On the Inversion of Certain Matrices

By Samuel Schechter

1. Introduction. Let $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n$ be $2n$ distinct, but otherwise arbitrary, complex numbers. For the matrix H, of order n,

$$H = \left\{\frac{1}{a_i - b_j}\right\} \qquad \text{with } 1 \leqq i, j \leqq n \tag{1}$$

let $G = H^{-1} = \{c_{ij}\}$. (The indices i, j, k will range from 1 to n unless it is specified otherwise.) If, for some constant p,

$$a_i - b_j = i + j - 1 + p \neq 0, \tag{2}$$

then H is a segment of the well known generalized Hilbert matrix, and in this case formulas for c_{ij} have been given by Savage and Lukacs [4], Smith [6] and Collar [1]. For

$$a_i - b_j = i - j + p, \tag{3}$$

Linfoot and Shepherd [3] and Collar [2] give formulas for c_{ij} and in both cases Collar exhibits diagonal matrices D, K such that $G = DH^TK$. Collar [2] and Smith also evaluate the quantities $\sum_{i,j} c_{ij}$, $\sum_j c_{ij}$.

These authors make use, in most cases, of the formula for the determinant [5]

$$\det H = \frac{\prod_{j>k} (a_j - a_k)(b_k - b_j)}{\prod_{j,k} (a_j - b_k)} \tag{4}$$

or require the evaluation of certain involved series.

The formulas of Collar and Smith are extended here to the general case (1). The method to be used does not depend on (4) but simply on Lagrange's interpolation formula. Indeed (4) comes out as a by-product of formula (17) given below.

2. Formula for the inverse. Let

$$A(x) = \prod_i (x - a_i), \qquad B(x) = \prod_i (x - b_i) \tag{5}$$

and denote the fundamental polynomials of the Lagrangian interpolation corresponding to the a_i and b_i, respectively, by

$$A_i(x) = \frac{A(x)}{A'(a_i)(x - a_i)}, \qquad B_i(x) = \frac{B(x)}{B'(b_i)(x - b_i)} \tag{6}$$

where prime denotes differentiation. We then have

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THEOREM 1. The elements of G are given by

$$c_{ij} = (a_j - b_i)A_j(b_i)B_i(a_j) \tag{7}$$

and if H is symmetric

$$c_{ij} = (a_j - b_i)A_j(b_i)A_i(b_j). \tag{8}$$

Proof: For any polynomial $p(x)$ of degree $n-1$ we may write

$$p(x) = \sum_i p(a_i)A_i(x) \tag{9}$$

or

$$\frac{p(x)}{A(x)} = \sum_i \frac{c_i}{x - a_i} \tag{10}$$

where

$$c_i = \frac{p(a_i)}{A'(a_i)}. \tag{11}$$

Now let $p(x) = -B_k(x)A(b_k) = p_k(x)$ and

$$c_{ki} = \frac{p_k(a_i)}{A'(a_i)}.$$

Then from (10) we obtain

$$\frac{B_k(x)A(b_k)}{A(x)} = \sum_i \frac{c_{ki}}{a_i - x}. \tag{12}$$

If we set $x = b_j$ then, since $B_k(b_j) = \delta_{kj}$, the Kronecker delta, we get that

$$\delta_{kj} = \sum_i \frac{c_{ki}}{a_i - b_j}.$$

Thus c_{ki} gives the desired inverse element, that is,

$$c_{ki} = \frac{-B_k(a_i)A(b_k)}{A'(a_i)} = (a_i - b_k)B_k(a_i)A_i(b_k).$$

If H is symmetric: $a_i - b_j = a_j - b_i$, it follows that $B_i(a_j) = A_i(b_j)$ and the theorem is proved.

We now obtain a simple formula for the sum of the c_{ij}. This quantity arises in problems of aerodynamics [2] and its simplicity allows it also to be used as a check on some alleged inverse.

