On the Inversion of Certain Matrices

By Samuel Schechter

1. Introduction. Let \( a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \) be \( 2n \) distinct, but otherwise arbitrary, complex numbers. For the matrix \( H \), of order \( n \),

\[
H = \begin{bmatrix}
1 \\
\frac{1}{a_i - b_j}
\end{bmatrix}
\end{bmatrix}
\end{equation}

with \( 1 \leq i, j \leq n \)

let \( G = H^{-1} = \{c_{ij}\} \). (The indices \( i, j, k \) will range from 1 to \( n \) unless it is specified otherwise.) If, for some constant \( p \),

\[
a_i - b_j = i + j - 1 + p \neq 0,
\]

then \( H \) is a segment of the well known generalized Hilbert matrix, and in this case formulas for \( c_{ij} \) have been given by Savage and Lukacs \[4\], Smith \[6\] and Collar \[1\]. For

\[
a_i - b_j = i - j + p,
\]

Linfoot and Shepherd \[3\] and Collar \[2\] give formulas for \( c_{ij} \) and in both cases Collar exhibits diagonal matrices \( D, K \) such that \( G = DHK \). Collar \[2\] and Smith also evaluate the quantities \( \sum_{i,j} c_{ij}, \sum c_{ij} \).

These authors make use, in most cases, of the formula for the determinant \[5\]

\[
det H = \prod_{j>k} (a_j - a_k)(b_j - b_k)
\end{equation}

or require the evaluation of certain involved series.

The formulas of Collar and Smith are extended here to the general case (1). The method to be used does not depend on (4) but simply on Lagrange's interpolation formula. Indeed (4) comes out as a by-product of formula (17) given below.

2. Formula for the inverse. Let

\[
A(x) = \prod (x - a_i), \quad B(x) = \prod (x - b_i)
\end{equation}

and denote the fundamental polynomials of the Lagrangian interpolation corresponding to the \( a_i \) and \( b_i \), respectively, by

\[
A_i(x) = \frac{A(x)}{A'(a_i)(x - a_i)}, \quad B_i(x) = \frac{B(x)}{B'(b_i)(x - b_i)}
\end{equation}

where prime denotes differentiation. We then have

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Theorem 1. The elements of $G$ are given by

$$
(7) \quad c_{ij} = (a_j - b_i)A_j(b_i)B_i(a_j)
$$

and if $H$ is symmetric

$$
(8) \quad c_{ij} = (a_j - b_i)A_j(b_i)A_i(b_j).
$$

Proof: For any polynomial $p(x)$ of degree $n - 1$ we may write

$$
(9) \quad p(x) = \sum_i p(a_i)A_i(x)
$$
or

$$
(10) \quad \frac{p(x)}{A(x)} = \sum_i \frac{c_i}{x - a_i}
$$

where

$$
(11) \quad c_i = \frac{p(a_i)}{A'(a_i)}.
$$

Now let $p(x) = -B_k(x)A(b_k) = p_k(x)$ and

$$
\xi_k = \frac{p_k(a_i)}{A'(a_i)}.
$$

Then from (10) we obtain

$$
(12) \quad \frac{B_k(x)A(b_k)}{A(x)} = \sum_i \frac{c_{ki}}{a_i - x}.
$$

If we set $x = b_j$ then, since $B_k(b_j) = \delta_{kj}$, the Kronecker delta, we get that

$$
\delta_{kj} = \sum_i \frac{c_{ki}}{a_i - b_j}.
$$

Thus $c_{ki}$ gives the desired inverse element, that is,

$$
(13) \quad c_{ki} = \frac{-B_k(a_i)A(b_k)}{A'(a_i)} = (a_i - b_k)B_k(a_i)A_i(b_k).
$$

If $H$ is symmetric: $a_i - b_j = a_j - b_i$, it follows that $B_i(a_j) = A_i(b_j)$ and the theorem is proved.

We now obtain a simple formula for the sum of the $c_{ij}$. This quantity arises in problems of aerodynamics [2] and its simplicity allows it also to be used as a check on some alleged inverse.

