Polynomial Expansions of Bessel Functions and Some Associated Functions

By Jet Wimp

1. Introduction. In this paper we first determine representations for the Anger-Weber functions $J_v(ax)$ and $E_v(ax)$ in series of symmetric Jacobi polynomials. (These include Legendre and Chebyshev polynomials as special cases.) If $v$ is an integer, these become expansions for the Bessel function of the first kind, since $J_k(ax) = J_k(ax)$. In Section 3, corresponding representations are found for $(ax)^{-v}J_v(ax)$. Convenient error bounds are obtained for the above expansions. In the fourth section we determine the similar type expansions for the Bessel functions $Y_k(ax)$ and $K_k(ax)$. In Section 5, the coefficients of some of our expansions are tabulated for particularly important values of the various parameters.

2. Symmetric Jacobi Expansions of Anger-Weber Functions. A function $f(x)$ satisfying certain conditions (for these consult [1]) may be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} C_n P_n^{(\alpha,\beta)}(x), \quad -1 \leq x \leq 1, \quad \alpha > -1,$$

where $P_n^{(\alpha,\beta)}(x)$ is called the symmetric Jacobi polynomial of degree $n$. For our present purposes we shall use a definition given in [2]:

$$2^n n! P_n^{(\alpha,\beta)}(x) = (-1)^n (1 - x^2)^{-\alpha} D^n[(1 - x^2)^{\alpha+n}].$$

Also

$$C_n = h_n^{-1} \int_{-1}^{1} f(x)(1 - x^2)^{\alpha} P_n^{(\alpha,\beta)}(x) \, dx,$$

$$h_n = \frac{2^n (n + 1)_{\alpha}}{(n + \alpha + 1/2)(n + \alpha + 1)_{\alpha}}; \quad (\nu)_n = \frac{\Gamma(\nu + \mu)}{\Gamma(\nu)}, \quad (\nu)_0 = 1.$$

Using the representation (2.2) in (2.3) and noticing that all derivatives of $(1 - x^2)^{\alpha+n}$ up to and including the $(n - 1)st$ vanish at $x = \pm 1$, we integrate (2.3) $n$ times by parts to get:

$$C_n = (2^n n! h_n)^{-1} \int_{-1}^{1} f^{(n)}(x)(1 - x^2)^{\alpha+n} \, dx.$$

Consider the integral definition of the Anger-Weber functions [2, v. 2, p. 35]

$$J_v(ax) + iE_v(ax) = \pi^{-1} \int_{0}^{\pi} e^{i(\nu\phi - ax \sin \phi)} \, d\phi = f(x).$$

When $v$ is an integer, $J_v(ax)$ coincides with the Bessel function of the first kind $J_v(ax)$ [2, v. 2, p. 4].

Now differentiate (2.6) $n$ times under the integral sign, substitute the result in

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(2.5) and interchange the order of integration (which is, of course, permissible). The inner integral is known [3] and after evaluating it we have

\[ C_n = (-i)^n(n + 1)(\pi^{1/2})^{-1} \]

\[ \cdot \int_0^\pi e^{i\phi} \left( \frac{a \sin \phi}{2} \right)^{-(n+1/2)} J_{n+1/2}^2(a \sin \phi) d\phi. \]

Use the power series expansion for the Bessel function in (2.7) and integrate term-by-term to get

\[ C_n = (-i)^n \left[ \cos \frac{\nu\pi}{2} + i \sin \frac{\nu\pi}{2} \right] \Lambda_n R_n(v, \alpha, a), \]

where

\[ \Lambda_n = \frac{a^n n!}{\Gamma \left( \frac{n}{2} + \frac{v}{2} + 1 \right) \Gamma \left( \frac{n}{2} - \frac{v}{2} + 1 \right)} (n + 2\alpha + 1)_n, \]

and \( R_n \) is conveniently described in hypergeometric notation [2, v. 1, p. 182] as

\[ R_n(v, \alpha, a) = _2F_1 \left[ \frac{n}{2} + 1, \frac{n}{2} + 1; \frac{\alpha + n + 3}{2}, \frac{\alpha + n + 1}{2} \right] - \frac{a^2}{4} \]

