A Note on the Solution of Quartic Equations

By Herbert E. Salzer

For any quartic equation with real coefficients,

\[ X^4 + AX^3 + BX^2 + CX + D = 0, \]

the following condensation of the customary algebraic solution is recommended as quickest and easiest for the computer to follow (no mental effort required). It works in every exceptional case.

Received December 22, 1959.
Denote the four roots of (1), by $X_1, X_2, X_3, \text{and } X_4$. With the aid of [1], solve the "resolvent cubic equation" $ax^3 + bx^2 + cx + d = 0$ for the real root $x_1$ only, where

$$a = 1, \quad b = -B, \quad c = AC - 4D, \quad \text{and} \quad d = D(4B - A^2) - C^2. $$

Find

$$m = +\sqrt[4]{\frac{1}{2}A^2 - B + x_1}, \quad n = \frac{Ax_1 - 2C}{4m}. $$

If $m = 0$, take $n = \sqrt[4]{\frac{1}{2}x_1^2 - D}$ and proceed according to the following Case I or Case II, depending upon whether $m$ is real or imaginary.

Case I: If $m$ is real, let $(\frac{1}{2}A^2 - x_1 - B) = \alpha, 4n - Am = \beta, \sqrt{\alpha + \beta} = \gamma, \sqrt{\alpha - \beta} = \delta$, and finally

$$X_1 = -\frac{1}{2}A + m + \gamma, \quad X_2 = -\frac{1}{2}A - m + \delta, $$

$$X_3 = -\frac{1}{2}A + m - \gamma, \quad \text{and} \quad X_4 = -\frac{1}{2}A - m - \delta. $$

Case II: If $m$ is imaginary, say $m = im'$, then $n$ is also imaginary, say $n = in'$. Let

$$\left(\frac{1}{2}A^2 - x_1 - B\right) = \alpha, \quad 4n' - Am' = \beta, \quad +\sqrt{\alpha^2 + \beta^2} = \rho, \quad \sqrt{\alpha + \beta} = \gamma, \quad \frac{\beta}{2\gamma} = \delta, $$

and finally

$$\left\{ \begin{aligned} X_1 &= -\frac{1}{2}A + \gamma + i(m' + \delta), \\
X_2 &= \bar{X}_1, \text{the complex conjugate of } X_1, \\
X_3 &= -\frac{1}{2}A - \gamma + i(m' - \delta), \\
X_4 &= \bar{X}_3, \text{the complex conjugate of } X_3. 
\end{aligned} \right. $$

If $\gamma = 0$, we must have $\alpha = -\alpha', \alpha' \geq 0$, and formula (41I) still holds provided that in it we replace $\delta$ by $+\sqrt{\alpha'}$.

As an example consider the quartic equation $X^4 + X^3 + X^2 + X + 1 = 0$, where $A = B = C = D = 1$, so that from (2) the resolvent cubic equation is $x^3 - x^2 - 3x + 2 = 0$. From [1] we find $x_1 = 0.61803 \ 400$. From (3), $m = +\sqrt{-0.13196 \ 600} = +0.36327 \ 125i$, so that $m' = +0.36327 \ 125$. Then $n = -1.38196 \ 600/1.45308 \ 500i = +0.95105 \ 655i$, so that $n' = +0.95105 \ 655$. Proceeding according to Case II, $\alpha = -1.11803 \ 400, \beta = 3.44095 \ 495, \rho = 3.61803 \ 41, \gamma = 1.11803 \ 40$ and $\delta = 1.53884 \ 18$. Then from (41I) we obtain $X_1 = 0.30901 \ 70 + 0.95105 \ 655i, \quad X_2 = \bar{X}_1 = 0.30901 \ 70 - 0.95105 \ 655i, \quad X_3 = -0.80901 \ 70 - 0.58778 \ 53i$ and
$X_4 = \bar{X}_3 = -0.8090170 + 0.5877853i$. These roots may be verified as correct, since they are known to be the complex fifth roots of unity, namely $X_1 = \cos 72^\circ + i \sin 72^\circ$, $X_2 = \cos 288^\circ + i \sin 288^\circ$, $X_3 = \cos 216^\circ + i \sin 216^\circ$, and $X_4 = \cos 144^\circ + i \sin 144^\circ$.

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**A Conjugate Factor Method for the Solution of a Cubic**

By D. A. Maguía

1. **Introduction.** This paper gives a simple method for computing the real roots of the reduced cubic equation with real coefficients,

\[ x^3 + Ax + B = 0, \]

having roots $a$, $b$, $c$. We assume $a$ to be real, since every cubic equation has at least one real root.

The method consists in factoring $B$, and setting one factor equal to $\pm \sqrt{m}$, the other $n$. For all pairs $m$, $n$ such that $m + n = -A$, $\pm \sqrt{m}$ is a root. If no such pair exists, a method of interpolation is shown.

2. **Proof of Method.** The reduced cubic equation (1) can be transformed, by using the relations between the roots and coefficients, into a complete cubic,

\[ p^3 + 6Ap^2 + 9A^2p + 4A^3 + 27B^2 = 0, \]

where

\[ p = (-3a^2 - 4A). \]

Equation (2) can be written in the form:

\[ (p + A)^2(-p - 4A) = 27B^2 \]

or

\[ \frac{(p + A)}{3} \sqrt{-p - 4A} = \pm B. \]

Let

\[ m = \frac{-p - 4A}{3} \quad \text{and} \quad n = \frac{p + A}{3} \]

and

\[ m + n = -A. \]