Corollary.

$$
(14) \quad \sum_{i,j} c_{ij} = \sum_k (a_k - b_k) = s
$$

Proof. If we apply (9) to $p(x) = B_j(x)$ and set $x = b_k$ we obtain that

$$
(15) \quad \sum_i B_j(a_i)A_i(b_k) = \delta_{jk}.
$$

By symmetry this is also valid if $A$, $a$ and $B$, $b$ are interchanged. Thus we get that

$$
(16) \quad s = \sum_j a_j \sum_i A_j(b_i)B_i(a_j) - \sum_i b_i \sum_j A_j(b_i)B_i(a_j) = \sum_i a_i - \sum_i b_i.
$$
For the special case of the Hilbert segment (2), formula (13) gives Smith's
[6] formula: \( s = n(p + n) \) and for the matrix of (3) we get the formula of Collar
[2]: \( s = pn \).

3. Row and column sums. Let the row and column sums of \( G \) be, respectively,
\[
\sum_j c_{ij} = \alpha_i, \quad \sum_i c_{ij} = \beta_j
\]
and define the diagonal matrices \( D_\alpha, D_\beta \) by
\[
D_\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n] \\
D_\beta = [\beta_1, \beta_2, \cdots, \beta_n].
\]

**Theorem 2.** The matrix \( H \) then satisfies the following relations:
\[
(16) \quad \alpha_i = - \frac{A(b_i)}{B'(b_i)}, \quad \beta_j = \frac{B(a_j)}{A'(a_j)},
\]
\[
(17) \quad H^{-1} = D_\alpha HTD_\beta.
\]

**Proof:** Assuming (16) to be true we note that \( c_{ij} \) can be written in the form
\[
(18) \quad c_{ij} = \frac{1}{b_i - a_j} \frac{A(b_i) B(a_j)}{B'(b_i) A'(a_j)} = \frac{\alpha_i \beta_j}{a_j - b_i},
\]
which gives (17).

To prove the formula for \( \alpha_i \) in (16) we need only show that
\[
(19) \quad \sum_j \frac{B(a_j)}{(a_j - b_i) A'(a_j)} = 1.
\]
However for any function \( f(x) \) we may write
\[
(20) \quad f[a_1, a_2, \cdots, a_n] = \delta^{(n-1)} f(x) = \sum_j \frac{f(a_j)}{A'(a_j)}
\]
where the left side of (20) is the \((n - 1)\)th divided difference of \( f(x) \) with respect
to the \( a_k \) (see Milne-Thomson [7] p. 9). If we apply (20) to the polynomial \( f(x) = B(x)/(x - b_i) \) of degree \( n - 1 \) we have that \( \delta^{(n-1)} f(x) \equiv \text{constant} \). Since the
coefficient of \( x^{n-1} \) in \( f(x) \) is 1, (19) is proved.

An alternate proof of (19) may be obtained, without divided differences, by
noting that the left hand side of (19) is the sum of the residues of the function
\( B(x)/A(x)(x - b_i) \) at the \( a_j \). (This, in fact, follows readily from (10).) How-
ever for a sufficiently large circle \( C \) about the origin in the complex \( x \)-plane
\[
\frac{1}{2\pi i} \int_C \frac{B(x)}{(x - b_i) A(x)} \, dx = 1
\]
which proves (19).

The proof for the \( \beta_j \) is obtained in the same manner from (19) with the roles
of \( A, a \) and \( B, b \) interchanged, and the theorem is proved. (The formula (19),
incidentally, represents an extension of one of the formulas of Collar [2] for the
sum of a series. Extensions of other formulas given in [2] can likewise be obtained from (20) by specializing \( f(x) \).