Equating real and imaginary parts of (2.6) and (2.1), we get

\[ J_v(ax) = \sum_{n=0}^\infty A_n P_n^{(\alpha, \alpha)}(x), \quad -1 \leq x \leq 1, \]

\[ E_v(ax) = \sum_{n=0}^\infty B_n P_n^{(\alpha, \alpha)}(x), \quad -1 \leq x \leq 1, \]

where

\[ A_n = \Lambda_n R_n(v, \alpha, a) \phi_n(v), \]

\[ B_n = \Lambda_n R_n(v, \alpha, a) \psi_n(v), \]

and

\[ \phi_n(v) = \begin{cases} (-)^{n/2} \cos \frac{\nu\pi}{2}, & n \text{ even}, \\ (-)^{(n-1)/2} \sin \frac{\nu\pi}{2}, & n \text{ odd}; \end{cases} \]

\[ \psi_n(v) = \begin{cases} (-)^{n/2} \sin \frac{\nu\pi}{2}, & n \text{ even}, \\ (-)^{(n+1)/2} \cos \frac{\nu\pi}{2}, & n \text{ odd}. \end{cases} \]

Equations (2.11) and (2.12) and the expansions in Section 3 may also be de-
rived from results in [4]. The present derivation is more satisfactory because it establishes a foundation for the work in Section 4.

When $\alpha = -\frac{1}{2}$,

$$P_n^{[-(1/2),-(1/2)]}(x) = (\frac{1}{2})^n (n!)^{-1} T_n(x), \quad n = 1, 2 \cdots,$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind of degree $n$. Also for this value of $\alpha$, $R_n$ simplifies to the product of two Bessel functions [2, v. 2, p. 11]. With $\alpha = -\frac{1}{2}$, then (2.11)-(2.14) become

$$J_\nu(ax) = \sum_{n=0}^{\infty} C_n T_n(x), \quad -1 \leq x \leq 1,$$

$$E_\nu(ax) = \sum_{n=0}^{\infty} D_n T_n(x), \quad -1 \leq x \leq 1,$$

where

$$C_n = \epsilon_n J_{(n+\nu)/2} \left( \frac{a}{2} \right) J_{(n-\nu)/2} \left( \frac{a}{2} \right) \phi_n(v),$$

$$D_n = \epsilon_n J_{(n+\nu)/2} \left( \frac{a}{2} \right) J_{(n-\nu)/2} \left( \frac{a}{2} \right) \psi_n(v),$$

and $\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0 \end{cases}$.

For integral $\nu$ we have the expansions

$$J_{2k}(ax) = \sum_{n=0}^{\infty} \epsilon_n J_{k+n} \left( \frac{a}{2} \right) J_{k-n} \left( \frac{a}{2} \right) T_{2n}(x), \quad -1 \leq x \leq 1,$$

$$J_{2k+1}(ax) = 2 \sum_{n=0}^{\infty} J_{k+n+1} \left( \frac{a}{2} \right) J_{k-n} \left( \frac{a}{2} \right) T_{2n+1}(x), \quad -1 \leq x \leq 1,$$

and $k = 0, 1, 2 \cdots$. Equation (2.22) is known [2, v. 2, p. 100].

Since

$$J_\nu(iz) = e^{(\nu+1)/2} I_\nu(z),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind [2, v. 2, p. 5], we may replace $a$ by $ia$ in (2.22) and (2.23) to get expansions for $I_{2k}(ax)$ and $I_{2k+1}(ax)$.

It is important to note that, although the above expansions are valid only for $x$ real and $|x| \leq 1$, (2.6) is entire in $a$ and $\nu$, and hence $a$ may be chosen arbitrarily to yield expansions valid anywhere in the finite complex plane.

