Note on "Approximation of Curves by Line Segments"

By N. Ream

The problem of obtaining a best fit of broken line segments to a curve over a given range has recently been investigated by Stone [1] who has prepared a general computer program to solve the least-squares equations.

The problem arose previously in designing diode function-generators for analog computers [2], [3], [4]. If \( f(x) \) is the given curve and \((u_0, u_N)\) is the range to be fitted by \( N \) segments, and if \( f(x) \) may be approximated by a parabola in each segment, then it may be shown [4] that the unweighted least-squares fit yields the following equation for the breakpoints \( u_1, \ldots, u_{N-1} \):

\[
\int_{u_0}^{u_j} \{f''(x)\}^{0.4} \, dx = \frac{1}{N} \int_{u_0}^{u_N} \{f''(x)\}^{0.4} \, dx,
\]

and that the ordinate \( v_j \) of each breakpoint is given by

\[
v_j - f(u_j) = -\frac{1}{12N^2} \{f''(u_j)\}^{0.2} \left[ \int_{u_0}^{u_N} \{f''(x)\}^{0.4} \, dx \right]^2.
\]

For \( f(x) = e^{-cx} \) fitted over \((0, 3)\), equations (1) and (2) become

\[
1 - e^{-0.4cu_j} = \frac{j}{N} (1 - e^{-1.2c}),
\]

\[
v_j - e^{-cu_j} = -\frac{25}{48N^2} e^{-0.2cu_j} (1 - e^{-1.2c})^2.
\]

Table 1 gives values of \( u_1 \) and maximum error \( E_{\text{max}} \) computed from (3) and (4) for \( N = 2 \); Stone’s values are shown in parentheses. \( E_{\text{max}} \) occurs at \( x = 0 \). The table also gives values of the r.m.s. error \( R \) which the least-squares analysis aims to minimize; \( R \) is computed from the formula

\[
(u_N - u_0)R^2 = \frac{1}{720N^4} \left[ \int_{u_0}^{u_N} \{f''(x)\}^{0.4} \, dx \right]^5,
\]

which for the chosen function becomes

\[
R = (6c)^{-0.5}E_{\text{max}}.
\]

The derivation of equations (1), (2), and (5) involves expanding \( f(x) \) in a Taylor series about the center of each segment and retaining the first three terms. Hence (i) the formulas are exact for a parabola—it follows immediately that the best fit to a parabola has equally-spaced breakpoints; (ii) the method fails where \( f''(x) = 0 \).

It may be mentioned that if the "best fit" is required to minimize the maximum

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Table 1

\[
f(x) = e^{-cx} \text{ fitted with 2 segments over } (0,3)
\]

<table>
<thead>
<tr>
<th>(c)</th>
<th>(s_0)</th>
<th>(E_{\max})</th>
<th>(R)</th>
<th>(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.454</td>
<td>(1.385) 0.00166</td>
<td>(0.0016) 0.00215</td>
<td>0.00121 ± 0</td>
</tr>
<tr>
<td>0.2</td>
<td>1.410</td>
<td>(1.400) 0.00593</td>
<td>(0.0059) 0.00541</td>
<td>0.00420 ± 0</td>
</tr>
<tr>
<td>0.3</td>
<td>1.366</td>
<td>(1.360) 0.0119</td>
<td>(0.0119) 0.00887</td>
<td>0.00821 ± 1</td>
</tr>
<tr>
<td>0.4</td>
<td>1.322</td>
<td>(1.316) 0.0189</td>
<td>(0.0189) 0.0122</td>
<td>0.0127 ± 0</td>
</tr>
<tr>
<td>0.5</td>
<td>1.278</td>
<td>(1.276) 0.0265</td>
<td>(0.0265) 0.0153</td>
<td>0.0174 ± 1</td>
</tr>
<tr>
<td>0.6</td>
<td>1.236</td>
<td>(1.235) 0.0343</td>
<td>(0.0344) 0.0181</td>
<td>0.0221 ± 1</td>
</tr>
<tr>
<td>0.7</td>
<td>1.194</td>
<td>(1.196) 0.0420</td>
<td>(0.0423) 0.0205</td>
<td>0.0264 ± 2</td>
</tr>
<tr>
<td>0.8</td>
<td>1.153</td>
<td>(1.155) 0.0496</td>
<td>(0.0500) 0.0226</td>
<td>0.0305 ± 3</td>
</tr>
<tr>
<td>0.9</td>
<td>1.113</td>
<td>(1.116) 0.0568</td>
<td>(0.0574) 0.0244</td>
<td>0.0343 ± 5</td>
</tr>
<tr>
<td>1.0</td>
<td>1.074</td>
<td>(1.080) 0.0636</td>
<td>(0.0645) 0.0260</td>
<td>0.0377 ± 7</td>
</tr>
<tr>
<td>1.2</td>
<td>1.001</td>
<td>(1.008) 0.0758</td>
<td>(0.0774) 0.0283</td>
<td>0.0435 ± 13</td>
</tr>
<tr>
<td>1.5</td>
<td>0.900</td>
<td>(0.912) 0.0907</td>
<td>(0.0936) 0.0302</td>
<td>0.0500 ± 26</td>
</tr>
</tbody>
</table>

error, the breakpoints are given by equations (1) with the index 0.4 replaced by 0.5 and \(f''(x)\) replaced by its absolute value. The maximum error \(E\) is then given by

\[
(7) \quad E = \left( \frac{1}{4N} \int_{x_0}^{x_N} |f''(x)|^{0.5} \, dx \right)^2.
\]

For the function under discussion (7) becomes

\[
(8) \quad E = \frac{1}{4N^2} (1 - e^{-1.5c})^2,
\]

and the error \(\delta E\) in \(E\) due to the approximations used in deriving (8) may be shown to be given by

\[
(9) \quad \delta E \cong \frac{1}{4} E^2 e^{1.5c}.
\]

Values of \(E\) and \(\delta E\) are included in the table.

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