Computing Error Bounds in Solving Linear Systems

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1. Introduction. Let there be given a system of \( m \) linear algebraic equations for \( m \) unknowns

\[
Gu = r
\]

and consider an iteration procedure

\[
u_{n+1} = Mu_n + s \quad (n = 0, 1, 2, \ldots)
\]

such that the system

\[
u = Mu + s
\]

is equivalent to (1.1). All elements of the \( m \)-dimensional vectors and the \( m \times m \)-matrices which occur are assumed to be real.

In the space of \( m \)-dimensional vectors \( u = (u^1, v, \ldots) \) we define an order relation and an absolute value by writing

\[
u \leq v \quad \text{if and only if} \quad u^i \leq v^i \quad (i = 1, 2, \ldots, m),
\]

and

\[
|u| = (|u^i|).
\]

Similarly, for \( m \times m \)-matrices \( A = (a_{ij}), B, \ldots \) we use the notation

\[
A \leq B \quad \text{if and only if} \quad a_{ij} \leq b_{ij} \quad (i, j = 1, 2, \ldots, m),
\]

and

\[
|A| = (|a_{ij}|).
\]

Let \( B \) denote a matrix such that

\[
|M| \leq B
\]

for \( M \) in equation (1.2) (use, for example, \( B = |M| \) if \( M \) is explicitly known). If there exists a vector \( v_0 \) such that

\[
|u_p - u_{p+1}| \leq v_0 - Bv_0 \quad \text{for some index } p,
\]

then the given equation has a solution \( u^* \) for which

\[
|u^* - u_{p+n}| \leq B^nv_0 \quad (n = 0, 1, 2, \ldots)
\]

holds (Theorem 1).

We present a method for computing a vector \( v_0 \) with the property (1.4). The calculation of \( v_0 \) and the vectors \( B^nv_0 \) in (1.5) constitutes an iterative procedure parallel to approximation procedure (1.2). This estimation procedure, described

Received August 2, 1961. This work was sponsored by the Mathematics Research Center, U.S. Army, Madison, Wisconsin.

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at the end of Section 2, can be programmed for computers as easily as approxima-
tion procedure (1.2).

If the matrix $B$ is irreducible and noncyclic and if the spectral radius $\rho(B)$ of
the matrix $B$ satisfies

\begin{equation}
\rho(B) < 1,
\end{equation}

then for some $p$ the method yields a vector $v_0$ which satisfies (1.4) (Theorem 3).

Section 4 is concerned with the methods of simultaneous and successive dis-
placements (with $B$ chosen as in (4.4)). Some numerical examples (up to 15 un-
knowns) are given. In these examples, the estimation procedure yields a suitable
vector $v_0$, i.e., an error bound, after at most three steps, thus the time needed for
estimation is considerably smaller than the time needed for solving the given
system by iteration (one need not start the estimation procedure and the approxi-
mation procedure simultaneously).

For many of the known estimation methods one has to calculate an upper bound
$\sigma$ of $\rho(B)$ and this bound has to satisfy the condition

\begin{equation}
\sigma < 1.
\end{equation}

For example, this inequality (1.7) may be the row-sum criterion for the method of
successive displacements. In this paper condition (1.7) is weakened to (1.6) where
$\rho(B)$ need not be known.

Of course, condition (1.6) still restricts the class of iteration procedures (1.2)
to which the estimation method can be applied. Note that the procedure (1.2)
converges for an arbitrary vector $u_0$ if and only if

$$
\rho(M) < 1
$$

where the spectral radius $\rho(M)$ of $M$ satisfies

$$
\rho(M) \leq \rho(|M|) \leq \rho(B).
$$

However, every convergence condition which works with upper bounds $b_{ij}$ of the
moduli $|m_{ij}|$ instead of with elements $m_{ij}$ cannot be weaker than condition (1.6).

For example, when solving difference equations for the Laplace equation by the
method of successive displacements, one has, in general, $M \succeq 0$ and $\rho(M) < 1,$
thus, $\rho(B) < 1$ for $B = M$. In the case of the biharmonic equation, however, the
method of successive displacements in general yields $\rho(|M|) > 1$.

2. Derivation of the Method. Let $R$ be the set of $m$-dimensional real vectors
$u = (u^i), v, \ldots$, and let $A = (a_{ij}), B, M, \ldots$ denote real $m \times m$-matrices. The
notation $u \preceq v$, $|u|$, $A \leq B$, and $|A|$ shall be defined as in the introduction.

Consider an equation

\begin{equation}
M u + s
\end{equation}

where $M$ denotes a given matrix, $s$ is a given vector and $u$ is unknown. With $Tu =
M u + s$ we can write this equation as $u = Tu.$

Let $B$ denote a fixed matrix such that

\begin{equation}
| M | \leq B,
\end{equation}

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then let \( H[u, v] \) be the function
\[
H[u, v] = \frac{1}{2}(B + M)u - \frac{1}{2}(B - M)v + s.
\]
This function is increasing in \( u \) and decreasing in \( v \), and for \( u = v \) we get \( H[u, u] = Tu \).