COROLLARY.

$$\sum_{i,j} c_{ij} = \sum_k (a_k - b_k) = s \tag{13}$$

Proof. If we apply (9) to $p(x) = B_j(x)$ and set $x = b_k$ we obtain that

$$\sum_i B_j(a_i)A_i(b_k) = \delta_{jk}. \tag{14}$$

By symmetry this is also valid if A, a and B, b are interchanged. Thus we get that

$$s = \sum_j a_j \sum_i A_j(b_i)B_i(a_j) - \sum_i b_i \sum_j A_j(b_i)B_i(a_j) = \sum_j a_j - \sum_i b_i.$$

For the special case of the Hilbert segment (2), formula (13) gives Smith's [6] formula: $s = n(p + n)$ and for the matrix of (3) we get the formula of Collar [2]: $s = pn$.

3. Row and column sums. Let the row and column sums of G be, respectively,

$$\sum_j c_{ij} = \alpha_i\,, \qquad \sum_i c_{ij} = \beta_j \tag{15}$$

and define the diagonal matrices D_α, D_β by

$$D_\alpha = [\alpha_1\,, \alpha_2\,, \cdots, \alpha_n]$$

$$D_\beta = [\beta_1\,, \beta_2\,, \cdots, \beta_n].$$

THEOREM 2. The matrix H then satisfies the following relations:

$$\alpha_i = -\frac{A(b_i)}{B'(b_i)}, \qquad \beta_j = \frac{B(a_j)}{A'(a_j)}, \tag{16}$$

$$H^{-1} = D_\alpha H^T D_\beta. \tag{17}$$

Proof: Assuming (16) to be true we note that c_{ij} can be written in the form

$$c_{ij} = \frac{1}{b_i - a_j}\frac{A(b_i)}{B'(b_i)}\frac{B(a_j)}{A'(a_j)} = \frac{\alpha_i\,\beta_j}{a_j - b_i} \tag{18}$$

which gives (17).

To prove the formula for α_i in (16) we need only show that

$$\sum_j \frac{B(a_j)}{(a_j - b_i)A'(a_j)} = 1. \tag{19}$$

However for any function $f(x)$ we may write

$$f[a_1, a_2, \cdots, a_n] \equiv \delta^{(n-1)}f(x) = \sum_j \frac{f(a_j)}{A'(a_j)} \tag{20}$$

where the left side of (20) is the $(n - 1)$th divided difference of $f(x)$ with respect to the a_k (see Milne-Thomson [7] p. 9). If we apply (20) to the polynomial $f(x) = B(x)/(x - b_i)$ of degree $n - 1$ we have that $\delta^{(n-1}f(x) \equiv$ constant. Since the coefficient of x^{n-1} in $f(x)$ is 1, (19) is proved.

An alternate proof of (19) may be obtained, without divided differences, by noting that the left hand side of (19) is the sum of the residues of the function $B(x)/A(x)(x - b_i)$ at the a_j. (This, in fact, follows readily from (10).) However for a sufficiently large circle C about the origin in the complex x-plane

$$\frac{1}{2\pi i}\int_C \frac{B(x)}{(x - b_i)A(x)}\,dx = 1$$

which proves (19).

The proof for the β_j is obtained in the same manner from (19) with the roles of A, a and B, b interchanged, and the theorem is proved. (The formula (19), incidentally, represents an extension of one of the formulas of Collar [2] for the sum of a series. Extensions of other formulas given in [2] can likewise be obtained from (20) by specializing $f(x)$.)

