1. Introduction. Here we discuss numerical integration formulas of the form

\[ \int_R f(x)w(x) \, dx \approx \sum_i a_i f(\nu_i) \]

where \( R \) is a region in \( n \)-dimensional, real, euclidean space; \( x = (x_1, x_2, \cdots, x_n) \); the \( a_i \) are constants; and the \( \nu_i \) are points in the space. Most previous authors have given formulas for special regions (for a bibliography see [4]). Thacher [7] has given a method for constructing formulas of degree 2 with \( n + 1 \) points for general regions and of degree 3 with \( 2n \) points for certain symmetric regions; with his method, however, each region must also be treated separately. Our main results are to obtain specific formulas of degree 2 with \( n + 1 \) points for a general region satisfying a certain condition of non-degeneracy, and to show that for these regions such formulas cannot be obtained with fewer points. We also give a specific \( 2n \) point formula of degree 3 for a general centrally symmetric region. These results are a generalization of those of Georgiev [1, 2, 3] who has obtained similar results (but gives no general formulas) for \( n = 2, 3 \) with \( w(x) = 1 \). Our results are obtained by a different method which was developed without knowledge of Georgiev’s work.

2. Formulas of degree 2. We assume at first that an integration formula of degree 2 for \( R \) with respect to \( w(x) \) can be obtained with \( n + 1 \) points

\[ \nu_i = (\nu_{i1}, \cdots, \nu_{in}), \quad i = 0, 1, \cdots, n. \]

Then the equations

\[
\begin{align*}
a_0 + a_1 + \cdots + a_n &= c_0 \\
a_0 \nu_{0j} + a_1 \nu_{1j} + \cdots + a_n \nu_{nj} &= c_{0j} \\
a_0 \nu_{0j} \nu_{0k} + a_1 \nu_{1j} \nu_{1k} + \cdots + a_n \nu_{nj} \nu_{nk} &= c_{jk}
\end{align*}
\]

must be solved for both the \( a_i \) and the \( \nu_i \), where

\[ c_0 = \int_R w(x) \, dx, \quad c_{0j} = \int_R x_j w(x) \, dx, \quad c_{jk} = \int_R x_j x_k w(x) \, dx. \]

We begin by writing (1) as the matrix equation

\[ U^T A U = C \]

where

\[
U = \begin{bmatrix} 1 & \nu_{01} & \cdots & \nu_{0n} \\ 1 & \nu_{11} & \cdots & \nu_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \nu_{n1} & \cdots & \nu_{nn} \end{bmatrix}, \quad A = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & c_{01} & \cdots & c_{0n} \\ c_{01} & c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0n} & c_{1n} & \cdots & c_{nn} \end{bmatrix}
\]

and where we assume \( 0 < c_0 < \infty \) and \( 0 < |\det C| < \infty \).

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Since $C$ is non-singular we can find a matrix $T$ such that

$$T^* U^* A U T = T^* C T = C_0 E$$

where $E$ is a diagonal matrix with entries $\pm 1$. The method for finding $T$ is well known (see [5], p. 56); we illustrate it using $n = 3$.

Since $C_0 \neq 0$ we define $t_{ii} = -c_{ii}/c_0$, $i = 1, 2, 3$, and form

$$T_1 = \begin{bmatrix} 1 & t_{01} & t_{02} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \quad C_1 = T_1^* C T_1 = \begin{bmatrix} c_0 & 0 & 0 \\ 0 & c_{11}^{(1)} & c_{12}^{(1)} & c_{13}^{(1)} \\ 0 & c_{12}^{(1)} & c_{22}^{(1)} & c_{23}^{(1)} \\ 0 & c_{13}^{(1)} & c_{23}^{(1)} & c_{33}^{(1)} \end{bmatrix}.$$ 

Now if $c_{11}^{(1)*} = 0$ some $c_{ii}^{(1)} \neq 0$ since $\det C \neq 0$. Assuming $c_{12}^{(1)} \neq 0$ we form

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \quad C_2 = T_2^* C_1 T_2 = \begin{bmatrix} c_0 & 0 & 0 \\ 0 & 2hc_{12}^{(1)} + h^2 c_{22}^{(1)} & c_{12}^{(1)} + hc_{22}^{(1)} & c_{13}^{(1)} + hc_{23}^{(1)} \\ 0 & c_{12}^{(1)} + hc_{22}^{(1)} & c_{22}^{(1)} & c_{23}^{(1)} \\ 0 & c_{13}^{(1)} + hc_{23}^{(1)} & c_{23}^{(1)} & c_{33}^{(1)} \end{bmatrix}$$

and choose $h$ so that $c_{11}^{(1)} = 2hc_{12}^{(1)} + h^2 c_{22}^{(1)} \neq 0$; if $c_{11}^{(1)*} \neq 0$ we take $h = 0$ so that $c_{11}^{(1)} = c_{11}^{(1)*}$. In this way we are assured that the element in the $1, 1$ position is $\neq 0$.