We consider the iteration procedure
\[
(2.3) \quad x_{n+1} = H[x_n, y_n], \quad y_{n+1} = H[y_n, x_n] \quad (n = 0, 1, 2, \cdots).
\]
Because of the described properties of \( H[u, v] \), for this procedure the following statements hold (see [5]).

Let the conditions
\[
x_0 \leq x_1, \quad x_0 \leq y_0, \quad y_1 \leq y_0
\]
be satisfied.

Then, the vectors \( x_n \) and \( y_n \) \((n = 0, 1, 2, \cdots)\) defined by (2.3) satisfy the inequalities
\[
(2.4) \quad x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y_2 \leq y_1 \leq y_0.
\]
Moreover, the sequences \( \{x_n\} \) and \( \{y_n\} \) converge to limit-vectors \( x^* \) and \( y^* \), respectively, such that
\[
(2.5) \quad x^* = H[x^*, y^*], \quad y^* = H[y^*, x^*]
\]
and
\[
(2.6) \quad x_n \leq x^* \leq y^* \leq y_n \quad (n = 0, 1, 2, \cdots).
\]
The inequalities (2.4) can be proved by induction, and the convergence of the sequences \( \{x_n\} \) and \( \{y_n\} \) then follows from the fact that these sequences are monotonic and bounded.

Adding the two equations in (2.5) and noting that (2.6) holds, we get, in addition, the following statement:

The vector
\[
u^* = \frac{1}{2}(x^* + y^*)
\]
is a solution of the given equation (2.1), and for this solution the estimate
\[
x_n \leq x^* \leq y_n \quad (n = 0, 1, 2, \cdots)
\]
holds.

Now let \( \{u_n\} \) and \( \{v_n\} \) be sequences of vectors which satisfy the equations
\[
(2.7) \quad u_{n+1} = Mu_n + s \quad (n = 0, 1, 2, \cdots)
\]
and
\[
(2.8) \quad v_{n+1} = Bv_n \quad (n = 0, 1, 2, \cdots),
\]
respectively. Then, the vectors \( x_n \) and \( y_n \) defined by
\[
u_n = \frac{1}{2}(x_n + y_n), \quad v_n = \frac{1}{2}(y_n - x_n) \quad (n = 0, 1, 2, \cdots),
\]
i.e.,
\[
x_n = u_n - v_n, \quad y_n = u_n + v_n \quad (n = 0, 1, 2, \cdots),
\]
satisfy (2.3). We reformulate the main results stated above in terms of the vectors $u_n$ and $v_n$ instead of $x_n$ and $y_n$.

Theorem 1. Let the conditions

\begin{equation}
(2.9) \quad v_0 \geq 0 \quad \text{and} \quad |u_0 - u_1| \leq v_0 - v_1
\end{equation}

be satisfied.

Then, the vectors $u_n$ and $v_n$ ($n = 0, 1, 2, \cdots$) defined by (2.7) and (2.8) satisfy the inequalities

\begin{equation}
(2.10) \quad v_n \geq 0, \quad |u_n - u_{n+1}| \leq v_n - v_{n+1} \quad (n = 0, 1, 2, \cdots).
\end{equation}

Moreover, the sequences $\{u_n\}$ and $\{v_n\}$ converge to vectors $u^*$ and $v^*$, respectively, such that

\[ u^* = Mu^* + s \quad \text{and} \quad v^* = Bv^*, \]

and for $u^*$ the error estimate

\begin{equation}
(2.11) \quad |u^* - u_n| \leq v_n \quad (n = 0, 1, 2, \cdots)
\end{equation}

holds.

From Theorem 1 we start to develop a method of error estimation for the iteration procedure (2.7). Clearly, we can replace the vectors $u_i$ in the theorem, which occur by $u_{i+p}$ ($p$ denoting a fixed non-negative integer). The assumption (2.9) then takes the form

\begin{equation}
(2.12) \quad v_0 \geq 0, \quad |u_p - u_{p+1}| \leq v_0 - v_1
\end{equation}

and the estimate (2.11) becomes

\begin{equation}
(2.13) \quad |u^* - u_{n+p}| \leq v_n \quad (n = 0, 1, 2, \cdots).
\end{equation}

When the vectors $u_p$ and $u_{p+1}$ are known one could try to construct a suitable vector $v_0$ using the special properties of the given problem. However, the error estimate would then, in general, be much more complicated than the calculation of the approximations $u_n$ because one can easily program the procedure (2.7) for computers. Therefore, we will establish a method of error estimation which also can be programmed quite easily.

A Method of Error Estimation for the Iteration Procedure $u_{n+1} = Mu_n + s$.