The expansions (2.11), (2.12), (2.18), (2.19), (2.22), and (2.23) are quite rapidly convergent, particularly in the Chebyshev cases [5]; consequently the last four expansions are eminently suitable for use on digital computers.* Such series

* The Bessel functions required to compute the coefficients in our expansions can be systematically generated on electronic computers with the aid of techniques discussed in [6, 7, 8]. There are numerous tables available for hand calculations. The words "accuracy," "error," and "convergence" in this paper always refer to the properties of the expansion when truncated after a finite number of terms.
may be truncated and rearranged in powers of \( x \). Clenshaw [9], though, by using the recursion formulas satisfied by the Chebyshev polynomials, has formulated a convenient nesting procedure which allows one to utilize such expansions directly. The scheme is as follows. Consider

\[
\begin{align*}
(2.25) & \quad f^{(1)}(x) = \sum_{n=0}^{N} A_n^{(1)} T_n\left(\frac{x}{a}\right), \quad 0 \leq x \leq a, \\
(2.26) & \quad f^{(2)}(x) = \sum_{n=0}^{N} A_n^{(2)} T_{2n}\left(\frac{x}{a}\right), \quad -a \leq x \leq a, \\
(2.27) & \quad f^{(3)}(x) = \sum_{n=0}^{N} A_n^{(3)} T_{2n+1}\left(\frac{x}{a}\right), \quad -a \leq x \leq a.
\end{align*}
\]

To evaluate the series (2.25), (2.26), or (2.27), respectively, we construct the following sequences:

\[
\begin{align*}
(2.28) & \quad b_n^{(1)} = \left[4 \left(\frac{x}{a}\right) - 2\right] b_{n+1}^{(1)} - b_{n+2}^{(1)} + A_n^{(1)}, \\
(2.29) & \quad b_n^{(2)} = \left[4 \left(\frac{x}{a}\right)^2 - 2\right] b_{n+1}^{(2)} - b_{n+2}^{(2)} + A_n^{(2)}, \\
(2.30) & \quad b_n^{(3)} = \left[4 \left(\frac{x}{a}\right)^2 - 2\right] b_{n+1}^{(3)} - b_{n+2}^{(3)} + A_n^{(3)},
\end{align*}
\]

for \( n = N, N - 1, N - 2, \ldots, 3, 2, 1, 0 \) with the initial values

\[
b_{N+1}^{(1)} = b_{N+2}^{(1)} = b_{N+1}^{(2)} = b_{N+2}^{(2)} = b_{N+1}^{(3)} = b_{N+2}^{(3)} = 0.
\]

\( f^{(1)}(x), f^{(2)}(x), \) and \( f^{(3)}(x) \) are then given by

\[
\begin{align*}
(2.31) & \quad f^{(1)}(x) = b_0^{(1)} + b_1^{(1)} \left[1 - 2 \left(\frac{x}{a}\right)\right], \\
(2.32) & \quad f^{(2)}(x) = b_0^{(2)} + b_1^{(2)} \left[1 - 2 \left(\frac{x}{a}\right)^2\right], \\
(2.33) & \quad f^{(3)}(x) = [b_0^{(3)} - b_1^{(3)}] \left(\frac{x}{a}\right).
\end{align*}
\]

The method is as direct as the ordinary nesting process used to evaluate polynomials.

We now derive error estimates for the expansions (2.11) and (2.12) for \(-1 \leq x \leq 1\). Notice that

\[
(2.34) \quad R_n(v, \alpha, a) = 1 + o\left(\frac{1}{n}\right)
\]

provided all other parameters are fixed, and consequently

\[
(2.35) \quad |A_n| \leq \frac{|a|^{n} n!}{\Gamma\left(\frac{n + v}{2} + 1\right) \Gamma\left(\frac{n - v}{2} + 1\right) (n + 2\alpha + 1)_n} \left|1 + o\left(\frac{1}{n}\right)\right|,
\]
and likewise for $B_n$. Also [2, v. 2, p. 206]

$$\max_{-1 \leq \varepsilon \leq 1} |P_n^{(\alpha, \alpha)}(x)| = \binom{n + \alpha}{n}, \quad \alpha \geq -\frac{1}{2}. \tag{2.36}$$