Similarly we may find matrices $T_3$, $T_4$ and $T_5$ such that

$$T_3 = T_4^* T_3^* C_2 T_3 T_4 = \begin{bmatrix} c_0 & 0 & 0 \\ 0 & c_{11}^{(2)} & 0 \\ 0 & 0 & c_{22}^{(2)} & c_{32}^{(2)} \\ 0 & 0 & c_{22}^{(2)} & c_{33}^{(2)} \end{bmatrix} \quad \quad T_5^* C_3 T_5 = \begin{bmatrix} c_0 & 0 & 0 \\ 0 & c_{11}^{(3)} & 0 \\ 0 & 0 & c_{22}^{(3)} \\ 0 & 0 & c_{22}^{(3)} \end{bmatrix},$$

where $c_{22}^{(2)}$ and $c_{32}^{(2)}$ are $\neq 0$. Defining $T_6$ as the diagonal matrix

$$[1, [c_0/|c_{11}^{(1)}|]^1, [c_0/|c_{22}^{(2)}|]^1, [c_0/|c_{33}^{(3)}|]^1]$$

we have finally $T = T_1 T_2 T_3 T_4 T_5 T_6$.

We can assume $E$ has the form $[1, 1, \ldots, 1, -1, \ldots, -1]$ since any other arrangement of $+1$'s and $-1$'s can be put into this form by a suitable interchange of the rows of $UT$ and the corresponding columns of $T^* U^*$. If $C$ is positive definite (for example if $w(x)$ is of constant sign on $R$) $E$ will be the identity. It should be noted that the first element of $E$ will always be positive.

In the following we write

$$UT = \begin{bmatrix} 1 & \xi_{01} & \cdots & \xi_{0n} \\ 1 & \xi_{11} & \cdots & \xi_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_{n1} & \cdots & \xi_{nn} \end{bmatrix} = \begin{bmatrix} 1 & \nu_{01} & \cdots & \nu_{0n} \\ 1 & \nu_{11} & \cdots & \nu_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \nu_{n1} & \cdots & \nu_{nn} \end{bmatrix} \begin{bmatrix} 1 & \tau_{01} & \cdots & \tau_{0n} \\ 0 & \tau_{11} & \cdots & \tau_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tau_{n1} & \cdots & \tau_{nn} \end{bmatrix}.$$ 

Because $UT$ is non-singular and $E^{-1} = E$ we easily obtain from (3)

$$(UT) E (UT)^T = c_0 A^{-1}.$$
In terms of the $\xi_i$ this equation is

$$1 + \xi_n \xi_{n-1} + \cdots + \xi_{ip} \xi_{i+1} \xi_{i,p+1} - \cdots - \xi_{in} \xi_{jn} = \frac{c_0}{a_i} \delta_{ij}$$

(4)

$i, j, = 0, 1, \cdots, n$.

where $p + 1$, $0 \leq p \leq n$, is the number of $+1$'s in $E$. We discuss the solution of (4); the $\nu_i$ are obtained from the $\xi_i$ by $\nu_{ij} = \tau_{ij} + \xi_{ii} \tau_{ij} + \cdots + \xi_{in} \tau_{ij}$, $i = 0, 1, \cdots, n, j = 1, \cdots, n$, where

$$T^{-1} = \begin{bmatrix} 1 & \tau_{01} & \cdots & \tau_{0n} \\ 0 & \tau_{11} & \cdots & \tau_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tau_{n1} & \cdots & \tau_{nn} \end{bmatrix}.$$ 

We are only interested in real solutions of (1) and therefore precisely $n - p + 1$ of the $a_i$ must be negative by Sylvester's "law of inertia" ([5], p. 56). If $E$ is the identity ($p = n$) clearly we must have $0 < a_i < c_0$; if $p < n$ the only condition for the $a_i$ that they be non-zero.

Table 1 gives a particular solution of (4); we have assumed $a_0, \cdots, a_{n-p}$ negative and $a_{n-p+1}, \cdots, a_n$ positive. In the places where a double sign occurs we mean to use the lower sign for the last $n - p$ components of each vector and the upper sign for the first $p$ components. Each $\xi_i$ is real.