Starting with some vector

\[ w_0 \geq 0 \quad \text{(for example } w_0 = o) \]

and using the formula

\begin{equation}
(2.14) \quad w_{n+1} = Bw_n + \delta_{n+1} \quad \text{with} \quad \delta_{n+1} = |u_{n+1} - u_n|
\end{equation}

calculate vectors $w_n$ up to the first index $n = p$ for which

\begin{equation}
(2.15) \quad w_p \geq w_{p+1},
\end{equation}

provided that such an index exists. Calculate in addition vectors $z_n$ ($n = p, p + 1, \cdots$) defined by

\[ z_p = w_p, \quad z_{n+1} = Bz_n \quad (n = p, p + 1, \cdots). \]
Theorem 2. If there exists an index \( p \) such that (2.15) holds, then the sequence \( \{u_n\} \) converges to a solution \( u^* = Mu^* + s \) and

\[
|u^* - u_n| \leq z_n \quad (n = p, p + 1, \ldots).
\]

**Proof.** Suppose \( p \) is an index such that (2.15) holds. Then, let \( \{v_n\} \) denote the sequence defined by (2.8) for \( v_0 = v_p \). Clearly, one has \( v_0 \geq o \). Moreover, also the second inequality in (2.12) is satisfied because this inequality is equivalent to the condition (2.15). Thus the sequence \( \{u_n\} \) converges to a solution \( u^* \) which satisfies the inequality (2.13), and this inequality is equivalent to (2.16).

3. Theoretical Investigation of the Method. We now investigate the conditions under which the method of error estimation described above will be successful.

For this, we assume that the matrix \( B \) is irreducible (non-decomposable) and noncyclic. Then, also because \( B \geq o \), according to Frobenius [3] the following statements hold.

Matrix \( B \) has an eigenvalue \( \lambda > 0 \), called the maximal root of \( B \), such that \( \lambda \) is greater than the modulus of each other eigenvalue of \( B \). Corresponding to \( \lambda \) there exists an eigenvector \( \varphi = (\varphi^1, \varphi^2, \ldots, \varphi^m) \) with

\[
\varphi^i > 0 \quad (i = 1, 2, \ldots, m),
\]

and there are no eigenvectors or principal vectors (generalized eigenvectors) corresponding to \( \lambda \) which are linearly independent of \( \varphi \).

Thus, each vector \( u \in \mathbb{R} \) can be written as a sum

\[
u = u^{(1)} \varphi + \psi
\]

where \( u^{(1)} \) is a constant and \( \psi = \psi(u) \) is a linear combination of the eigenvectors and principal vectors of \( B \) belonging to eigenvalues different from \( \lambda \).

Then let \( P_1 \) and \( P_2 \) denote the two projection matrices defined by

\[
P_1u = u^{(1)} \varphi \quad (\text{for } u \in \mathbb{R}), \quad P_2 = I - P_1,
\]

and moreover let

\[
B_1 = BP_1, \quad B_2 = BP_2.
\]

Then the equation

\[
B = B_1 + B_2
\]

represents a spectral decomposition of matrix \( B \) with \( B_1 \) belonging to \( \lambda \) and \( B_2 \) belonging to the remainder of the spectrum of \( B \). We have

\[
B^n u = B_1^n u + B_2^n u = \lambda^n u^{(1)} \varphi + B_2^n u.
\]

Let \( \rho(A) \) denote the spectral radius of a matrix \( A \), i.e., the maximum of the moduli of its eigenvalues. Then, we have \( \rho(B_1) = \rho(B) = \lambda \) and

\[
\rho(B_2) < \lambda.
\]

If \( u \) is a vector such that \( u \geq o \), then \( B^n u \geq o \ (n = 0, 1, 2, \cdots) \) and it follows from (3.3) that

\[
u^{(1)} \varphi + (\lambda^{-1} B_2)^n u \geq o.
\]
Because of (3.4) the inequality (3.5) yields $u^{(i)} \varphi \geq 0$ for $n \to \infty$, hence $u^{(i)} \geq 0$. This proves the statement that

$$u \geq 0 \quad \text{implies} \quad u^{(i)} \geq 0.$$  

If $u^i > 0$ for all $i = 1, 2, \cdots, n$, then there exists a constant $\alpha > 0$ such that $u \geq \alpha \varphi$ and we now derive from (3.3)

$$u^{(i)} \varphi + (\lambda^{-i}B_2)^{n} u = \lambda^{-n}B^n \varphi \geq \lambda^{-n} \alpha B^n \varphi = \alpha \varphi.$$  

Hence, for $n \to \infty$ we obtain the result

$$u^i > 0 \quad \text{(for all} \ i = 1, 2, \cdots, n) \quad \text{implies} \quad u^{(i)} > 0.$$  

We write

$$u > 0 \quad \text{if and only if} \quad u \geq 0 \quad \text{and} \quad u \neq 0,$$

and we now also prove that

$$u > 0 \quad \text{implies} \quad u^{(i)} > 0.$$  

Let $\mu > \lambda$, then the equation

$$\sum_{i=0}^\infty \mu^{-i}B^i = (I - \mu^{-1}B)^{-1}$$

holds, and the matrix $(I - \mu^{-1}B)^{-1}$ has all elements positive because $B$ is irreducible. (This follows from Theorem 2.2 in [7], applied to $A = \mu I$, $z = \varphi$ and $B$ as defined above.) Therefore, if $u > 0$, all components of the vector

$$v = (I - \mu^{-1}B)^{-1} u = \sum_{i=0}^\infty \mu^{-i}B^i u$$

are positive, hence it follows from (3.7) that $v^{(i)} > 0$. Finally, we conclude from the equation

$$v^{(i)} \varphi = P \nu = P \sum_{i=0}^\infty \mu^{-i}B^i u = \sum_{i=0}^\infty \mu^{-i} \lambda^i u^{(i)} \varphi = u^{(i)} (1 - \lambda \mu^{-1})^{-1} \varphi$$

and that $u^{(i)} > 0$.