4. Remarks.

1.) We note that if \( H \) is symmetric then \( \alpha_i = \beta_i \) and if we set \( D = D_\alpha = D_\beta \) then

\[
G = DHD
\]

2.) From (16) and (17) one immediately obtains the formula for the determinant of \( H \) up to a sign. That is

\[
(\det H)^2 = \prod_i \frac{1}{\alpha_i \beta_i}
\]

and for \( H \) symmetric we get that

\[
\det H = (-1)^{\lfloor n(n-1)/2 \rfloor} \prod_i \frac{A'(a_i)}{B(a_i)}.
\]

The signs may readily be determined, by induction, by using (6) and the formula for the \((n + 1, n + 1)\) element of \( G_{n+1} \):

\[
c_{n+1, n+1} \cdot \det H_{n+1} = \det H_n
\]

where

\[
H_n = H \quad \text{and} \quad H_{n+1} = \left\{ \frac{1}{a_i - b_j} \right\} \text{ with } 1 \leq i, j \leq n + 1.
\]

3.) Formula (8) may also be applied to the problem of obtaining a least-squares fit of a function \( f(x) \) on, say, \((0, 1)\) by a function of the form \( \sum \gamma_i x^{a_i} \). (All variables are here assumed to be real.) The normal equations for this problem yield a matrix of the form (1) with \( b_i = -a_i - 1 \), and the \( \gamma_i \) may be obtained by applying \( G \) to the moments of \( f(x) \) with respect to the \( x^{a_i} \).

For fitting a function of two variables the problem to determine \( \gamma_{ij} \) such that

\[
\int_0^1 \int_0^1 [f(x, y) - \sum_{i,j} \gamma_{ij} x^{a_i} y^{a_j}]^2 dx dy = \min
\]

is solved by

\[
K = DHDFD'H'D'
\]

where

\[
H = \left\{ \frac{1}{a_i + a_j + 1} \right\}, \quad H' = \left\{ \frac{1}{a_i' + a_j' + 1} \right\}, \quad K = \left\{ \gamma_{ij} \right\},
\]

\[
F = \left\{ \int_0^1 \int_0^1 f(x, y) x^{a_i} y^{a_j} dx dy \right\},
\]

and \( D, D' \) are the diagonal matrices corresponding to \( H, H' \), respectively, defined above.
Although the elements $c_{ij}$ get quite large in the case of the Hilbert matrix, it
can happen that for suitable choices of the $a_i$ this may not be the case.

An explicit solution for $\gamma_i$ can also be obtained from (7) for the equations
\[
\int_0^1 \left( f(x) - \sum_i \gamma_i x^i \right) x^{n-k} dx = 0, \quad k = 1, 2, \ldots, n
\]
given the moments of $f(x)$ with respect to the $x^{a_i}$.

4.) If $H$ is real and symmetric and if $a_i > b_i$, $i = 1, 2, \ldots, n$ then it follows
from (4) that all the principal minor determinants are positive and $H$ is positive
definite. In this case $a_i > b_j$ for all $i$ and $j$; that is, all the elements of $H$ are positive,
since if $a_i < a_j$, then $0 < a_i - b_i < a_j - b_i$. Thus $B(a_i) > 0$ and, since
$A(x)$ has $n$ simple zeros at the $a_i$, $A'(a_i)$ alternate in sign. From (8) it then
follows that, if $a_i < a_j$ for $i < j$ then $(-1)^{i+j}c_{ij} > 0$ so that $G$ has the same checker-
board distribution of signs as the inverse of the Hilbert segment. From (16) it
then follows that in this case the $a_i$ also alternate in sign and that $(-1)^{i+j}a_j > 0$.

Let $\lambda$ and $\mu$ denote the smallest and largest eigenvalue of $H$ respectively, and let
\[
M_i = \min_{k \neq i} \left( \frac{a_k - b_i}{a_k - a_i} \right).
\]

Then it follows readily that $|A_i(b_i)| \geq M_i^{n-1} > 1$ and that
\[
\lambda \leq \min_i \frac{1}{c_{ii}} \leq \min_i \left( \frac{M_i^{2n-2}}{a_i - b_i} \right)
\]

If the $a_i$ increase with $i$ then
\[
\min_i (\sum_j (-1)^{i+j}c_{ij})^{-1} \leq \lambda.
\]

We then get for the $P$-condition [8] of $H$:
\[
\frac{\mu}{\lambda} \geq \max_i \left( \frac{1}{a_i - b_i} \right) \cdot \max_i c_{ii} \geq \max_i [A_i(b_i)]^2 \geq \max_i M_i^{2n-2}
\]

so that the $P$-condition of $H$ may get very large. This number has been estimated
for the Hilbert segment (2) with $p = 0$ by Todd [8].

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