Products of Laguerre Polynomials

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1. Problems occasionally arise in theoretical physics where one wishes to express the product of two linear combinations of Laguerre polynomials as a linear series of these same polynomials. The purpose of this note is to investigate the coefficients C_{rst} where the following definitions are used:

$$L_n(x) = \sum_{r=0}^{n} (-1)^n \binom{n}{r} x^r/r! \tag{1}$$

$$L_r(x)L_s(x) = \sum_{t=|r-s|}^{r+s} C_{rst} L_t(x) \tag{2}$$

The limits of the sum in (2) follow, in fact, from repeated application of the recurrence formula for the L_n's. This can be written [1] in the form

$$xL_n(x) = -(n+1)L_{n+1}(x) + (2n+1)L_n(x) - nL_{n-1}(x). \tag{3}$$

It follows from the orthogonality properties of Laguerre polynomials that

$$C_{rst} = \int_0^\infty e^{-x} L_r(x) L_s(x) L_t(x)\, dx \tag{4}$$

and, in particular, is symmetric in r, s, t. A closed formula has been obtained for C_{rst} by Watson [4]. We begin by obtaining the same formula by a very simple argument. In §3 we derive a simple recurrence relation suitable for rapidly generating the coefficients as needed when working with a high speed computing machine. This will be seen to be more useful in practice than the formal expression in equation (7).

2. It is known [2] that the Laplace transform of $L_n(x)$ is $p^{-n-1}(p-1)^n$ while that of $L_h(x)L_k(x)$ is

$$\binom{h+k}{h} \frac{(p-1)^{h+k}}{p^{h+k+1}} {}_2F_1\left[-h, -k; -h-k; \frac{p(p-2)}{(p-1)^2}\right].$$

Hence, taking Laplace transforms of both sides of (2), we get

$$\binom{r+s}{r} p^{-r-s-1}(p-1)^{r+s} {}_2F_1\left[-r, -s; -r-s; \frac{p(p-2)}{(p-1)^2}\right] = \sum_t C_{rst} p^{-t-1}(p-1)^t. \tag{5}$$

With $q = 1 - p^{-1}$ we have

$$\sum_t C_{rst} q^t = \binom{r+s}{r} q^{r+s} {}_2F_1[-r, -s; -r-s; (2q-1)/q^2]. \tag{6}$$

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Comparing coefficients of q^t in (6) gives us

$$
\begin{aligned}
C_{rst} &= \sum_n \frac{(r+s-n)!}{n!(r-n)!(s-n)!}(-2)^{2n-r-s+t}\binom{n}{2n-r-s+t} \\
&= (-\tfrac{1}{2})^p \sum_n 2^{2n} \frac{(r+s-n)!}{(r-n)!(s-n)!(2n-p)!(p-n)!}
\end{aligned} \tag{7}
$$

where we have written $p = r + s - t$. This is equivalent to Watson's formula.

The limits of the sum in (7) are defined by the requirement that none of the arguments of the factorials can be negative. It is easily confirmed from this, incidentally, that there can be no terms in the sum if t lies outside the range $(|r - s|, r + s)$.

3. To establish a recurrence relation we base ourselves on the well-known result [3] that, if $u(x)$, $v(x)$ satisfy the normalized differential equations

$$
\begin{aligned}
u''(x) + I(x)u(x) &= 0 \\
v''(x) + J(x)v(x) &= 0,
\end{aligned} \tag{8}
$$

then $y = uv$ satisfies the equation

$$
\frac{d}{dx}\left[\frac{y''' + 2(I+J)y' + (I'+J')y}{I-J}\right] + (I-J)y = 0 \tag{9}
$$

provided that $I \not\equiv J$. In case $I \equiv J$, y satisfies the third order equation

$$
y''' + 4Iy' + 2I'y = 0. \tag{10}
$$

Applying this result to the differential equation satisfied by Laguerre polynomials, after normalization, we obtain the following equation for $y = L_r(x)L_s(x)(r \geqq s)$

$$
\begin{aligned}
\mathsf{D}(y) &= x^2y^{(\mathrm{iv})} + x(5-4x)y''' + [4 + (2\sigma - 15)x + 5x^2]y'' \\
&\quad + [(3\sigma - 8) - 4(\sigma - 3)x - 2x^2]y' + [\delta^2 - 3(\sigma - 1) + 2(\sigma - 1)x]y = 0
\end{aligned} \tag{11}
$$

where

$$
\begin{cases} \sigma = r + s + 1 \\ \delta = r - s. \end{cases} \tag{12}
$$

In fact, equation (11) holds whether or not $r = s$. We now substitute from (2) into $x\mathsf{D}(y)$, making use of the following formulas:

$$
xL_t'' = -(1-x)L_t' - tL_t, \tag{13}
$$

$$
x^2L_t''' = [2 - (t+2)x + x^2]L_t' + t(2-x)L_t, \tag{14}
$$

$$
\begin{aligned}
x^3L_t^{(\mathrm{iv})} = -\{6 - 2(2t+3)x + (2t+3)x^2 - x^3\}L_t' \\
- t\{6 - (t+3)x + x^2\}L_t.
\end{aligned} \tag{15}
$$