**Theorem 3.** Suppose that matrix $B$ is irreducible and noncyclic, and assume moreover that the maximal root $\lambda = \rho(B)$ satisfies

$$\lambda < 1.$$  

Then there exists an index $p$ such that (2.15) holds.

**Proof.** Consider the sequence $\{\omega_n\}$ defined by

$$\omega_0 = w_0, \quad \omega_{n+1} = B\omega_n + \delta_{n+1} \quad \text{for} \ n = 0, 1, 2, \cdots.$$  

We shall prove that

$$\omega_n - \omega_{n+1} \geq 0 \quad \text{for} \ n \text{ large enough},$$

which is sufficient for the existence of an index $p$ with the desired property.

We first suppose that $\omega_0 = w_0 = 0$. Then, if $u_0 = u$, the inequality (2.15) is satisfied for $p = 0$. Therefore, we will assume that $\delta_1 = | u_1 - u_0 | > 0$. 

The difference $\omega_n - \omega_{n+1}$ can be written as

$$\omega_n - \omega_{n+1} = (I - B)(B^{n-1}\delta_1 + B^{n-2}\delta_2 + \cdots + \delta_n) - \delta_{n+1} \quad (n = 1, 2, \cdots)$$

with

$$\delta_j = |M^{j-1}| \varepsilon_1 \quad (j = 1, 2, \cdots)$$

and

$$\varepsilon_1 = u_1 - u_0.$$

In the following we consider two cases described by $\rho(M) < \lambda$ and $\rho(M) = \lambda$, respectively.

**Case I.** Let

$$\rho(M) < \lambda.$$ 

Using (3.3), we decompose $\omega_n - \omega_{n+1}$ into two summands

$$\omega_n - \omega_{n+1} = S_n^1 + S_n^2 \quad (n = 1, 2, \cdots)$$

with

$$S_n^1 = (1 - \lambda)(\lambda^{n-1}\delta_1^{(1)} + \lambda^{n-2}\delta_2^{(1)} + \cdots + \delta_n^{(1)})\varphi,$$

$$S_n^2 = (P_2 - B_2)\eta_n - \delta_{n+1}$$

and

$$\eta_n = B_2^{n-1}\delta_1 + B_2^{n-2}\delta_2 + \cdots + \delta_n.$$

Because of (3.6), all coefficients $\delta_j^{(1)}$ are non-negative. Therefore, we get

$$\lambda^{-n+1}S_n^1 \geq (1 - \lambda)\delta_1^{(1)}\varphi \quad (n = 1, 2, \cdots),$$

and from (3.1), (3.8), and (3.9) we show that the vector on the right side of (3.14) has all components positive.

Because the spectral radii of $B_2$ and $M$ are smaller than $\lambda$ and because $\delta_j$ is of the form (3.11), the series $\sum_{n=0}^{\infty} \lambda^{-n}B_2^n$ and

$$\sum_{n=1}^{\infty} \lambda^{-n+1}\delta_n$$

converge. Therefore, the product series

$$\sum_{n=1}^{\infty} \lambda^{-n+1}\eta_n = \sum_{i=0}^{\infty} \lambda^{-i}B_2^i \cdot \sum_{j=1}^{\infty} \lambda^{-j+1}\delta_j$$

also converges. In particular, the summands $\lambda^{-n+1}\delta_n$ and $\lambda^{-n+1}\eta_n$ of the series (3.15) and (3.16) converge to the null vector. Thus, we have

$$\lim_{n \to \infty} \lambda^{-n+1}S_n^2 = 0,$$

and this relation, together with the inequality (3.14), indicates that (3.10) holds in Case I.

**Case II.** Suppose now that $\rho(M) = \lambda$. Then, according to a result of Wielandt [8], $M$ can be written as

$$M = e^{i\alpha}D^{-1}BD$$
where $\alpha$ denotes a real number and $D$ is a diagonal matrix with diagonal elements of modulus 1. Because $M$ is supposed to be real the relation (3.18) holds for $\alpha = 0$ or $\alpha = \pi$ and a diagonal matrix $D$ with diagonal elements $\pm 1$.

In this case the vectors $\delta_j$ take the form
\[
\delta_j = |M^{-1}e_1| = |D^{-1}B^{-1}D e_1| = |B^{-1}\xi| = |\lambda^{-1}\xi (\varphi + B_2 j^{-1}\xi)|
\]
with $\xi = D e_1$ ($j = 1, 2, \ldots$).

If $\xi^{(1)} = 0$, then we start again with the decomposition (3.13), for which (3.14) holds. As in Case I, we can prove again that (3.17) holds. For this, we now use $\delta_j = |B_2 j^{-1}\xi|$ and (3.4) instead of (3.11) and (3.12). As in Case I, (3.14) and (3.17) together yield (3.10).