### Table 1

$$\begin{array}{c}
\xi_0 = \left(0, 0, \cdots, 0, 0, \left[ \frac{c_0 - d_0}{\pm a_0} \right]^{1/2} \right) \\
\xi_1 = \left(0, 0, \cdots, 0, \left[ \frac{c_0(c_0 - d_0 - a_1)}{\pm (c_0 - a_0)a_1} \right]^{1/2}, \mp \left[ \frac{\pm a_0}{c_0 - a_0} \right]^{1/2} \right) \\
\xi_2 = \left(0, 0, \cdots, \left[ \frac{c_0(c_0 - d_0 - a_1 - a_2)}{\pm (c_0 - a_0 - a_2)a_2} \right]^{1/2}, \mp \left[ \frac{\pm a_0 a_2}{(c_0 - a_0 - a_2)(c_0 - a_0 - a_1)} \right]^{1/2}, \mp \left[ \frac{\pm a_0}{c_0 - a_0} \right]^{1/2} \right) \\
\xi_{n-2} = \left(0, \left[ \pm a_0 (c_0 - a_0 - \cdots - a_{n-2}) \right]^{1/2}, \cdots \right) \\
\xi_{n-1} = \left(\left[ \pm a_0 (c_0 - a_0 - \cdots - a_{n-1}) \right]^{1/2}, \cdots \right) \\
\xi_n = \left(\left[ \pm a_0 (c_0 - a_0 - \cdots - a_n) \right]^{1/2}, \cdots \right)
\end{array}$$
From a particular solution \( \xi_{ij} \) of (4) other solutions may be obtained as follows. If

\[
S = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \sigma_{11} & \cdots & \sigma_{1n} \\
\vdots & & \ddots & \vdots \\
0 & \sigma_{n1} & \cdots & \sigma_{nn}
\end{bmatrix}
\]

is a cogredient automorph of \( E \), that is if \( SES^T = E \), then

\[
\xi'_{ij} = \xi_{ij} \sigma_{ij} + \cdots + \xi_{in} \sigma_{nj}
\]

is also a solution. If \( Q \) is an arbitrary skew matrix of order \( n + 1 \), with first row and column entirely zero, such that \( \det (E + Q)(E - Q) \neq 0 \), then

\[
S = (E + Q)^{-1}(E - Q)
\]

is a cogredient automorph of \( E \) (see [5], p. 65) of the above form. If \( E \) is the identity \( S \) is orthogonal. We remark that in this latter case (4) determines the distances \( d(\xi, 0) \) and \( d(\xi, \xi_j) \), \( i, j = 0, 1, \cdots, n, i \neq j \).

\[
d(\xi, 0) = \left[(c_0 - a_i)/a_i\right]^j \\
d(\xi, \xi_j) = [c_0(a_i + a_j)/a_i a_j]^j
\]

The formulas discussed above are minimal; that is, similar formulas cannot be obtained with fewer points. For if a formula could be obtained with \( m + 1 \) points \( \nu_i, i = 0, 1, \cdots, m, m < n \), then equation (2) would still hold, where \( C \) is the same as before and

\[
U = \begin{bmatrix}
1 & \nu_{01} & \cdots & \nu_{0n} \\
1 & \nu_{11} & \cdots & \nu_{1n} \\
\vdots & & \ddots & \vdots \\
1 & \nu_{m1} & \cdots & \nu_{mn}
\end{bmatrix} \\
A = \begin{bmatrix}
a_0 & 0 & \cdots & 0 \\
0 & a_1 & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & a_m
\end{bmatrix}
\]

that is, \( U \) is a rectangular matrix. Since \( U \) and \( A \) have rank at most \( m + 1 \), then \( U^T A U \) has rank at most \( m + 1 \) and therefore \( \det (U^T A U) = 0 \). By assumption \( \det C \neq 0 \) and thus (2) cannot hold for \( m < n \).

3. Formulas of degree 3 for centrally symmetric regions. We assume \( R \) to be centrally symmetric with respect to the origin; then if \( x \) is in \( R \), \( -x \) is also in \( R \). Let us further assume \( w(-x) = w(x) \) for \( x \) in \( R \). Then

\[
\int_R x_i w(x) \, dx = \int_R x_i x_j x_k w(x) \, dx = 0, \quad i, j, k = 1, \cdots, n.
\]