Let $\varepsilon_n$ denote the error incurred by taking just $N$ terms of (2.11) or (2.12). Because of the rapidity of convergence of the expansions, as shown by (2.35), the $(N + 1)$th term furnishes us with a convenient error estimate

$$|\varepsilon_n| = \frac{|a|^N N^{\alpha+(1/2)} N^{1/2}}{2^{2N+2n} \Gamma \left( \frac{N + v}{2} + 1 \right) \Gamma \left( \frac{N - v}{2} + 1 \right) \Gamma(\alpha + 1)} \left| 1 + 0 \left( \frac{1}{N} \right) \right|, \tag{2.37}$$

where $\alpha \geq -\frac{1}{2}, \quad N > v, \quad -1 \leq x \leq 1$.

Among the values of $\alpha$ considered, it follows from (2.37) that the choice $\alpha = -\frac{1}{2}$, i.e., the Chebyshev case, yields the smallest error term for large $N$.

3. Expansions of Bessel Functions of the First Kind of Nonintegral Order.

Results in the previous section gave symmetric Jacobi polynomial expansions for $J_v(ax)$ and $I_v(ax)$ for integral $v$. When $v$ is nonintegral, these functions are no longer entire functions of $x$, and it is convenient to derive an expansion for the entire function

$$\Gamma(v + 1)(ax/2)^{-v} J_v(ax) = \,\!_0F_1 \left( v + 1; -\frac{a^2 x^2}{4} \right). \tag{3.1}$$

Corresponding expansions for $\Gamma(v + 1)(ax/2)^{-v} I_v(ax)$ then follow, as before, from (2.24).

Let $f(x)$ in (2.5) be the right-hand side of (3.1). Then we have

$$J_v(ax) = (ax)^v \sum_{n=0}^{\infty} A_n P^{(\alpha, \alpha)}_n(x), \quad -1 \leq x \leq 1, \tag{3.2}$$

where

$$A_n = \frac{(-)^n (2a)^{2n}}{2^{\alpha+1/2} (2n + 2\alpha + 1) \Gamma(n + \frac{1}{2}) \Gamma(v + \frac{1}{2})}. \tag{3.3}$$

These equations also follow from a result in [4]. Indeed, using a general expansion given there, an alternative formula for (3.3) can be stated. We have

$$\,\!_1F_2 \left[ \rho; \sigma, \tau; -\frac{a^2}{4} \right] = \Gamma(\sigma) (z/2)^{1-\sigma} \sum_{k=0}^{\infty} \frac{(z/2)^k (\tau - \rho)^k}{k!} J_{k+\sigma-1}(z), \tag{3.4}$$

$$A_n = \frac{(-)^n 2^{(\alpha+1/2) - \sigma - v} \Gamma(n + \frac{1}{2}) (2n + \alpha + \frac{1}{2}) (2n + \alpha + v) \Gamma(v + \frac{1}{2})}{a^{(\alpha+1/2)}} \tag{3.5}$$

$$\cdot \sum_{k=0}^{\infty} \frac{(a/2)^k (v + \frac{1}{2})_k}{k! \Gamma(v + n + k + 1)} J_{2n+\alpha+\alpha+1/2}(a).$$
For the Chebyshev case of (3.2) \( \alpha = -\frac{1}{2} \) and

\[
J_v(ax) = (ax)^v \sum_{n=0}^{\infty} C_n T_{2n}(x), \quad -1 \leq x \leq 1,
\]

where

\[
C_n = \frac{e^{-n(a/4)^2}}{2^{n+1} \Gamma(n+1)} {}_1F_2 \left[ \begin{array}{c} n + \frac{1}{2}; \quad v + n + 1, 2n + 1; \\ - \frac{a^2}{4} \end{array} \right].
\]

Notice that when \( v = -\frac{1}{2} \), (3.3) simplifies. Also, since

\[
\int_{-1}^{1} (x^2)^{v/2} dx = \frac{1}{2} a^{1/2} \cos (ax),
\]

we infer the expansion

\[
J_{-(1/2)}(ax) = \left( \frac{\pi ax}{2} \right)^{(1/2)} \cos (ax),
\]

which can be derived in a number of different ways.