Now let $\xi^{(1)} \neq 0$. Then we have
\[
\delta_j = |\xi^{(1)}| \lambda^{-1} |\psi_j| \quad \text{with} \quad \psi_j = \varphi + (\xi^{(1)})^{-1}(\lambda^{-1}B_2)j^{-1}\psi (j = 1, 2, \ldots )
\]
Because of (3.1) and since the second summands of the $\psi_j$ converge to 0 for $j \to \infty$ there exists a number $j_0$ such that $\psi_j \geq 0$ for $j \geq j_0$; hence
\[
(3.19) \quad \delta_j = |\xi^{(1)}| \lambda^{-1}\varphi + (\text{sgn } \xi^{(1)})B_2 j^{-1}\xi \quad \text{for} \quad j \geq j_0.
\]

Suppose now that $n > j_0 + 1$. Then we write
\[
\omega_n - \omega_{n+1} \quad \text{as a sum} \quad \omega_n - \omega_{n+1} = S_n^3 + S_n^4 \quad (n > j_0 + 1)
\]
where
\[
S_n^3 = (I - B)(B^{-1}\delta_1 + B^{-2}\delta_2 + \cdots + B^{-n}\delta_n)
\]
and
\[
S_n^4 = (I - B)(B^{-1}\delta_{j_0+1} + \cdots + B^{-n}) - \delta_{n+1}.
\]
Using (3.3) we decompose $S_n^3$ further into a sum
\[
S_n^3 = S_n^{31} + S_n^{32}
\]
with
\[
S_n^{31} = (1 - \lambda)(\lambda^{-1}\delta^{(1)}_1 + \lambda^{-2}\delta^{(1)}_2 + \cdots + \lambda^{-n}\delta^{(1)}_{j_0}),
\]
\[
S_n^{32} = (I - B_2)B_2^{-1}\delta_{j_0+1} + \cdots + \delta_{j_0} + \delta_{j_0}.
\]
Since the coefficients $\delta^{(1)}_j$ are non-negative, we get
\[
(3.20) \quad \lambda^{-n+1}S_n^{31} \geq (1 - \lambda)\delta^{(1)}_1 \varphi \quad (n > j_0 + 1),
\]
where the vector on the right side of the inequality has all components positive.

The second summands $S_n^{32}$ satisfy
\[
(3.21) \quad \lim_{n \to \infty} \lambda^{-n+1}S_n^{32} = 0.
\]
This follows from (3.4) because $j_0$ is a fixed number.

Using (3.19), we also split up $S_n^4$ into the following sum:
\[
S_n^4 = S_n^{41} + S_n^{42}
\]
with
\[ S_n^{41} = |\xi^{(i)}| [(1 - B)(\lambda^i B^{n-1}) + \cdots + \lambda^{n-1}]\varphi - \lambda^n \varphi \]

and
\[ S_n^{42} = (\text{sgn } \xi^{(i)})[(1 - B)(B^{n-1}B^{n-1} + \cdots + B^{n-1})\xi - B^{n-1}\xi] \]

We have
\[ (3.22) \lambda^{-n+1}S_n^{41} \geq o \text{ for } n \text{ large enough}, \]

and from (3.4) we deduce that
\[ (3.23) \lim_{n \to \infty} \lambda^{-n+1}S_n^{42} = o. \]

Altogether, from the relations (3.20), (3.21), (3.22), and (3.23) it follows that (3.10) holds in Case II also.

Finally, let \( w_0 = w_0 > o \). Then, the difference \( \omega_n - \omega_{n+1} \) gets an additional summand
\[ \tilde{S}_n = (1 - B)B^{n}w_0 = (1 - \lambda)\lambda^n w_0^{(i)}\varphi + (1 - B)B^{n}w_0 \quad (n = 1, 2, \cdots). \]

These summands \( \tilde{S}_n \) satisfy
\[ \lim_{n \to \infty} \lambda^{-n+1}\tilde{S}_n = (1 - \lambda)\lambda w_0^{(i)}\varphi \]

where the vector on the right side has all components positive. Thus, (3.10) also holds in this case \( w_0 > o \) (even if \( u_1 - u_0 = o \), i.e., \( \delta_1^{(i)} = 0 \) in (3.14) and (3.20)).

**Corollary.** Suppose that matrix \( B \) is irreducible and let \( u_1 \neq u_0 \). Then, condition (3.9) is necessary for the existence of a vector \( v_0 \) which satisfies
\[ (3.24) v_0 \geq o \quad \text{and} \quad |u_0 - u_1| \leq v_0 - Bv_0. \]

**Remark.** This corollary, together with Theorem 3, says, roughly speaking, that if \( B \) is irreducible and noncyclic, the method of estimation in Section 2 is always successful if one can get an estimate with Theorem 2.

**Proof** of the corollary. Since \( u_1 \neq u_0 \), it follows from (3.24) that
\[ v_0 > o \quad \text{and} \quad (I - B)v_0 > o, \]

hence, in view of (3.8)
\[ v_0^{(i)} > o \quad \text{and} \quad [(I - B)v_0]^{(i)} = (1 - \lambda)v_0^{(i)} > o. \]

These two inequalities can hold only if \( \lambda < 1 \).

