A Note on Gaussian Twin Primes

By Daniel Shanks

If \( m^2 + 1 \) is a prime, then \( m + i \) is a Gaussian prime and conversely. If \( (n - 1)^2 + 1 \) and \( (n + 1)^2 + 1 \) are both prime, then \( n - 1 + i \) and \( n + 1 + i \) form a pair of Gaussian twin primes [1, p. 82]. This is the case for \( n = 3, 5, 15, 25, 55, \ldots, 184705, 184745, 184755, \ldots \); the corresponding (rational) primes being 5 and 17 for \( n = 3 \), and 34134040517 and 34134779537 for \( n = 184755 \). Let \( g(N) \) be the number of such pairs for \( 4 \leq n + 1 \leq N \).

Similarly, let \( z(N) \) be the number of pairs of rational twin primes, \( n - 1 \) and \( n + 1 \), (such as \( n = 4, 6, 12, 18, \ldots \)), for \( 5 \leq n + 1 \leq N \). Hardy and Littlewood [2] conjectured that

\[
(1) \quad z(N) \sim 1.32032 \int_2^N \frac{dn}{(\log n)^2},
\]

where

\[
(2) \quad 1.32032 \cdots = 2 \prod_{p=3}^\infty \left(1 - \frac{1}{(p - 1)^2}\right),
\]

the product being taken over all odd primes.

By the use of a sieve argument very similar to that recently presented [3] in support of another Hardy-Littlewood conjecture, the following asymptotic relation was obtained:

\[
(3) \quad g(N) \sim 0.36932z(N),
\]

where

\[
(4) \quad 0.36932 \cdots = \frac{1}{4} \prod_{p=3}^\infty \left[1 - 2\left(-\frac{1}{p}\right)(p - 2)^{-1}\right].
\]

Here \((-1/p)\) is the Legendre symbol. Assuming the truth of both (3) and (1), we have

\[
(5) \quad g(N) \sim 0.48762 \int_2^N \frac{dn}{(\log n)^2}.
\]

We may compute the constant in (5), and therefore also that in (4), from

\[
(6) \quad 0.48762 \cdots = \frac{\pi^2}{8} \prod_{p=4m+1} \left(1 - \frac{4}{p}\right)^2 \left(\frac{p + 1}{p - 1}\right)^2,
\]

the product being taken over all primes of the form \( 4m + 1 \). The evaluation of the right side of (6) is facilitated by a transformation similar to that previously used [3] in computing the Hardy-Littlewood constants, \( h_n \).

The number of Gaussian twin pairs, \( g(N) \), was determined for \( N = 500(500)185000 \), by counting these pairs in a recently computed table [1, p. 81] of the greatest prime factor of \( n^2 + 1 \) for \( n = 1(1)185000 \). A short summary is shown in Table 1 together with a comparison of \( g(N) \) and

Received December 31, 1959.
Table 1
Gaussian and Rational Twin Primes

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<th>(N)</th>
<th>(\tau(N))</th>
<th>(\kappa(N))</th>
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</table>

(7) \[ \gamma(N) = 0.48762 \int_2^N dn / (\log n)^2. \]

Also shown are Glaisher's counts [4] of \(z(N)\) to \(N = 10^5\) and

(8) \[ \zeta(N) = 1.32032 \int_2^N dn / (\log n)^2. \]

In this range of \(N\) the deviations from unity of \(g(N)/\gamma(N)\) and \(z(N)/\zeta(N)\) are about equal in magnitude, [5].

The slow oscillations of \(g(N)/\gamma(N)\) around one have two significant consequences.

1. They make improbable any value of the constant in (5) which differs more than slightly from the theoretical value, (6).

2. They make possible a sensitive test for the correctness of the function of \(N\) assigned to the proposed asymptote, \(\int_2^N dn / (\log n)^2\). For since \(| g(N) - \gamma(N) | \ll \gamma(N)\), even small functional modifications in \(\gamma(N)\) would greatly alter the phase, frequency, and amplitude of the corresponding oscillations of \(g(N)/\gamma(N)\) around one. Now consider \(P(N)\), the total number of Gaussian primes of the form \(m + i\), [1, see Table on p. 78; p. 81]. The corresponding Hardy-Littlewood conjecture reads

(9) \[ P(N) \sim \hat{P}(N) = 0.68641 \int_2^N dn / \log n, \]

and similar remarks are applicable to the function \(\hat{P}(N)\) and to any oscillations of \(P(N)/\hat{P}(N)\). But if (9) and (5) are both valid, we must expect that any slow oscillations of \(g(N)/\gamma(N)\) and \(P(N)/\hat{P}(N)\) will agree in phase and frequency. For, where
there is an excess of primes, there should generally also be an excess of twins, and
if the oscillations are slow, then any complicating higher frequency fluctuations in
the local density will largely disappear by integration. In Figure 1 we compare
graphs of $g(N)/\bar{g}(N)$ and $P(N)/\bar{P}(N)$ for $N$ to 185000. Very close agreement is
seen in the phase and frequency of the slow oscillations. Since such an agreement
would seem improbable if either or both of (5) and (9) were false, it may be re-
garded as providing further evidence in their favor.

The difficulties that stand in the way of a proof of (3), (assuming it to be true)
are similar to those previously discussed for other problems [3]. Thus it is unlikely
that (3) will be proven without a simultaneous solution of the long outstanding
Goldbach, twin prime, and $n^2 + 1$ prime problems.

In conclusion it should be noted that the Gaussian twins on the line $n + i$ are
by no means the only “twins” in the Gauss plane. On the line $n + 2i$, for instance,
we not only have twins, $n = (179983, 179985)$, and triplets, $n = (423, 425, 427)$,
but even one octuplet, $n = (-7, -5, -3, -1, +1, +3, +5, +7)$.

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3. Daniel Shanks, “On the conjecture of Hardy and Littlewood concerning the number
A forthcoming paper with the same title will give an expanded version of this report.
28–33.
5. The empirical evidence for (1) is much more extensive. D. H. Lehmer has computed
$z(N) = 183728$, $\bar{z}(N) = 183582$, and $z/\bar{z}(N) = 1.0008$ for $N = 37\cdot10^6$. See the review, UMT 3,