Using an analysis similar to that of Section 2, we may derive the estimate for the error incurred when just \( N \) terms of (3.2) are used.

\[
| \epsilon_N | = \frac{\pi^{1/2} a^{v+2N} \left| \frac{ax}{2} \right| N^{1/2} \Gamma(N + v + 1) \Gamma(\alpha + 1)}{2^{N+\alpha+v-1} \alpha^{N+\alpha} N!} \left( \frac{1}{N} \right)^{1/2}, \quad -1 \leq x \leq 1, \quad \alpha \geq -\frac{1}{2}, \quad N > v.
\]

Concerning the optimum choice of \( \alpha \) in (3.2), see the discussion surrounding (2.37).

4. Expansions of Bessel Functions of the Second Kind. The Bessel function and modified Bessel function of the second kind are denoted by \( Y_v(z) \) and \( K_v(z) \), respectively, and a treatment of them can be found in [2, v. 2, Ch. VII]. If \( v \) is non-integral, then

\[
Y_v(z) = [\sin (v\pi)]^{-1} \left\{ J_v(z) \cos (v\pi) - J_{-v}(z) \right\},
\]

and

\[
K_v(z) = \frac{\pi}{2} [\sin (v\pi)]^{-1} \left\{ I_v(z) - I_{-v}(z) \right\},
\]

so for such values of \( v \) expansions for the functions follow directly from the results of Section 3.

If \( v \) is an integer, it can be shown that

\[
Y_{\pm}(ax) = \frac{2}{\pi} \left[ \gamma + \ln \left( \frac{ax}{2} \right) \right] J_{\pm}(ax) + N_{\pm-1}(ax) - \frac{1}{\pi} W_{\pm}(ax),
\]
and

\begin{equation}
K_k(ax) = (-1)^{k+1} \left[ \gamma + \ln \left( \frac{ax}{2} \right) \right] I_k(ax) - \frac{\pi}{2} i^k N_{k-1}(i ax) + \frac{i^k}{2} W_k(i ax),
\end{equation}

where

\begin{equation}
N_{k-1}(ax) = \begin{cases} 
- \frac{1}{\pi} \sum_{m=0}^{k-1} \frac{(ax)^{2m-k}}{2^m m!} \frac{(k - m - 1)!}{m!}, & k > 0 \\
0, & k = 0,
\end{cases}
\end{equation}

and

\begin{equation}
W_k(ax) = \sum_{n=0}^{\infty} \left( \frac{ax}{2} \right)^{k+2m} \frac{[h_{m+k} + h_m]}{m!(k + m)!}.
\end{equation}

In the above \( \gamma = 0.57721 \ldots \) = Euler's constant and

\begin{equation}
h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \quad h_0 = 1.
\end{equation}

We assume the value of \( \log (ax/2) \) is known. Then, since expansions for \( J_k(ax) \) and \( I_k(ax) \) were found in Section 2, and since \( N_{k-1}(ax) \) is simply a polynomial in \( 1/(ax) \), we need expand only the entire part of (4.3), i.e., \( W_k(ax) \), in symmetric Jacobi polynomials.

Using the representation (4.6) as \( f(x) \) in formula (2.5), a straightforward derivation gives the series

\begin{equation}
W_k(ax) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha, \beta)}(x), \quad -1 \leq x \leq 1,
\end{equation}

where

\begin{equation}
A_n = \frac{((-1)^k + (-1)^n)(n + \alpha + 1) \alpha (n + \alpha + \frac{1}{2})}{2^{n+2a+1}}.
\end{equation}

\begin{equation}
\sum_{n=0}^{\infty} \left( \frac{-m(-k - 2m)_m}{m + k - n + 1} \frac{a^{k+2m}}{2} \right)^{n+\alpha+1} \frac{[h_{m+k} + h_m]}{m!(k + m)!}.
\end{equation}

We note that the expansion for \( Y_0(ax) \) may also be obtained by partially differentiating (3.2) with respect to \( v \) since

\begin{equation}
Y_0(ax) = 2 \pi^{-1} \left( \frac{\partial J_v(ax)}{\partial v} \right)_{v=0}.
\end{equation}

A similar procedure yields the expansion for \( K_0(ax) \). The Jacobi series for \( Y_k(ax) \) and \( K_k(ax) \) for \( k > 0 \), however, are not so easily obtained in this manner.