**4. Applications to Numerical Examples.** A given system of \( m \) linear equations
\[ (4.1) Gu = r \quad \text{with} \quad g_{ii} > 0 \quad (i = 1, 2, \cdots, m) \]

can be written in the form (2.1) as follows. Let \( G = D - C_1 - C_2 \) where \( D \) is the diagonal matrix with diagonal elements \( g_{ii} \) and \( C_1 \) is some lower triangular matrix.
Then equation (4.1) is equivalent to (2.1) with 
\( s = (D - C_1)^{-1}r \) and
\[
M = (D - C_1)^{-1}C_2.
\]

In case \( C_1 = 0 \) the iteration procedure
\[
(4.3) \quad u_{n+1} = Mu_n + s \quad (n = 0, 1, 2, \ldots)
\]
is the method of simultaneous displacements of the system (4.1). On the other hand, if \( C_2 \) is an upper triangular matrix (4.3) represents the method of successive displacements for equation (4.1).

For matrix \( M \) given in equation (4.2) inequality (2.2) is satisfied for
\[
(4.4) \quad B = (D - |C_1|)^{-1} |C_2|.
\]

Clearly, one need not start the estimation procedure described at the end of Section 2, and the approximation procedure (4.3) simultaneously. One can define \( w_0 = w_1 = \cdots = w_q \) (\( q \) denoting some non-negative integer), then start to calculate further vectors by equation (2.14). In this case, the estimation method is described by the following formulas for matrix \( B \) in equation (4.4):
\[
w_q \geq o \quad (\text{with } q \text{ a given non-negative integer});
\]
\[
(D - |C_1|)r_{n+1} = |C_2| w_n, \quad w_{n+1} = r_{n+1} + \delta_{n+1} \quad (n = q, q + 1, \ldots, p)
\]
where \( \delta_{n+1} = |u_{n+1} - u_n| \) and \( p \) is the smallest index such that \( w_p \geq w_{p+1} \);
\[
z_p = w_p, \quad (D - |C_1|)z_{n+1} = |C_2| z_n \quad (n = p, p + 1, \ldots).
\]

This procedure has been programmed for the CDC 1604 Computer. Of course the index \( p \) and the bounds \( z_n \) depend on the value of the chosen \( q \). To indicate this we write \( p = p(q) \) and \( z_n = z_n(q) \). According to Theorem 2 we have
\[
|u^* - u_n| \leq z_n(q) \quad \text{for} \quad n \geq p(q).
\]

The following examples have been calculated using the program mentioned above with \( w_q = o \).

**Example 1.** The system \( Gu = r \) with \( G \) and \( r \) as given in Table 1 consists of difference equations which approximate a certain boundary value problem for the Laplace equation [4]. We solve this system by the method of successive displacements. In the present case, we have \( C_1 = |C_1| \) and \( C_2 = |C_2| \), thus
\[
(4.5) \quad M = B.
\]
The starting vector \( u_0 \) as given in Table 2 is the (rounded) solution of difference equations for a smaller mesh width. In the same table are listed the exact errors \( \xi_n = u_n - u^* \) for some indices \( n \) and the corresponding bounds \( z_n(q) \) for \( q = 0, 10, \) and 15. The indices \( p(q) \) belonging to these values of \( q \) are
\[
p(0) = 3, \quad p(10) = 11, \quad p(15) = 16.
\]
This indicates that the estimation method was successful for \( q = 0 \) after three steps and for \( q = 10 \) and \( q = 15 \) after one step. Table 2 shows that the bounds for \( q = 10 \) and \( q = 15 \) are equal (up to nine decimals) but sharper than the bounds for \( q = 0 \). We learn from this that the estimation procedure should not be started...
immediately \((q = 0)\), but rather after the iteration process \((4.3)\) has become "steady." That the bounds for \(q = 10\) and \(q = 15\) are almost equal is a consequence of \((4.5)\) (in this connection, see also the next example). It would have been sufficient to start the estimation procedure at \(q = 15\).

**Example 2.** Matrix \(G\) and vector \(r\) of the second example are given in Table 3. The corresponding equation \(Gu = r\) consists of difference equations approximating a boundary value problem for the differential equation \(\Delta \Delta u = \varphi(x, y)\) [1]. In general, condition \(\rho(B) < 1\) is not satisfied for such problems. However, the estimation method works in this case with a few unknowns. We calculate this example in order to test how the bounds \(z_n(q)\) behave if \(\rho(M) < \rho(B) < 1\).

Numerical results for \(q = 0, 10,\) and \(25\) are given in Table 4. The corresponding indices \(p\) are

\[
p(0) = 2, \quad p(10) = 12, \quad p(25) = 27.
\]

In this example, the largest index \(q\) gives the sharpest bounds. This can always be expected if \(\rho(M) < \rho(B)\), because in this case the vectors \(|\xi_n|\) decrease faster than the bounds \(z_n\). For example, if the equation \(\det(M - \kappa I) = 0\) has a simple root \(\kappa_1\) such that \(|\kappa_1| = \rho(M)\) and all other eigenvalues of \(M\) have moduli smaller than \(|\kappa_1|\), then we have in general

\[
|\xi_{n+1}| \approx \rho(M) |\xi_n| \quad \text{for all sufficiently large } n
\]

while

\[
|z_{n+1}| \approx \rho(B) z_n \quad \text{for all sufficiently large } n.
\]

The number of steps \(p(q) - q\) from the beginning of the estimation procedure until success is achieved is the same for the two larger values of \(q\). This phenomenon also occurred in Example 1. It can be explained by the fact that, in general, \(\delta_{n+1} \approx \rho(M) \delta_n\) for \(n\) large enough.

