Calculation of Gamma Functions to High Accuracy

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The two constants, $\Gamma\left(\frac{1}{2}\right)$ and $\Gamma\left(\frac{3}{2}\right)$ have been calculated to 35 decimal places on the Cambridge Computer by means of an interpretive routine that treats floating-point numbers of 37 significant digits [1, 2].

Stirling's asymptotic expansion for $\ln \Gamma(x)$ can be written

$$\ln \Gamma(x) \sim (x - \frac{1}{2}) \ln x - x + \frac{1}{2} \ln 2\pi + \sum_{r=1}^{\infty} \frac{C_r}{x^{2r-1}}.$$  

With the remainder of the series made very small, the accuracy of $\ln \Gamma(x)$ depends primarily on the accuracy of $\ln 2\pi$, $\ln x$, and the $C_r$. Uhler [3, 4] has published the $\ln x$ for all primes through 101, as well as the $\ln \pi$, to more than 100 significant figures. Uhler has also calculated the $C_r$ to over 100 significant figures [5].

The recursion formula for the Gamma function

$$(2) \quad \Gamma(x + 1) = x\Gamma(x)$$

can be extended to

$$(3) \quad \Gamma(x + n) = (x)_n \Gamma(x)$$

where $(x)_n$ is the Pochhammer-Barnes symbol

$$\text{(4)} \quad (x)_n = (x + n - 1)(x + n - 2) \cdots (x + 1).$$

If we take the logarithm of Eq. (3),

$$\text{(5)} \quad \ln \Gamma(x + n) = \ln \Gamma(x) + \sum_{j=0}^{n-1} \ln (x + j).$$

Substituting Eq. (1) into Eq. (5) gives, after solving for $\ln \Gamma(x)$,

$$\text{(6)} \quad \ln \Gamma(x) \sim \lambda(x) + \sum_{r=1}^{\infty} \frac{C_r}{(x + n)^{2r-1}},$$

where

$$\text{(7)} \quad \lambda(x) = (x + n - \frac{1}{2}) \ln (x + n) - (x + n) + \frac{1}{2} \ln 2\pi - \sum_{j=0}^{n-1} \ln (x + j).$$

The term $\lambda(x)$ is calculated quite easily by hand, and it is not difficult to calculate $\ln \Gamma(x)$ from Eq. (6) with a digital computer.

To calculate $\Gamma(x)$, let

$$\text{(8)} \quad \Gamma(x) = e^{\lambda(x)} e^{\ln \Gamma(x) - x}$$

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where

\[
\psi = \pm 0.1n
\]

is the exponent of \( e \) nearest in value to \( \ln \Gamma(x) \). Values of \( e^{\pm 0.1n} \), \( n \) being any integer from 0 to 10, have been tabulated by Van Orstrand [6].

(10) Since

\[
| \ln \Gamma(x) - \psi | \leq 0.05
\]

it will always be possible to calculate \( e^{\ln\Gamma(x)} \) by means of the exponential power series expansion using very few terms.

These techniques led to the following values:

\[
\begin{align*}
\Gamma(\frac{1}{2}) &= 2.6789385347077476336556929409746776 \\
\Gamma(\frac{3}{4}) &= 1.3541179394264004169452880281545138 \\
\ln \Gamma(\frac{1}{2}) &= .98542064692776706918717403697796139 \\
\ln \Gamma(\frac{3}{4}) &= .30315027514752356867586281737201104.
\end{align*}
\]

It was possible to examine round-off and truncation errors at each step in the calculation. The final relative error was less than \( \pm 3 \times 10^{-35} \) in each case.

As a final independent check, the values of \( \Gamma(\frac{1}{2}) \) and \( \Gamma(\frac{3}{4}) \) were put into

\[
\Gamma(\frac{1}{2})\Gamma(\frac{3}{4}) = \frac{2\pi}{\sqrt{3}}.
\]

The error analysis was consistent with this identity.

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1. B. Zondek, "The values of \( \Gamma(\frac{1}{2}) \) and \( \Gamma(\frac{3}{4}) \) and their logarithms accurate to 28 decimals," \textit{MTAC}, v. 9, 1955, p. 24-25.