

integrated numerically to obtain the value $y_1(1) = y_2(0)$. The three equations (2.1)(0), (2.1)(1), (2.1)(2) are integrated numerically to obtain the initial value $y_2(1) = y_3(0)$, etc. Since $y_j(x) = y(x + j)$, $j = 0, 1, \dots, N$, we obtain in this way the solution $y(x)$ in $0 \leq x \leq N + 1$.

3. Numerical Results of Digital Computer Experiments. Using an IBM 7090 Fortran program with integration subroutines INT and INTM from the IBM Share Library D2 RWFINT and a fixed grid size H , our results agreed to eight significant figures up to $N = 5$ for $H = 2^{-7}$ and $H = 2^{-8}$. As N increased, the agreement got poorer and at $x = 20$ there was agreement to only three significant figures with the initial value $y_1(0) = 1$.

4. Stability. Examining the related system of differential equations, we note that the characteristic values of the matrix of coefficients are all zero. Consequently, we are on the borderline of stability, and progressive loss of accuracy is to be expected as N increases. Using finer grids and more precise methods, we could, of course, decrease the rate of loss of accuracy. In an earlier computation of the solution of differential-difference equations [3] this effect was not present, and more accurate results were obtained.

5. Tables. In this section we present two tables of values. The first presents values of $y(x)$ at intervals of $\frac{1}{16}$ accurate to six or more significant figures. The second presents subsequent values to the degree of accuracy we possess. Observe that as x increases, the number of significant figures decreases.

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The Coefficients of the Lemniscate Function

By L. Carlitz

Let $\wp(u)$ denote the special Weierstrass \wp -function that satisfies the differential equation

$$\wp'^2(u) = 4\wp^3(u) - 4\wp(u).$$

Hurwitz [4] put

$$\wp(u) = \frac{1}{u^2} + \frac{2^4 E_1}{4} \frac{u^2}{2!} + \frac{2^8 E_2}{8} \frac{u^6}{6!} + \dots + \frac{2^{4n} E_n}{4n} \frac{u^{4n-2}}{4n-2} + \dots$$

Received March 26, 1962. The research reported in this paper was supported in part by the National Science Foundation.

He showed that the E_n satisfy the recurrence

$$(1) \quad E_n = \frac{3}{(2n-3)(16n^2-1)} \sum_{k=1}^{n-1} (4k-1)(4n-4k-1) \binom{4n}{4k} E_k E_{n-k}$$

from which it follows that the E_n are positive rational numbers.

Hurwitz proved also that

$$\sum' \frac{1}{(r+si)^{4n}} = \frac{(2\omega)^{4n}}{(4n)!} E_n,$$

where the summation is over all complex integers $r+si$ except 0 and

$$\omega = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

The most interesting property proved by Hurwitz for the E_n is the following analog of the Staudt-Clausen theorem. Let p be a prime of the form $4k+1$ so that $p = a^2 + b^2$; we take a odd and such that

$$a \equiv b + 1 \pmod{4}.$$

Then

$$(2) \quad E_n = G_n + \frac{1}{2} + \sum \frac{(2a)^{4n/(p-1)}}{p},$$

where the summation is over all primes p of the form $4k+1$ such that $p-1 \mid 4n$; G_n is an integer.

Hurwitz computed the first twelve values of E_n both in the form (2) and also in the form

$$(3) \quad E_n = N_n/D_n \quad (N_n, D_n) = 1,$$

where N_n denotes the numerator and D_n the denominator of E_n in reduced form.

The present writer [2] has discussed arithmetic properties of the coefficients of singular elliptic functions. In particular, he showed that if p is a prime of the form $4k+3$ then

$$(4) \quad E_m \equiv 0 \pmod{p^{r+s}}$$

when $p^s \mid m, p^2 - 1 \nmid 4m$ and $4m > pr + 1$, while

$$(5) \quad E_m \equiv 0 \pmod{p^{r-1}}$$

when $p^s \mid m, p^2 - 1 \mid 4m$ and $4m > pr + 1$. In some cases, but not all, (4) and (5) predict the correct power of the primes p dividing E_m . However, the formulas give no information about the occurrence in the numerator of E_n about primes of the form $4k+1$. An interesting question is whether infinitely many primes

$$p \equiv 1 \pmod{4}$$

occur in the numerators of E_m .

In a later paper [3] the writer obtained congruences (mod 2^r) for the E_n as well as for the coefficients of certain related functions. In particular, he showed that

$$(6) \quad \sum_{s=0}^r (-1)^s \binom{r}{s} 2E_{n+sz} \equiv 0 \pmod{2^{re}},$$

provided $2^{s-1} \mid z, r \geq 0$ and $n > re$.

In order to get additional information about the numerators N_n of E_n , it was thought desirable to compute some additional values of N_n . The following values were computed by R. Carlitz in the Duke University Computing Laboratory, making use of the recurrence (1). We remark that the first twelve values are in agreement with the results given by Hurwitz. We write N_n for the numerator of E_n as defined in (3).