**Further Examples.** In Example 3 we have solved a system of 15 difference equations which approximate the same boundary value problem as the equations in Example 1 and which are of the same type as those equations; however, in Example 3 the row-sum criterion is not satisfied. Compared with Example 1 there has been no essential difference in the behavior of the estimation procedure. Therefore, we give only a few numerical results in Table 5. Furthermore, we applied our method to a system \(Gu = r\) with a 2-cyclic matrix \(G\) (Example 4) using the

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Coefficients of Example 1</strong></td>
</tr>
<tr>
<td>Matrix (G)</td>
</tr>
<tr>
<td>12</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-4</td>
</tr>
</tbody>
</table>

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### Table 2

<table>
<thead>
<tr>
<th>( i )</th>
<th>Results of Example 1 ((u_0 = \text{starting vector}, \tilde{z}_i = u^* - u^* = \text{exact error, } z_i(0) = \text{bounds for } \tilde{z}_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
<td>( \tilde{z}_i )</td>
</tr>
<tr>
<td>1</td>
<td>0.3896</td>
</tr>
<tr>
<td>2</td>
<td>0.4655</td>
</tr>
<tr>
<td>3</td>
<td>0.215</td>
</tr>
<tr>
<td>4</td>
<td>0.1509</td>
</tr>
<tr>
<td>5</td>
<td>0.1091</td>
</tr>
<tr>
<td>6</td>
<td>0.0822</td>
</tr>
<tr>
<td>7</td>
<td>0.0582</td>
</tr>
<tr>
<td>8</td>
<td>0.0447</td>
</tr>
</tbody>
</table>

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Table 3

Coefficients of Example 2

<table>
<thead>
<tr>
<th>Matrix G</th>
<th>Vector r</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 -3 -3 1</td>
<td>1</td>
</tr>
<tr>
<td>-3 10 -2 -3 1</td>
<td></td>
</tr>
<tr>
<td>-3 -2 10 -3 1</td>
<td></td>
</tr>
<tr>
<td>2 -6 -6 11 1</td>
<td></td>
</tr>
</tbody>
</table>

Table 4

Results of Example 2 ($u_o = \text{starting vector}$, $\xi_n = u_n - u^* = \text{exact error}$, $z_n(q) = \text{bounds for } \xi_n$)

<table>
<thead>
<tr>
<th>i</th>
<th>$u_o^i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1^i$</td>
<td>0.079 260 342</td>
<td>0.097 059 129</td>
<td>0.175 309 129</td>
<td>0.117 789 897</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(0)$</td>
<td>0.275</td>
<td>0.327 954 545</td>
<td>0.273 545 455</td>
<td>0.334 426 997</td>
<td></td>
</tr>
<tr>
<td>$\xi_2^i$</td>
<td>0.000 474 178</td>
<td>0.000 677 978</td>
<td>0.000 601 444</td>
<td>0.000 611 652</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(10)$</td>
<td>0.000 861 331</td>
<td>0.001 076 452</td>
<td>0.000 967 820</td>
<td>0.001 140 014</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(0)$</td>
<td>0.009 386 534</td>
<td>0.011 627 371</td>
<td>0.010 937 345</td>
<td>0.014 014 669</td>
<td></td>
</tr>
<tr>
<td>$\xi_3^i$</td>
<td>0.000 000 096</td>
<td>0.000 000 137</td>
<td>0.000 000 122</td>
<td>0.000 000 123</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(25)$</td>
<td>0.000 000 174</td>
<td>0.000 000 217</td>
<td>0.000 000 196</td>
<td>0.000 000 230</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(10)$</td>
<td>0.000 006 455</td>
<td>0.000 007 996</td>
<td>0.000 007 522</td>
<td>0.000 009 638</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(0)$</td>
<td>0.000 076 075</td>
<td>0.000 094 237</td>
<td>0.000 088 644</td>
<td>0.000 113 585</td>
<td></td>
</tr>
<tr>
<td>$\xi_3^i$</td>
<td>0.000 000 018</td>
<td>0.000 000 025</td>
<td>0.000 000 022</td>
<td>0.000 000 023</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(25)$</td>
<td>0.000 000 062</td>
<td>0.000 000 076</td>
<td>0.000 000 072</td>
<td>0.000 000 092</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(10)$</td>
<td>0.000 002 464</td>
<td>0.000 003 053</td>
<td>0.000 002 871</td>
<td>0.000 003 679</td>
<td></td>
</tr>
<tr>
<td>$z_2^i(0)$</td>
<td>0.000 029 040</td>
<td>0.000 033 973</td>
<td>0.000 033 838</td>
<td>0.000 043 358</td>
<td></td>
</tr>
</tbody>
</table>

procedure of simultaneous displacements. In this case matrix $B$ in (4.4) was 2-cyclic also, and the estimation procedure was unsuccessful.