Of these equations, (13) is simply the differential equation of L_t, while (14) and (15) are obtained from it by differentiation followed by substitution from (13) itself. We thus obtain, after some reduction,

$$
x\mathsf{D}(L_t) = \tau x(1-2x)L_t' + \{2\tau x^2 + x[\delta^2 - t^2 - \tau(2t+3)]\}L_t \tag{16}
$$

where

$$\tau = \sigma - t - 1 = r + s - t \tag{17}$$

But, [1],

$$xL_t' = (t + 1)L_{t+1} - (t + 1 - x)L_t . \tag{18}$$

Substituting from (18) into (16) and making repeated use of (3) leads finally to

$$\begin{aligned} x\mathsf{D}(L_t) = {} & 2\tau(t + 1)(t + 2)L_{t+2} - (t + 1)[\tau(4t + 5) + (\delta^2 - t^2)]L_{t+1} \\ & + (2t + 1)[(t + 1)\tau + \delta^2 - t^2]L_t - t(\delta^2 - t^2)L_{t-1} . \end{aligned} \tag{19}$$

For fixed r, s write $C_{rst} = A_t$. It follows from (19) that

$$x\mathsf{D}(\sum_t A_t L_t) = \sum B_t L_t \tag{20}$$

where

$$\begin{aligned} B_t = {} & -(t + 1)[\delta^2 - (t + 1)^2]A_{t+1} \\ & + (2t + 1)[\delta^2 - t^2 + (t + 1)\sigma - (t + 1)^2]A_t \\ & - t[\delta^2 - (t - 1)^2 + (4t + 1)(\sigma - t)]A_{t-1} + 2t(t - 1)(\sigma - t + 1)A_{t-2} . \end{aligned} \tag{21}$$

We thus obtain the recurrence relation

$$\begin{aligned} (t + 1)[\delta^2 - (t + 1)^2]A_{t+1} = {} & (2t + 1)[\delta^2 - t^2 + (t + 1)(\sigma - t - 1)]A_t \\ & - t[\delta^2 - (t - 1)^2 + (4t + 1)(\sigma - t)]A_{t-1} \\ & + 2t(t - 1)(\sigma - t + 1)A_{t-2} . \end{aligned} \tag{22}$$

We know that $A_t = 0$ for $t < \delta$. However (22) becomes indeterminate for $t = \delta - 1$ and so A_δ has to be calculated independently. This was to have been expected since equation (11) is homogeneous. It follows immediately from (7) that

$$A_\delta = C_{r,s,r-s} = \binom{r}{s} \tag{23}$$

When working with an electronic computer it will nearly always be more efficient to use (22) and (23) than (7). All that one need store is the binomial coefficients (23) and a sub-routine for effecting (22).

4. It may be of interest to consider some values of C_{rst} for special values of r, s, t. The value of $C_{r,s,r-s}$ is given by (23) and we deduce, by means of (22), that

$$\begin{cases} C_{r,s,r-s+1} = -2s\binom{r}{s} & \text{and} \\ C_{r,s,r-s+2} = (r - s + 1)^{-1}s[(2s - 1)(r + 1) - 2s^2]\binom{r}{s} . \end{cases} \tag{24}$$

We can calculate directly from (7) that

$$\begin{cases} C_{r,s,r+s} = \binom{r+s}{r} \\ C_{r,s,r+s-1} = -2(r + s - 1)\binom{r+s-2}{r-1} \\ C_{r,s,r+s-2} = (2rs - r - s + 1)\binom{r+s-1}{r-1} . \end{cases} \tag{25}$$

One might also draw attention to some elementary arithmetical properties of the coefficients C_{rst}. In the first place, it is clear from (7) that the sign of C_{rst} is that of $(-1)^p$ and so of $(-1)^{r+s+t}$. Again, it follows from the same formula that C_{rst} is always an integer. For the expression

$$\frac{(r+s-n)!}{(r-n)!(s-n)!\,(2n-p)!\,(p-n)!}$$

is an integer, being a multinomial coefficient. Also, the occurrence of the term $(2n-p)!$ in the denominator imposes the limitation $2n \geqq p$ on the range of n. The result is immediate. Finally, we deduce, by putting $x = 0$ in (1) and (2) that

$$\sum_t C_{rst} = 1 \tag{26}$$

and, from symmetry, that the summation in (26) could equally be taken over s or r.

5. The relation (22) is, as we have said, well adapted to machine work. It is rather complicated for hand computation and the authors are indebted to a referee who drew their attention to the alternative relation

$$(r+1)C_{r+1,s,t} = (t+1)C_{r,s,t+1} + 2(t-r)C_{r,s,t} - rC_{r-1,s,t} + tC_{r,s,t-1}. \tag{27}$$

This follows immediately from the orthogonality and recurrence relations of the Laguerre polynomials, and is very much simpler arithmetically than (22). There is no doubt that many other relations of this type could be found. However, for machine computation they would all share the disadvantage of (27), namely, the increased programming complications involved in varying two of the subscripts. They would also be slower to generate since, to arrive at a given r, s, t, one would have to progress through a much larger number of intermediate terms.

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