A NOTE ON THE SOLUTION OF QUARTIC EQUATIONS

\[ \frac{1}{2} \cdot 89 (n^4 + 1) \] for \( n = 2747, 2771, 2885. \\
\[ \frac{1}{2} \cdot 97 (n^4 + 1) \] for \( n = 2669, 2683, 2749. \\

New factorizations are as follows:

\[ 938^4 + 1 = 809273 \cdot 956569 \]
\[ 1060^4 + 1 = 847577 \cdot 1489513 \]
\[ 1348^4 + 1 = 940169 \cdot 3511993 \]
\[ 1512^4 + 1 = 926617 \cdot 5640361 \]
\[ 1874^4 + 1 = 914561 \cdot 1348547 \]
\[ 2100^4 + 1 = 17 \cdot 873553 \cdot 1309601 \]
\[ 2838^4 + 1 = 868841 \cdot 74663657 \]
\[ 2908^4 + 1 = 41 \cdot 940369 \cdot 1854793 \]
\[ \frac{1}{2} (1155^4 + 1) = 830233 \cdot 1071761 \]
\[ \frac{1}{2} (1191^4 + 1) = 935353 \cdot 1075577 \]
\[ \frac{1}{2} (1509^4 + 1) = 872369 \cdot 2971849 \]
\[ \frac{1}{2} (2635^4 + 1) = 857569 \cdot 28107577 \]
\[ \frac{1}{2} (2765^4 + 1) = 908353 \cdot 32173321 \]
\[ \frac{1}{2} (2977^4 + 1) = 17 \cdot 809041 \cdot 2855393 \]

The following factorization was omitted from my original table [1]:

\[ \frac{1}{2} (2055^4 + 1) = 17 \cdot 572233 \cdot 916633. \]

The least integers still incompletely factored correspond to \( n = 1038 \) and 1229, for even and odd values of \( n \), respectively.

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1. A. GLODEN, "Table de factorisation des nombres \( n^4 + 1 \) dans l'intervalle \( 1000 < n < 3000, \)" Institut Grand-Ducal de Luxembourg, Archives, Tome XVI, Luxembourg, 1946, p. 71-88.

2. A. GLODEN, Table des Solutions de la Congruence \( x^4 + 1 = 0 \) (mod \( p \)) pour 800,000 < \( p \) < 1,000,000, published by the author, rue Jean Jaurès, 11, Luxembourg, 1959.

A Note on the Solution of Quartic Equations

By Herbert E. Salzer

For any quartic equation with real coefficients,

\[ X^4 + AX^3 + BX^2 + CX + D = 0, \]

the following condensation of the customary algebraic solution is recommended as quickest and easiest for the computer to follow (no mental effort required). It works in every exceptional case.

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Denote the four roots of (1), by $X_1$, $X_2$, $X_3$, and $X_4$. With the aid of [1], solve the "resolvent cubic equation" $ax^3 + bx^2 + cx + d = 0$ for the real root $x_1$ only, where

(2) $a = 1, \quad b = -B, \quad c = AC - 4D, \quad$ and $\quad d = D(4B - A^2) - C^2$.

Find

(3) $m = +\sqrt{\frac{1}{4}A^2 - B + x_1}, \quad n = \frac{Ax_1 - 2C}{4m}$.

If $m = 0$, take $n = \sqrt{\frac{1}{4}x_1^2 - D}$ and proceed according to the following Case I or Case II, depending upon whether $m$ is real or imaginary.

Case I: If $m$ is real, let $(\frac{1}{4}A^2 - x_1 - B) = \alpha, \quad 4n - Am = \beta, \quad \sqrt{\alpha + \beta} = \gamma,$ \[\sqrt{\alpha - \beta} = \delta, \quad and \quad finally\]

$X_1 = \frac{-\frac{1}{2}A + m + \gamma}{2}, \quad X_2 = \frac{-\frac{1}{2}A - m + \delta}{2},$

$X_3 = \frac{-\frac{1}{2}A + m - \gamma}{2}, \quad$ and $\quad X_4 = \frac{-\frac{1}{2}A - m - \delta}{2}$.