We may obtain an integration formula of degree 3 for \( R \) with respect to \( w(x) \) with \( 2n \) points as follows. Take the points to be \( \nu_i, -\nu_i, i = 1, \cdots, n \), and take \( \nu_k, -\nu_k \) to have common weight \( a_k \). Any \( 2n \) points chosen in this way integrate exactly the monomials \( x_i, x_i x_j x_k \) with respect to \( w(x) \) over \( R \). In addition we must solve

\[
a_1 + a_2 + \cdots + a_n = \frac{1}{2} c_0 \\
a_{ij} \nu_{1j} \nu_{1k} + a_{2j} \nu_{2j} \nu_{2k} + \cdots + a_{nij} \nu_{nj} \nu_{nk} = \frac{1}{2} c_{jk}, \quad j, k = 1, \cdots, n.
\]
The second of these may be written as the matrix equation (2) where now

\[
U = \begin{bmatrix}
\nu_{11} & \nu_{12} & \cdots & \nu_{1n} \\
\nu_{21} & \nu_{22} & \cdots & \nu_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\nu_{n1} & \nu_{n2} & \cdots & \nu_{nn}
\end{bmatrix}, \quad A = \begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & a_n
\end{bmatrix}, \quad C = \frac{1}{2} \begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
c_{n1} & c_{n2} & \cdots & c_{nn}
\end{bmatrix}
\]

and where we assume \(-\infty < \alpha < \infty\) and \(0 < |\det C| < \infty\).

We solve this equation by a method similar to that of the preceding section. We find a non-singular matrix \(T\) such that

\[
T^T U^T A U T = T^T C T = E
\]

where \(E\) is diagonal with elements \(\pm 1\). Again it is convenient to assume

\[
E = [1, \cdots, 1, -1, \cdots, -1]
\]

where the first \(p\) elements are \(+1\), \(0 \leq p \leq n\). Now writing

\[
UT = \begin{bmatrix}
\xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\xi_{n1} & \xi_{n2} & \cdots & \xi_{nn}
\end{bmatrix}, \quad A = \begin{bmatrix}
\nu_{11} & \nu_{12} & \cdots & \nu_{1n} \\
\nu_{21} & \nu_{22} & \cdots & \nu_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\nu_{n1} & \nu_{n2} & \cdots & \nu_{nn}
\end{bmatrix}, \quad C = \frac{1}{2} \begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
c_{n1} & c_{n2} & \cdots & c_{nn}
\end{bmatrix}
\]

the \(\xi_{ij}\) may be solved for in terms of the \(a_i\). This gives

\[
(5) \quad \xi_{11} + \cdots + \xi_{1p} \xi_{jp} - \xi_{1,p+1} \xi_{1,p+1} - \cdots - \xi_{1n} \xi_{2n} = \frac{1}{a_i} \delta_{ij}, \quad i, j = 1, \cdots, n
\]

precisely \(n - p\) of the \(a_i\) must be negative in order that the \(\xi_i\) be real.

If \(a_1, \cdots, a_p\) are positive and \(a_{p+1}, \cdots, a_n\) negative a particular solution of (5) is

\[
\xi_i = (0, \cdots, 0, \sqrt{1/|a_i|}, 0, \cdots, 0) \quad i = 1, \cdots, n
\]

where the \(i\)th component of \(\xi_i\) is non-zero. If \(S = (\sigma_{ij})\) is any cogredient automorph of \(E\) then \(\xi_i = \xi_i + \sigma_{i1} + \cdots + \sigma_{in} + \sigma_{i1} + \cdots + \sigma_{in} + \sigma_{ii}\) is also a solution of (5). If \(E\) is the identity, that is, \(C\) is positive-definite, the solutions of (5) correspond to the sets of \(n\) orthogonal vectors in the space having the property that the \(i\)th vector of each set is a distance \(\sqrt{1/a_i}\) from the origin.

4. Concluding remarks. The importance of the result given in this paper for formulas of degree 2 is that it is the first result (other than the trivial one point formula, the centroid of \(R\), which integrates any linear function) which holds for an arbitrary region in \(n\)-dimensional space and which gives all such formulas containing the minimum number of points.

A question, which may have some practical importance, which may be asked about the above formulas of degree 2 concerns the conditions \(R\) must satisfy, say for \(w(x) = 1\), in order that such a formula will exist with all of its points interior to \(R\). For example, can a formula interior to \(R\) be found if \(R\) is convex? if \(R\) is star-like about its centroid?

The error bound of von Mises [6] for \(n\)-dimensional integration formulas is very well suited for use with the formulas developed in this paper. In a later paper we will give specific values of this error bound for various known formulas.
I am especially indebted to Dr. P. C. Hammer for many discussions concerning this subject.

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3. G. Georgiev, “Formulas of mechanical cubature with minimal numbers of terms,” Rozprawy Matematyczne, No. 8, 1955, p. 72. (Russian; English summary)