Two Theorems on Inverses of Finite Segments of the Generalized Hilbert Matrix

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1. Introduction. A quick check on the inverse $S_n$ of finite segments of the generalized Hilbert matrix $H_n = (h_{ij})$, $h_{ij} = 1/(p + i + j - 1)$, $i, j = 1, 2, \ldots, n$, $p \neq -1, -2, \ldots, -(2n - 1)$, may be made by summation of the $n^2$ elements of the inverse. The summation requires complete accuracy.

2. Inversion of the Matrix. The inverse of this matrix has been derived by Savage and Lukacs [1] for $p = 0$ and by Collar [2] for nonnegative integer values of $p$. By using the method employed by Savage and Lukacs which is based on a formula in [4], the element in the $i$th row and $j$th column of $S_n$ is

$$S_n^{ij} = \frac{(-1)^{i+j}}{p + i + j - 1} \sum_{k=0}^{n-1} \frac{(p + i + k)(p + j + k)}{(i - 1)!(n - i)!(j - 1)!(n - j)!}$$

where $p \neq -1, -2, \ldots, -(2n - 1)$.

3. Summation Identity. Let binomial coefficients of the form $C_s^r = 0$ for $s < 0$ and $s > r$. Then

$$\sum_{j=0}^{n} (-1)^j C_j^n C_{j+i}^n = (-1)^n C_{n+i}^n$$

follows from formula (27) in [5].

4. Theorem I. Let $S_n^{ij}$ be defined by (1), then

$$\sum_{i,j=1}^{n} S_n^{ij} = n(p + n)$$

where $p \neq -1, -2, \ldots, -(2n - 1)$.

When $n = 1$ or 2 equation (3) is easily verified. By assuming (3) for $n$ it remains to be shown that

$$\sum_{i,j=1}^{n+1} S_{n+1}^{ij} = (n + 1)(p + n + 1)$$

$$= n(p + n) + p + 2n + 1,$$

or

$$\sum_{i,j=1}^{n+1} S_{n+1}^{ij} - \sum_{i,j=1}^{n} S_n^{ij} = p + 2n + 1$$

where $p \neq -1, -2, \ldots, -[2(n + 1) - 1]$. By substitution of (1) for $S_{n+1}^{ij}$ and $S_n^{ij}$ in (4) we have

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\[
\sum_{i,j=1}^{n} \frac{(-1)^{i+j}}{p + i + j - 1} \frac{1}{(i-1)!(n-i)!(j-1)!(n-j)!} \prod_{k=0}^{n-1} (p + i + k)(p + j + k) \cdot \left[ \frac{(p + n + i)(p + n + j)}{(n+1-i)(n+1-j)} - 1 \right]
+ 2(-1)^{n+1} \frac{\prod_{k=0}^{n-1} (p + n + k)}{n!} \sum_{i=1}^{n} (-1)^i \frac{\prod_{k=0}^{n-1} (p + i + k)}{(i-1)!(n+1-i)!}
+ (p + 2n + 1) \left[ \prod_{k=1}^{n} (p + n + k) \right]^{2n} \cdot \left[ \sum_{i=1}^{n} (-1)^i \frac{\prod_{k=0}^{n-1} (p + i + k)}{(i-1)!(n+1-i)!} + (-1)^{n+1} \frac{\prod_{k=0}^{n} (p + n + k)}{n!} \right].
\]

Let \(i - 1 = j\) and this becomes

\[
(p + 2n + 1) \left[ \sum_{j=0}^{n} (-1)^j \frac{\prod_{k=1}^{n} (p + j + k)}{j!(n-j)!} \right].
\]

Take the sum from (5) as the coefficient of \(x^{p+j}\) and consider

\[
f(x) = \sum_{j=0}^{n} (-1)^j \frac{\prod_{k=1}^{n} (p + j + k)}{j!(n-j)!} x^{p+j} = \sum_{j=0}^{n} (-1)^j \frac{D^{(n)} x^{p+j+n}}{j!(n-j)!},
\]

where \(D = d/dx\). Differentiating \(x^{p+j+n}\) as \(x^p x^{j+n}\) we have

\[
f(x) = \sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!} \sum_{i=0}^{n} C_i^n \{D^{(i)} x^p\} \{D^{(n-i)} x^{j+n}\}
= \sum_{i=0}^{n} C_i^n \{D^{(i)} x^p\} \sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!} \{D^{(n-i)} x^{j+n}\}
= x^p \sum_{j=0}^{n} (-1)^j \frac{(j+n)!}{j!j!(n-j)!} x^j
+ \sum_{i=0}^{n} C_i^n \prod_{k=0}^{i-1} (p-k) \sum_{j=0}^{n} (-1)^j \frac{(j+n)!}{j!(n-j)!(j+i)!} x^{j+i}.
\]

Let \(x = 1\). Then

\[
f(1) = \sum_{j=0}^{n} (-1)^j C_j^n C_j^{j+n} + \sum_{i=0}^{n} \prod_{k=0}^{i-1} (p-k) \sum_{j=0}^{n} (-1)^j C_j^n C_j^{j+i},
\]

and considering (2) we have \(f(1) = (-1)^n\). Thus (5) becomes \(p + 2n + 1\).
5. **Theorem II.** Let $S_n^{ij}$ be defined by (1), then

$$
\sum_{i=1}^{n} S_n^{ij} = \sum_{i=1}^{n} S_n^{i\bar{j}} = (-1)^{n+j} \prod_{k=0}^{n-1} (p+j+k) \frac{1}{(j-1)!(n-j)!}
$$

where $j = 1, 2, \ldots, n$ and $p \neq -1, -2, \ldots, -(2n - 1)$.

The proof is similar to the one given for Theorem I.