Note on "Approximation of Curves by Line Segments"

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The problem of obtaining a best fit of broken line segments to a curve over a given range has recently been investigated by Stone [1] who has prepared a general computer program to solve the least-squares equations.

The problem arose previously in designing diode function-generators for analog computers [2], [3], [4]. If \( f(x) \) is the given curve and \( (u_0, u_N) \) is the range to be fitted by \( N \) segments, and if \( f(x) \) may be approximated by a parabola in each segment, then it may be shown [4] that the unweighted least-squares fit yields the following equation for the breakpoints \( u_1, \ldots, u_{N-1} : \)

\[
\int_{u_0}^{u_j} f''(x) \, dx = \frac{j}{N} \int_{u_0}^{u_N} f''(x) \, dx,
\]

and that the ordinate \( v_j \) of each breakpoint is given by

\[
v_j - f(u_j) = -\frac{1}{12N^2} \left[ f''(u_j) \right]^{0.4} \left[ \int_{u_0}^{u_N} f''(x) \, dx \right]^2.
\]

For \( f(x) = e^{-cx} \) fitted over \((0, 3)\), equations (1) and (2) become

\[
1 - e^{-0.4cu_j} = \frac{j}{N} (1 - e^{-1.2c}),
\]

\[
v_j - e^{-cu_j} = -\frac{25}{48N^2} e^{-0.2cu_j} (1 - e^{-1.2c})^2.
\]

Table 1 gives values of \( u_1 \) and maximum error \( E_{\text{max}} \) computed from (3) and (4) for \( N = 2 \); Stone’s values are shown in parentheses. \( E_{\text{max}} \) occurs at \( x = 0 \). The table also gives values of the r.m.s. error \( R \) which the least-squares analysis aims to minimize; \( R \) is computed from the formula

\[
(u_N - u_0) R^2 = \left( \frac{1}{720N^4} \right) \left[ \int_{u_0}^{u_N} f''(x) \, dx \right]^2,
\]

which for the chosen function becomes

\[
R = (6c)^{-0.5} E_{\text{max}}.
\]

The derivation of equations (1), (2), and (5) involves expanding \( f(x) \) in a Taylor series about the center of each segment and retaining the first three terms. Hence (i) the formulas are exact for a parabola—it follows immediately that the best fit to a parabola has equally-spaced breakpoints; (ii) the method fails where \( f''(x) = 0 \).

It may be mentioned that if the “best fit” is required to minimize the maximum

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error, the breakpoints are given by equations (1) with the index 0.4 replaced by 0.5 and \( f''(x) \) replaced by its absolute value. The maximum error \( E \) is then given by

\[
E = \left[ \frac{1}{4N} \int_{u_0}^{u_N} |f''(x)|^{0.5} \, dx \right]^2. 
\]

For the function under discussion (7) becomes

\[
E = \frac{1}{4N^2} (1 - e^{-1.6c})^2,
\]

and the error \( \delta E \) in \( E \) due to the approximations used in deriving (8) may be shown to be given by

\[
\delta E \cong \frac{1}{4} E e^{1.6c}.
\]

Values of \( E \) and \( \delta E \) are included in the table.

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