Two Formulas Relating to Elliptic Integrals of the Third Kind

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Using Legendre's notation, the normal elliptic integral of the third kind is defined by the equation

\[ \Pi(\phi, \alpha^2, k) = \int_{0}^{\phi} \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}. \]

For \( k^2 < 1 \), the following expansion holds uniformly over the closed interval \( 0 \leq \theta \leq \frac{\pi}{2} \):

\[ \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} = \sum_{m=0}^{\infty} \left( -\frac{1}{2} \right)^m (-1)^m k^{2m} \sin^{2m} \theta, \]

where \( \left( -\frac{1}{2} \right)^m = \frac{(-\frac{1}{2})(-\frac{3}{2}) \cdots (\frac{1}{2} - m + 1)}{m!} \) for \( m > 0 \), and \( \left( -\frac{1}{2} \right)^m = 1 \).

The factor \( \frac{1}{1 - \alpha^2 \sin^2 \theta} \) in the integrand is bounded for \( -\infty < \alpha^2 < \frac{1}{\sin^2 \phi} \) and \( 0 \leq \theta \leq \phi \); consequently, the expanded integrand may be integrated term by term. Such integration yields the series

\[ \Pi(\phi, \alpha^2, k) = \sum_{m=0}^{\infty} b_m k^{2m}, \]

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where
\[ b_m = \left( -\frac{1}{m} \right) (-1)^m \int_0^\phi \frac{\sin^{2m} \theta}{1 - \alpha^2 \sin^2 \theta} d\theta, \quad m > 0, \]
and
\[ b_0 = \int_0^\phi \frac{d\theta}{1 - \alpha^2 \sin^2 \theta} = \frac{1}{\sqrt{1 - \alpha^2}} \tan^{-1}[\sqrt{1 - \alpha^2} \tan \phi], \text{ for } -\infty < \alpha^2 < 1, \]
\[ = \tan \phi, \text{ for } \alpha^2 = 1, \]
\[ = \frac{1}{\sqrt{\alpha^2 - 1}} \tan^{-1}[\sqrt{\alpha^2 - 1} \tan \phi], \text{ for } 1 < \alpha^2 < \frac{1}{\sin^2 \phi}. \]

In general, the coefficients \( b_m \) satisfy the recurrence relation
\[ 2(m + 1) \alpha^2 b_{m+1} = (-1)^{m+1} (2m + 1) \left( -\frac{1}{m} \right) t_m(\phi) + (2m + 1) b_m, \]
where \( t_m(\phi) = \int_0^\phi \sin^{2m} \theta d\theta. \)

Byrd and Friedman [1] give [formula (902.00)] the recurrence relation
\[ t_{2m}(\phi) = \frac{2m - 1}{2m} t_{2m-2}(\phi) - \frac{1}{2m} \sin^{2m-1} \phi \cos \phi \]
and explicit expressions for \( t_0(\phi), t_1(\phi), \) and \( t_2(\phi). \) Corresponding to these we find
\[ b_1 = \frac{b_0 - \phi}{2\alpha^2}, \]
\[ b_2 = \frac{1}{16 \alpha^4} \left[ 3\alpha^2 \sin \phi \cos \phi + b_0 - 3(2 + \alpha^2) \phi \right] \]
\[ b_3 = \frac{5}{128 \alpha^4} \left[ 2\alpha^4 \sin^3 \phi \cos \phi + \alpha^2 (3\alpha^2 + 4) \sin \phi \cos \phi + 8b_0 - (8 + 4\alpha^2 + 3\alpha^4) \phi \right]. \]

When \( \phi = \frac{\pi}{2}, \) \( -\infty < \alpha^2 < 1, \) and \( k^2 < 1, \) we deduce the following expansion
of the complete elliptic integral of the third kind:
\[ \Pi(\alpha^2, k) = \Pi \left( \frac{\pi}{2}, \alpha^2, k \right) = \sum_{m=0}^\infty c_m k^{2m}, \]
where
\[ c_0 = \frac{\pi}{2\sqrt{1 - \alpha^2}}, \]
\[ c_1 = \frac{\pi}{4 \alpha^2} \left[ \frac{1}{\sqrt{1 - \alpha^2}} - 1 \right], \]
\[ c_2 = \frac{3\pi}{32 \alpha^4} \left[ \frac{2}{\sqrt{1 - \alpha^2}} - 2 - \alpha^2 \right], \]
\[ c_3 = \frac{5\pi}{256\alpha^4} \left[ -4\alpha^2 - 3\alpha^4 - 8 + \frac{8}{\sqrt{1 - \alpha^2}} \right]; \]

and, in general, the coefficients satisfy the recurrence formula

\[ 2(m + 1)\alpha^2 c_{m+1} = -\left(m + \frac{1}{2}\right) \pi \left(\frac{-\frac{1}{2}}{m}\right)^2 + (2m + 1)c_m, \]

which follows from the recurrence formula for \( b_m \) when use is made of the definite integral

\[ t_{2m} \left( \frac{\pi}{2} \right) = \int_0^{\pi/2} \sin^{2m} \theta \, d\theta = \frac{1}{2} \cdot \frac{\Gamma(m + \frac{1}{2})\Gamma(\frac{1}{2})}{m!} \]

\[ = \frac{\pi}{2} (-1)^m \left( \frac{-\frac{1}{2}}{m} \right). \]

The expansions obtained above for \( \prod (\phi, \alpha^2, k) \) and \( \prod (\alpha^2, k) \) constitute extensions and simplifications of formulas (906.01) and (906.00), respectively, in the book already cited, by Byrd and Friedman. Furthermore, the coefficient of \( \alpha^2 \) has been corrected here in the expression for \( c_3 \) appearing in (906.00).

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