For \( k = 0 \) and 1, the Chebyshev cases of (4.3) and (4.4) are

\begin{equation}
Y_0(ax) = \frac{2}{\pi} \left[ \gamma + \ln \left( \frac{ax}{2} \right) \right] J_0(ax) + \sum_{n=0}^{\infty} v_n T_{m+n}(x), \quad 0 < x \leq 1,
\end{equation}

\begin{equation}
Y_1(ax) = \frac{2}{\pi} \left[ \gamma + \ln \left( \frac{ax}{2} \right) \right] J_1(ax) - \frac{2}{\pi ax} + \sum_{n=0}^{\infty} F_n T_{m+n}(x), \quad 0 < x \leq 1,
\end{equation}

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(4.13) \[ K_0(ax) = -\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_0(ax) + \sum_{n=0}^{\infty} G_n T_{2n}(x), \quad 0 < x \leq 1, \]

(4.14) \[ K_1(ax) = \left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_1(ax) + \sum_{n=0}^{\infty} H_n T_{2n+1}(x), \quad 0 < x \leq 1, \]

where

\[ E_n = \frac{2\epsilon_n (a/4)^{2n}}{\pi(n!)^2} \sum_{k=0}^{\infty} \frac{(-)^k (a/2)^{2k} (n + 1/2)^k}{(n + 1)_k(2n + 1)_k} k!, \]

\[ F_n = \frac{2(-)^{n+1} (a/4)^{2n+1}}{\pi n!(n + 1)!} \sum_{k=0}^{\infty} \frac{(-)^k (a/2)^{2k} (n + 3/2)_k}{(n + 2)_k(2n + 2)_k} k!, \]

\[ G_n = \frac{\epsilon_n (a/4)^{2n}}{(n!)^2} \sum_{k=0}^{\infty} \frac{(a/2)^{2k} (n + 1/2)_k}{(n + 1)_k(2n + 1)_k} k!, \]

\[ H_n = -\frac{(a/4)^{2n+1}}{n!(n + 1)!} \sum_{k=0}^{\infty} \frac{(a/2)^{2k} (n + 3/2)_k}{(n + 2)_k(2n + 2)_k} k!. \]

5. Tables. Tables 1 through 3 are based on the Chebyshev polynomial cases of the expansions given in the previous sections of this paper. The entries in Tables 1 and 2 were computed on the UNIVAC 1103-A and those in Table 3 on the IBM 7090 at ASD. The calculations were designed so that the error incurred in using the expansions whose coefficients are tabulated here will not exceed five units in the 15th decimal place. Spot checks indicate the error is even less. Because all entries are to 16 significant figures, the expansions may be rearranged in powers of \( x \) with no loss of accuracy.

The number in parentheses after each entry is the power of ten by which the entry is to be multiplied. We have chosen coefficients corresponding to \( a = 5 \), but the coefficients for other values of \( a \) from one through ten are available on request.

Note that the expansions in this paper are valid not only for \(-1 \leq x \leq 1\) but for complex \( x \) in a region which can be determined by a theorem of Szegö [1, p. 238]. More specifically, a Jacobi series representing an entire function converges everywhere in the finite complex plane. However, the further \( x \) lies away from \(-1 \leq x \leq 1\), the more the accuracy of the expansion deteriorates. This is so because \( P_n^{(\alpha, \beta)}(x) \) for complex \( x \) can no longer be bounded by a simple power of \( n \) but behaves in the following manner [10]

\[ P_n^{(\alpha, \beta)}(z) = \frac{\Gamma(n + \alpha + 1)}{n! \pi^{1/2}} N^{2\pi} \left(\sin\frac{\theta}{2}\right)^{2\gamma} \left(\cos\frac{\theta}{2}\right)^{-2\gamma - 2n - 1} \]

\[ \cdot \cos \left[N\theta + \pi\gamma\right] \left\{1 + 0 \left(\frac{1}{N}\right)\right\} \]

valid in the \( z \) plane cut from \(-1\) to \(-\infty\) and from 1 to \( \infty \). In (5.1), \( \cos \theta = z \),
### Table 1

