A Note on the Solution of Quartic Equations

By Herbert E. Salzer

For any quartic equation with real coefficients,

\[ X^4 + AX^3 + BX^2 + CX + D = 0, \]

the following condensation of the customary algebraic solution is recommended as quickest and easiest for the computer to follow (no mental effort required). It works in every exceptional case.

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Denote the four roots of (1), by \(X_1, X_2, X_3,\) and \(X_4\). With the aid of [1], solve the "resolvent cubic equation" \(ax^3 + bx^2 + cx + d = 0\) for the real root \(x_1\) only, where

\[
\begin{align*}
(2) \quad & a = 1, \quad b = -B, \quad c = AC - 4D, \quad \text{and} \quad d = D(4B - A^2) - C^2.
\end{align*}
\]

Find

\[
(3) \quad m = +\sqrt{\frac{1}{4}A^2 - B + x_1}, \quad n = \frac{Ax_1 - 2C}{4m}.
\]

If \(m = 0\), take \(n = \sqrt{\frac{1}{4}x_1^2 - D}\) and proceed according to the following Case I or Case II, depending upon whether \(m\) is real or imaginary.

**Case I:** If \(m\) is real, let \((\frac{1}{4}A^2 - x_1 - B) = \alpha, 4n - Am = \beta, \sqrt{\alpha + \beta} = \gamma, \sqrt{\alpha - \beta} = \delta,\) and finally

\[
\begin{align*}
X_1 &= -\frac{\alpha}{2} + m + \gamma, \quad X_2 = -\frac{\alpha}{2} - m + \delta, \\
X_3 &= -\frac{\alpha}{2} - m - \gamma, \quad \text{and} \quad X_4 = -\frac{\alpha}{2} + m - \delta.
\end{align*}
\]

**Case II:** If \(m\) is imaginary, say \(m = \imath m'\), then \(n\) is also imaginary, say \(n = \imath n'\). Let

\[
(\frac{1}{4}A^2 - x_1 - B) = \alpha, \quad 4n' - Am' = \beta, \quad +\sqrt{\alpha^2 + \beta^2} = \rho,
\]

\[
\sqrt{\frac{\alpha + \rho}{2}} = \gamma, \quad \frac{\beta}{2\gamma} = \delta,
\]

and finally

\[
(4I) \quad \begin{cases} 
X_1 = -\frac{\alpha}{2} + \gamma + \imath(m' + \delta) \\
X_2 = \bar{X}_1, \text{ the complex conjugate of } X_1, \\
X_3 = -\frac{\alpha}{2} - \gamma + \imath(m' - \delta) \\
X_4 = \bar{X}_3, \text{ the complex conjugate of } X_3.
\end{cases}
\]

If \(\gamma = 0\), we must have \(\alpha = -\alpha', \alpha' \geq 0,\) and formula (4II) still holds provided that in it we replace \(\delta\) by \(+\sqrt{\alpha'}\).

As an example consider the quartic equation \(X^4 + X^3 + X^2 + X + 1 = 0,\) where \(A = B = C = D = 1,\) so that from (2) the resolvent cubic equation is \(x^3 - x^2 - 3x + 2 = 0.\) From [1] we find \(x_1 = 0.61803\ 400.\) From (3), \(m = +\sqrt{-0.13196\ 600} = +0.36327\ 125\imath,\) so that \(m' = +0.36327\ 125.\) Then \(n = -1.38196\ 600 = +0.95105\ 655\imath,\) so that \(n' = +0.95105\ 655.\) Proceeding according to Case II, \(\alpha = -1.11803\ 400, \beta = 3.44095\ 495, \rho = 3.61803\ 41, \gamma = 1.11803\ 40\) and \(\delta = 1.53884\ 18.\) Then from (4II) we obtain \(X_1 = 0.30901\ 70 + 0.95105\ 655\imath, X_2 = \bar{X}_1 = 0.30901\ 70 - 0.95105\ 655\imath, X_3 = -0.80901\ 70 - 0.58778\ 53\imath\) and
Conjugate Factor Method for the Solution of a Cubic

By D. A. Maguía

1. Introduction. This paper gives a simple method for computing the real roots of the reduced cubic equation with real coefficients,

\[ x^3 + Ax + B = 0, \]

having roots \( a, b, c \). We assume \( a \) to be real, since every cubic equation has at least one real root.

The method consists in factoring \( B \), and setting one factor equal to \( \pm \sqrt{m} \), the other \( n \). For all pairs \( m, n \) such that \( m + n = -A \), \( \pm \sqrt{m} \) is a root. If no such pair exists, a method of interpolation is shown.

2. Proof of Method. The reduced cubic equation (1) can be transformed, by using the relations between the roots and coefficients, into a complete cubic,

\[ p^3 + 6Ap^2 + 9A^2p + 4A^3 + 27B^2 = 0, \]

where

\[ p = (-3a^2 - 4A). \]

Equation (2) can be written in the form:

\[ (p + A)^3(-p - 4A) = 27B^2 \]

or

\[ \frac{(p + A)}{3} \sqrt{\frac{(-p - 4A)}{3}} = \pm B. \]

Let

\[ m = \frac{-p - 4A}{3} \quad \text{and} \quad n = \frac{p + A}{3} \]

and

\[ n\sqrt{m} = \pm B \]

(8) \[ m + n = -A. \]

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