- $N_1 = 1$
- $N_2 = 3$
- $N_3 = 3^4 \cdot 7$
- $N_4 = 3^4 \cdot 7^2 \cdot 11$
- $N_5 = 3^6 \cdot 7^2 \cdot 11$
- $N_6 = 3^7 \cdot 7^3 \cdot 11^2 \cdot 19$
- $N_7 = 3^9 \cdot 7^4 \cdot 11^2 \cdot 19 \cdot 23$
- $N_8 = 3^{10} \cdot 7^4 \cdot 11^2 \cdot 19 \cdot 23 \cdot 223$
- $N_9 = 3^{14} \cdot 7^5 \cdot 11^3 \cdot 19 \cdot 23 \cdot 31 \cdot 61$
- $N_{10} = 3^{13} \cdot 7^5 \cdot 11^3 \cdot 19^2 \cdot 23 \cdot 31 \cdot 2381$
- $N_{11} = 3^{15} \cdot 7^6 \cdot 11^4 \cdot 19^4 \cdot 23 \cdot 31$
- $N_{12} = 3^{16} \cdot 7^5 \cdot 11^4 \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 1162253$
- $N_{13} = 3^{18} \cdot 7^7 \cdot 11^4 \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 13 \cdot 8887$
- $N_{14} = 3^{19} \cdot 7^8 \cdot 11^5 \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 61 \cdot 52289$
- $N_{15} = 3^{22} \cdot 7^8 \cdot 11^5 \cdot 19^3 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 2630966033$
- $N_{16} = 3^{22} \cdot 7^9 \cdot 11^5 \cdot 19^3 \cdot 23^2 \cdot 31^2 \cdot 43 \cdot 47 \cdot 59 \cdot 109 \cdot 814903$
- $N_{17} = 3^{24} \cdot 7^9 \cdot 11^6 \cdot 19^3 \cdot 23^2 \cdot 31^2 \cdot 43 \cdot 47 \cdot 59 \cdot 17 \cdot 80232721$
- $N_{18} = 3^{25} \cdot 7^{10} \cdot 11^6 \cdot 19^3 \cdot 23^3 \cdot 31^2 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 48316510111193$
- $N_{19} = 3^{27} \cdot 7^{10} \cdot 11^6 \cdot 19^4 \cdot 23^3 \cdot 31^2 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 3469 \cdot 1330177$
- $N_{20} = 3^{28} \cdot 7^{11} \cdot 11^7 \cdot 19^4 \cdot 23^3 \cdot 31^2 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 503 \cdot 1389248989981$
- $N_{21} = 3^{31} \cdot 7^{13} \cdot 11^7 \cdot 19^4 \cdot 23^3 \cdot 31^2 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 79 \cdot 221430324996967$
- $N_{22} = 3^{31} \cdot 7^{12} \cdot 11^8 \cdot 19^4 \cdot 23^3 \cdot 31^2 \cdot 43^2 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 79 \cdot 83 \cdot 6195097 \cdot 111333763$
- $N_{23} = 3^{33} \cdot 7^{13} \cdot 11^8 \cdot 19^4 \cdot 23^4 \cdot 31^2 \cdot 43^2 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 79 \cdot 83 \cdot 474230496549203$
- $N_{24} = 3^{34} \cdot 7^{12} \cdot 11^8 \cdot 19^5 \cdot 23^4 \cdot 31^3 \cdot 43^2 \cdot 47^2 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 79 \cdot 83 \cdot 859 \cdot 3137 \cdot 7753 \cdot 70880471$

In every case the N_n has been factored completely.

It will be noted that $N_{13} \equiv 0 \pmod{13}$ and $N_{17} \equiv 0 \pmod{17}$. This suggests the following

THEOREM. *If p is a prime greater than 5, we have*

$$(7) \quad N_p \equiv 0 \pmod{p}.$$

More generally if

$$(8) \quad p^r \mid m, \quad p - 1 \nmid 4m, \quad (-1)^m 2^{2m-1} \not\equiv 1 \pmod{p}$$

then

$$(9) \quad N_m \equiv 0 \pmod{p^r}.$$

Proof. If we put

$$\frac{u}{\phi(u)} = \sum_{m=0}^{\infty} \beta_{4m} \frac{x^{4m}}{(4m)!} \quad (\phi(u) = (\wp(u))^{-1/2})$$

then, as Hurwitz showed,

$$(10) \quad \beta_{4m} = (1+i)^{4m} \{ (1+i)^{4m} - 2 \} E_m.$$

The writer [1, Theorem 9] has proved that if $p^r \mid m$, $p-1 \nmid 4m$, then

$$\beta_{4m} \equiv 0 \pmod{p^r}.$$

Thus (10) yields

$$(11) \quad \{ (1+i)^{4m} - 2 \} E_m \equiv 0 \pmod{p^r}.$$

Since $p-1 \nmid 4m$, the denominator of E_m is not divisible by p , so that (11) implies

$$(12) \quad \{ (1+i)^{4m} - 2 \} N_m \equiv 0 \pmod{p^r}.$$

In the next place, since

$$(1+i)^{4m} = (-4)^m,$$

it is evident that

$$(1+i)^{2m} - 2 \equiv 0 \pmod{p}$$

if and only if

$$(13) \quad (-1)^m 2^{2m-1} \equiv 1 \pmod{p}.$$

If, therefore, (13) is not satisfied, it is clear from (12) that

$$N_m \equiv 0 \pmod{p^r}.$$

Finally we note that if $p = m$, then (13) is not satisfied and (7) follows at once.

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All Factors $q < 10^8$ in All Mersenne Numbers $2^p - 1$, p Prime $< 10^4$

By H. Riesel

During the year 1960 the author made additional investigations respecting factors of Mersenne numbers $M_p = 2^p - 1$, where p is a prime. The author has earlier examined the least factor q of M_p , for $p < 10^4$, if $q < 10 \cdot 2^{20} \approx 10^7$. This inquiry was secondary to an effort to ascertain Mersenne primes [1].

The present examination, which resulted in 255 new factors of the numbers M_p , was done in the following manner: All primes $s < 10^8$, for which $s \equiv 1 \pmod{2p}$ and $s \equiv 1 \pm \pmod{8}$ were tested as factors of M_p for $1200 < p < 10000$. The range $p < 1200$ has been examined by Brillhart and Johnson [2].

Received February 10, 1961.