Coefficients for the Series

\[ J_0(x) = \sum_{n=0}^{\infty} A_n T_{2n}(x/5) \quad J_1(x) = \sum_{n=0}^{\infty} B_n T_{2n+1}(x/5) \quad I_0(x) = \sum_{n=0}^{\infty} C_n T_{2n}(x/5) \quad I_1(x) = \sum_{n=0}^{\infty} D_n T_{2n+1}(x/5) \]

\[-5 \leq x \leq 5\]

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### Table 2

Coefficients for the Series

\[ Y_0(x) = (2/\pi)\gamma + \ln(x/2) J_0(x) + \sum_{n=0}^{\infty} E_n T_{2n}(x/5) \quad Y_1(x) = (2/\pi)\gamma + \ln(x/2) J_1(x) + \sum_{n=0}^{\infty} F_n T_{2n+1}(x/5) - 2/\pi x \]

\[ K_0(x) = |\gamma + \ln(x/2)| I_0(x) + \sum_{n=0}^{\infty} G_n T_{2n}(x/5) \quad K_1(x) = |\gamma + \ln(x/2)| I_1(x) + \sum_{n=0}^{\infty} H_n T_{2n+1}(x/5) - 1/x \]

\[ 0 < x \leq 5 \]

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Table 3
Coefficients for the Series

\[ x^{-1}J_{1/2}(x) = \sum_{n=0}^{\infty} A_n^{(1/2)} T_n(x/5) \]
\[ x^{-1}J_{-1/2}(x) = \sum_{n=0}^{\infty} B_n^{(-1/2)} T_n(x/5) \]

\(-5 \leq x \leq 5\)
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456
$N = [n(n + 2a + 1)]^{1/2}$, $\gamma = -(1 + 2a)/4$. In general, if values of $f(x)$ for complex $x$ are desired, it is wisest to choose $a$ such that the expansions are interpolatory along a suitable ray in the complex $x$-plane and to stay as close as possible to this ray.

Suppose we have the truncated expansion

\begin{equation}
(5.2) \quad f(x) = \sum_{n=0}^{N} A_n T_n(x) + \epsilon_{N+1} = \phi_N(x) + \epsilon_{N+1}, \quad -1 \leq x \leq 1,
\end{equation}

and

\begin{equation}
(5.3) \quad \epsilon_{N+1} = \sum_{n=N+1}^{\infty} A_n T_n(x)
\end{equation}

Then $\phi_N(x)$ is not generally the Chebyshev approximation of degree $N$ to $f(x)$ in the sense of [11], i.e., the polynomial $\Phi_N(x)$ of degree $N$ uniquely characterized by the fact that in the interval $[-1, 1]$ the number of consecutive points at which the difference $f(x) - \Phi_N(x)$ with alternate changes in sign assumes the value

$$\max_{-1 \leq x \leq 1} |f(x) - \Phi_N(x)|,$$

is not less than $N + 2$; but $\phi_N(x)$ may closely approximate $\Phi_N(x)$. How closely, of course, depends on the coefficients $A_n$. If $A_n$ goes quite rapidly to zero as $n \to \infty$, then $A_{N+1}$ is small compared to $A_{N+1}$ and consequently

\begin{equation}
(5.4) \quad \epsilon_{N+1} \sim A_{N+1} T_{N+1}(x)
\end{equation}

and the error curve is practically uniform, i.e., $\phi_N(x)$ is nearly $\Phi_N(x)$. Such is the case in our expansions, and, consequently, we must expect the approximation $\Phi_N(x)$ for moderate values of $a$ to offer a negligible improvement over the Chebyshev polynomial expansions derived in this paper and truncated after $N + 1$ terms.

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