General Remarks Concerning Practical Application. The sharpness of the bounds $z_n(q)$ depends on the chosen index $q$. In general, for larger $q$ one gets sharper bounds after the estimation procedure has been successful, i.e., for $n \geq p(q)$. Only if $p(M) = p(B)$ can one expect that the bounds are almost equal for different $q$, provided $q$ is so large that the iteration process has become "steady." Moreover, the time needed for the estimation becomes smaller for larger $q$. Therefore, the best way might be to start the estimation procedure with an index $q$ such that $\xi_q$ is smaller than a suitably chosen bound. Then, the approximations are improved still more for $n > q$.

There are several further possible ways to verify the program. For example, one may use the fact that in general the differences $p(q) - q$ become equal for $q$ large enough. One may start at some index $q_1$ in order to find $p(q_1) - q_1$, then stop the estimation procedure and start it again with a suitable index $q_2$ such that $p(q_2) \approx q_2 + (p(q_1) - q_1)$.

In our examples we used the starting vector $u_o = o$. In other cases however,
Table 5
Some Results of Example 3

<table>
<thead>
<tr>
<th></th>
<th>u_0</th>
<th>( u_{10} )</th>
<th>( z_{10}^{(15)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.39256</td>
<td>0.384 711 925</td>
<td>0.000 000 026</td>
</tr>
<tr>
<td></td>
<td>0.384 707 090</td>
<td>0.000 000 145</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.384 707 035</td>
<td>0.000 000 067</td>
<td></td>
</tr>
</tbody>
</table>

for a vector

\[ w_q = \beta \delta_q \]

with \( \beta > 0 \) the corresponding number \( p \) might be much smaller than for \( w_q = \delta_q \).

For example, let

\[ M \geq O \quad \text{and} \quad B = M. \]

Then, for \( q \) large enough the difference \( u_q - u_{q+1} \) in general is approximately proportional to \( \phi \). Suppose that \( u_q - u_{q+1} = \phi \) and choose \( w_q = \beta \delta_q = \beta \phi \). Then the first index \( n = p \) for which \( w_p \geq w_{p+1} \) holds is the smallest integer such that

\[ p \geq \lambda (1 - \lambda)^{-1} \beta + q \quad \text{and} \quad p \geq q. \]

For \( \beta = 0 \) and \( \lambda \) very close to 1 this is a large number, for \( \beta > 1 - \lambda \) however, one has \( p = q \). Of course, for \( \beta \) much larger than \( 1 - \lambda \) the bound \( w_p = w_q \) is not sharp.

For example, solving large systems of difference equations for the Laplace equation by the method of successive displacements one may choose

\[ w_q = (1 - \kappa)^{-1} \delta_q \]

where \( \kappa \) approximates the corresponding eigenvalue \( \lambda \). Such an approximation \( \kappa \) is known in many cases.

Round-off errors have not been considered in our program. However, we believe that it is not difficult to do this if suitable subroutines are available. It certainly is not necessary to take into consideration all round-off errors which occur in the entire approximation and estimation procedure. One has to do this only for the last step.

Let \( n_0 \) be the index up to which the approximations and bounds have been calculated. Then, consider the vectors \( u_{n_0-1} \) and \( z_{n_0-1} \) as they are computed with a certain number of digits. If one can show that

\[ z_{n_0-1} \geq 0 \quad \text{and} \quad |u_{n_0-1} - u_{n_0}| \leq z_{n_0-1} - z_{n_0} \] (4.7)

hold, then as a consequence of Theorem 1, applied to \( u_{n_0-1} \) and \( z_{n_0-1} \) instead of \( u_0 \).
and \( v_0 \), there exists a solution \( u^* \) such that \( |u^* - u_{n_0}| \leq z_{n_0} \) and \( |u^* - u_{n_0-1}| \leq z_{n_0-1} \).

If the entire approximation and estimation process could be done without round-off errors, then, certainly, inequalities (4.7) would be satisfied. This follows from the statement (2.10) in Theorem 1. Therefore, in general, one can expect that (4.7) can be proved also for the vectors \( u_{n_0-1} \) and \( z_{n_0-1} \) which are actually computed.

In (4.7), \( u_{n_0} \) and \( z_{n_0} \) do not denote the vectors numerically calculated, but the exact vectors defined by

\[
(4.8) \quad u_{n_0} = M u_{n_0-1} + s, \quad z_{n_0} = B z_{n_0-1}.
\]

Thus, one has to estimate the round-off errors which occur in computing \( u_{n_0} \) and \( z_{n_0} \) by (4.8).

In order to do this, one may, for example, reckon with pairs of numbers (instead of numbers) in (4.8). For example, a subroutine for calculation with pairs of numbers has been written for the IBM 650 Computer and this subroutine has been successfully applied for error estimation for certain differential equations [6]. A similar subroutine has been developed by G. E. Collins for the IBM 704 Computer (1959). The method of calculating with pairs of numbers is called "interval arithmetic" by Collins. This method has been used for desk computers by Dwyer [2] under the name "range arithmetic."

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