4. Remarks.

1.) We note that if H is symmetric then $\alpha_i = \beta_i$ and if we set $D = D_\alpha = D_\beta$ then

$$G = DHD \tag{21}$$

2.) From (16) and (17) one immediately obtains the formula for the determinant of H up to a sign. That is

$$(\det H)^2 = \prod_i \frac{1}{\alpha_i \, \beta_i} \tag{22}$$

and for H symmetric we get that

$$\det H = (-1)^{[n(n-1)]/2} \prod_i \frac{A'(a_i)}{B(a_i)}. \tag{23}$$

The signs may readily be determined, by induction, by using (6) and the formula for the $(n + 1, n + 1)$ element of G_{n+1} :

$$c_{n+1,\, n+1} \cdot \det H_{n+1} = \det H_n$$

where

$$H_n = H \quad \text{and} \quad H_{n+1} = \left\{\frac{1}{a_i - b_j}\right\} \text{ with } 1 \leqq i,j \leqq n + 1.$$

3.) Formula (8) may also be applied to the problem of obtaining a least-squares fit of a function $f(x)$ on, say, (0, 1) by a function of the form $\sum_i \gamma_i x^{a_i}$. (All variables are here assumed to be real.) The normal equations for this problem yield a matrix of the form (1) with $b_i = -a_i - 1$, and the γ_i may be obtained by applying G to the moments of $f(x)$ with respect to the x^{a_i}.

For fitting a function of two variables the problem to determine γ_{ij} such that

$$\int_0^1 \int_0^1 [f(x, y) - \sum_{i,j} \gamma_{ij} \, x^{a_i} y^{a_i'}]^2 dxdx = \min$$

is solved by

$$K = DHDFD'H'D'$$

where

$$H = \left\{\frac{1}{a_i + a_j + 1}\right\}, \qquad H' = \left\{\frac{1}{a_i' + a_j' + 1}\right\}, \qquad K = \{\gamma_{ij}\},$$

$$F = \left\{\int_0^1 \int_0^1 f(x, y) x^{a_i} y^{a_i'} dxdy\right\},$$

and D, D' are the diagonal matrices corresponding to H, H', respectively, defined above.

Although the elements c_{ij} get quite large in the case of the Hilbert matrix, it may happen that for suitable choices of the a_i this may not be the case.

An explicit solution for γ_i can also be obtained from (7) for the equations

$$\int_0^1 \left(f(x) - \sum_i \gamma_i x^{a_i}\right) x^{a_k'} dx = 0, \qquad k = 1, 2, \cdots, n$$

given the moments of $f(x)$ with respect to the $x^{a_i'}$.

4.) If H is real and symmetric and if $a_i > b_i$, $i = 1, 2, \cdots, n$ then it follows from (4) that all the principal minor determinants are positive and H is positive definite. In this case $a_i > b_j$ for all i and j; that is, all the elements of H are positive, since if $a_i < a_j$, then $0 < a_i - b_i < a_j - b_i$. Thus $B(a_i) > 0$ and, since $A(x)$ has n simple zeros at the a_i, $A'(a_i)$ alternate in sign. From (8) it then follows that, if $a_i < a_j$ for $i < j$ then $(-1)^{i+j}c_{ij} > 0$ so that G has the same checkerboard distribution of signs as the inverse of the Hilbert segment. From (16) it follows that in this case the α_i also alternate in sign and that $(-1)^{j+n}\alpha_j > 0$.

Let λ and μ denote the smallest and largest eigenvalue of H respectively, and let

$$M_i = \min_{k, k \neq i} \left(\frac{a_k - b_i}{| a_k - a_i |}\right).$$

Then it follows readily that $| A_i(b_i) | \geqq M_i^{n-1} > 1$ and that

$$\lambda \leqq \min_i \frac{1}{c_{ii}} \leqq \min_i \left(\frac{M_i^{2-2n}}{a_i - b_i}\right)$$

If the a_i increase with i then

$$\min_i (\sum_j (-1)^{i+j} c_{ij})^{-1} \leqq \lambda.$$

We then get for the P-condition [8] of H:

$$\frac{\mu}{\lambda} \geqq \max_i \left(\frac{1}{a_i - b_i}\right) \cdot \max_i c_{ii} \geqq \max_i [A_i(b_i)]^2 \geqq \max_i M_i^{2n-2}$$

so that the P-condition of H may get very large. This number has been estimated for the Hilbert segment (2) with $p = 0$ by Todd [8].

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