Case II: If $m$ is imaginary, say $m = im'$, then $n$ is also imaginary, say $n = in'$. Let

(4) $\alpha = (\frac{1}{4}A^2 - x_1 - B), \quad 4n' - Am' = \beta, \quad +\sqrt{\alpha^2 + \beta^2} = \rho,$

and finally

\[\begin{cases} X_1 = \frac{-\frac{1}{2}A + \gamma + i(m' + \delta)}{2}, \\
X_2 = \bar{X}_1, \quad \text{the complex conjugate of } X_1, \\
X_3 = \frac{-\frac{1}{2}A - \gamma + i(m' - \delta)}{2}, \\
X_4 = \bar{X}_3, \quad \text{the complex conjugate of } X_3. \end{cases}\]

If $\gamma = 0$, we must have $\alpha = -\alpha'$, $\alpha' \geq 0$, and formula (4II) still holds provided that in it we replace \(\delta\) by $+\sqrt{\alpha'}$.

As an example consider the quartic equation $X^4 + X^3 + X^2 + X + 1 = 0$, where $A = B = C = D = 1$, so that from (2) the resolvent cubic equation is $x^3 - x^2 - 3x + 2 = 0$. From [1] we find $x_1 = 0.61803 + 400$. From (3), $m = +\sqrt{-0.13196 600} = +0.36327 125i$, so that $m' = +0.36327 125$. Then $n = 1.45308 500i = +0.95105 655i$, so that $n' = +0.95105 655i$. Proceeding according to Case II, $\alpha = -1.11803 400, \beta = 3.44095 495, \rho = 3.61803 41, \gamma = 1.11803 40$ and $\delta = 1.53884 18$. Then from (4II) we obtain $X_1 = 0.30901 70 + 0.95105 65i, \quad X_2 = \bar{X}_1 = 0.30901 70 - 0.95105 65i, \quad X_3 = -0.80901 70 - 0.58778 53i$ and
$X_4 = X_3 = -0.8090170 + 0.5877853i$. These roots may be verified as correct,
since they are known to be the complex fifth roots of unity, namely $X_1 = \cos 72^\circ + i \sin 72^\circ$, $X_2 = \cos 288^\circ + i \sin 288^\circ$, $X_3 = \cos 216^\circ + i \sin 216^\circ$, and $X_4 = \cos 144^\circ + i \sin 144^\circ$.

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A Conjugate Factor Method for the Solution of a Cubic

By D. A. Maguía

1. Introduction. This paper gives a simple method for computing the real roots
of the reduced cubic equation with real coefficients,

\[ x^3 + Ax + B = 0, \]

having roots $a, b, c$. We assume $a$ to be real, since every cubic equation has at least one real root.

The method consists in factoring $B$, and setting one factor equal to $\pm \sqrt{m}$, the other $n$. For all pairs $m, n$ such that $m + n = -A$, $\pm \sqrt{m}$ is a root. If no such pair exists, a method of interpolation is shown.

2. Proof of Method. The reduced cubic equation (1) can be transformed, by
using the relations between the roots and coefficients, into a complete cubic,

\[ p^3 + 6Ap^2 + 9A^2p + 4A^3 + 27B^2 = 0, \]

where

\[ p = (-3a^2 - 4A). \]

Equation (2) can be written in the form:

\[ (p + A)^2(-p - 4A) = 27B^2 \]

or

\[ \left( \frac{p + A}{3} \right)^2 \sqrt{\left( -p - 4A \right) \frac{3}{3}} = \pm B. \]

Let

\[ m = \frac{-p - 4A}{3} \quad \text{and} \quad n = \frac{p + A}{3} \]

and

\[ n\sqrt{m} = \pm B \]

and

\[ m + n = -A. \]