (p + 2n + 1) \left[ \prod_{k=1}^{n} (p + n + k) \right]^{2} = (p + 2n + 1)
\]

\[
\cdot \left[ \sum_{i=1}^{n} (-1)^{i} \frac{\prod_{k=0}^{i+j-1} (p + i + k)}{(i - 1)(n + 1 - i)!} + (-1)^{n+1} \frac{\prod_{k=1}^{n} (p + n + k)}{n!} \right].
\]

Let \( i - 1 = j \) and this becomes

\[
(5) \quad (p + 2n + 1) \left[ \sum_{j=0}^{n} (-1)^{j} \frac{\prod_{k=1}^{j+1} (p + j + k)}{j!(n - j)!} \right]^{2}.
\]

Take the sum from (5) as the coefficient of \( x^{p+j} \) and consider

\[
f(x) = \sum_{j=0}^{n} (-1)^{j} \frac{\prod_{k=1}^{j+1} (p + j + k)}{j!(n - j)!} x^{p+j} = \sum_{j=0}^{n} (-1)^{j} \frac{D^{(n)}x^{p+j+n}}{j!(n - j)!},
\]

where \( D = d/dx \). Differentiating \( x^{p+j+n} \) as \( x^{p} \cdot x^{j+n} \) we have

\[
f(x) = \sum_{j=0}^{n} (-1)^{j} \frac{1}{j!(n - j)!} \sum_{i=0}^{n} C_{i}^{n} \left\{ D^{(i)}x^{p} \right\} \left\{ D^{(n-i)}x^{j+n} \right\}
\]

\[
= \sum_{i=0}^{n} C_{i}^{n} \left\{ D^{(i)}x^{p} \right\} \sum_{j=0}^{n} (-1)^{j} \frac{1}{j!(n - j)!} \left\{ D^{(n-i)}x^{j+n} \right\}
\]

\[
= x^{p} \sum_{j=0}^{n} (-1)^{j} \frac{(j + n)!}{j!(n - j)!} x^{j}
\]

\[
+ \sum_{i=1}^{n} C_{i}^{n} \prod_{k=0}^{i-1} (p - k) x^{p-i} \sum_{j=0}^{n} (-1)^{j} \frac{(j + n)!}{j!(n - j)!(j + i)!} x^{j+i}.
\]

Let \( x = 1 \). Then

\[
f(1) = \sum_{j=0}^{n} (-1)^{j} C_{j}^{n} C_{j}^{j+n} + \sum_{i=1}^{n} \prod_{k=0}^{i-1} (p - k) \sum_{j=0}^{n} (-1)^{j} C_{j}^{n} C_{j}^{j+i},
\]

and considering (2) we have \( f(1) = (-1)^{n} \). Thus (5) becomes \( p + 2n + 1 \).
5. **Theorem II.** Let $S_n^{ij}$ be defined by (1), then

$$
\sum_{i=1}^{n} S_n^{ij} = \sum_{i=1}^{n} S_n^{ji} = (-1)^{n+j} \prod_{k=0}^{n-1} \frac{(p+j+k)}{(j-1)!(n-j)!}
$$

where $j = 1, 2, \ldots, n$ and $p \neq -1, -2, \ldots, -(2n - 1)$.

The proof is similar to the one given for Theorem I.

Westinghouse Electric Corporation
Bettis Atomic Power Division
Pittsburgh, Pennsylvania


