integrated numerically to obtain the value \( y_1(1) = y_2(0) \). The three equations
\((2.1)(0), (2.1)(1), (2.1)(2)\) are integrated numerically to obtain the initial value
\( y_2(1) = y_3(0) \), etc. Since \( y_j(x) = y(x + j), \quad j = 0, 1, \ldots, N \), we obtain in
this way the solution \( y(x) \) in \( 0 \leq x \leq N + 1 \).

Fortran program with integration subroutines INT and INTM from the IBM
Share Library D2 RWFINT and a fixed grid size \( H \), our results agreed to eight signifi-
cant figures up to \( N = 5 \) for \( H = 2^{-7} \) and \( H = 2^{-8} \). As \( N \) increased, the agreement
got poorer and at \( x = 20 \) there was agreement to only three significant figures with
the initial value \( y_1(0) = 1 \).

4. Stability. Examining the related system of differential equations, we note that
the characteristic values of the matrix of coefficients are all zero. Consequently, we
are on the borderline of stability, and progressive loss of accuracy is to be expected
as \( N \) increases. Using finer grids and more precise methods, we could, of course,
decrease the rate of loss of accuracy. In an earlier computation of the solution of
differential-difference equations \[3\] this effect was not present, and more accurate
results were obtained.

5. Tables. In this section we present two tables of values. The first presents
values of \( y(x) \) at intervals of \( \frac{1}{16} \) accurate to six or more significant figures. The
second presents subsequent values to the degree of accuracy we possess. Observe
that as \( x \) increases, the number of significant figures decreases.

The RAND Corporation
Santa Monicas, California

1. N. G. de Bruijn, "On the number of positive integers \(< x \) and free of prime factors \(> y\),"
2. R. Bellman, "On the computational solution of differential-difference equations,"
3. R. Bellman, & B. Kotkin, On the Computational Solution of a Class of Nonlinear

The Coefficients of the Lemniscate Function

By L. Carlitz

Let \( \wp(u) \) denote the special Weierstrass \( \wp \)-function that satisfies the differential
equation
\[ \wp'^2(u) = 4\wp^3(u) - 4\wp(u). \]

Hurwitz \[4\] put
\[ \wp(u) = \frac{1}{u^2} + \frac{2^4 E_1 u^2}{4 2!} + \frac{2^8 E_2 u^6}{8 6!} + \cdots + \frac{2^{4n} E_n u^{4n-2}}{4n 4n-2} + \cdots. \]

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National Science Foundation.
He showed that the $E_n$ satisfy the recurrence

$$E_n = \frac{3}{(2n-3)(16n^2-1)} \sum_{k=1}^{n-1} (4k-1)(4n-4k-1) \left(\begin{array}{c} 4n \\ 4k \end{array}\right) E_k E_{n-k}$$

from which it follows that the $E_n$ are positive rational numbers.

Hurwitz proved also that

$$\sum_{r} \frac{1}{(r+si)^{2n}} = \frac{(2\omega)^{4n}}{(4n)!} E_n,$$

where the summation is over all complex integers $r + si$ except 0 and $s = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

The most interesting property proved by Hurwitz for the $E_n$ is the following analog of the Staudt-Clausen theorem. Let $p$ be a prime of the form $4k + 1$ so that $p = a^2 + b^2$; we take $a$ odd and such that $a = b + 1 \pmod{4}$.

Then

$$E_n = G_n + \frac{1}{2} + \sum \frac{(2a)^{4n/(p-1)}}{p},$$

where the summation is over all primes $p$ of the form $4k + 1$ such that $p - 1 \mid 4n$; $G_n$ is an integer.

Hurwitz computed the first twelve values of $E_n$ both in the form (2) and also in the form

$$E_n = N_n/D_n \quad (N_n, D_n) = 1,$$

where $N_n$ denotes the numerator and $D_n$ the denominator of $E_n$ in reduced form.

The present writer [2] has discussed arithmetic properties of the coefficients of singular elliptic functions. In particular, he showed that if $p$ is a prime of the form $4k + 3$ then

$$E_m \equiv 0 \pmod{p^{r+*}}$$

when $p^r \mid m$, $p^2 - 1 \not\mid 4m$ and $4m > pr + 1$, while

$$E_m \equiv 0 \pmod{p^{r-1}}$$

when $p^r \mid m$, $p^2 - 1 \mid 4m$ and $4m > pr + 1$. In some cases, but not all, (4) and (5) predict the correct power of the primes $p$ dividing $E_n$. However, the formulas give no information about the occurrence in the numerator of $E_n$ about primes of the form $4k + 1$. An interesting question is whether infinitely many primes $p \equiv 1 \pmod{4}$ occur in the numerators of $E_m$.

In a later paper [3] the writer obtained congruences (mod $2^r$) for the $E_n$ as well as for the coefficients of certain related functions. In particular, he showed that

$$\sum_{i=0}^{r} (-1)^i \left(\begin{array}{c} r \\ i \end{array}\right) 2E_{n+zi} = 0 \pmod{2^{r+*}},$$

provided $2^{r-1} \mid z$, $r \geq 0$ and $n > re$. 
In order to get additional information about the numerators \(N_n\) of \(E_n\), it was thought desirable to compute some additional values of \(N_n\). The following values were computed by R. Carlitz in the Duke University Computing Laboratory, making use of the recurrence (1). We remark that the first twelve values are in agreement with the results given by Hurwitz. We write \(N_n\) for the numerator of \(E_n\) as defined in (3).

\[ N_1 = 1 \]
\[ N_2 = 3 \]
\[ N_3 = 3^2 \cdot 7 \]
\[ N_4 = 3^3 \cdot 7^2 \cdot 11 \]
\[ N_5 = 3^4 \cdot 7^3 \cdot 11 \]
\[ N_6 = 3^5 \cdot 7^4 \cdot 11^2 \cdot 19 \]
\[ N_7 = 3^6 \cdot 7^5 \cdot 11^3 \cdot 19 \cdot 23 \cdot 23 \]
\[ N_8 = 3^7 \cdot 7^6 \cdot 11^4 \cdot 19 \cdot 23 \cdot 31 \]
\[ N_9 = 3^8 \cdot 7^7 \cdot 11^5 \cdot 19 \cdot 23 \cdot 31 \cdot 61 \]
\[ N_{10} = 3^9 \cdot 7^8 \cdot 11^6 \cdot 19 \cdot 23 \cdot 31 \cdot 61 \cdot 121 \]
\[ N_{11} = 3^{10} \cdot 7^9 \cdot 11^7 \cdot 19 \cdot 23 \cdot 31 \cdot 61 \cdot 121 \cdot 127 \]
\[ N_{12} = 3^{11} \cdot 7^{10} \cdot 11^8 \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 1162253 \]
\[ N_{13} = 3^{12} \cdot 7^{11} \cdot 11^9 \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 13 \cdot 8887 \]
\[ N_{14} = 3^{13} \cdot 7^{12} \cdot 11^{10} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 61 \cdot 52289 \]
\[ N_{15} = 3^{14} \cdot 7^{13} \cdot 11^{11} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 61 \cdot 2630966033 \]
\[ N_{16} = 3^{15} \cdot 7^{14} \cdot 11^{12} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 109 \cdot 814903 \]
\[ N_{17} = 3^{16} \cdot 7^{15} \cdot 11^{13} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 17 \cdot 80232721 \]
\[ N_{18} = 3^{17} \cdot 7^{16} \cdot 11^{14} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 4831651011193 \]
\[ N_{19} = 3^{18} \cdot 7^{17} \cdot 11^{15} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 3469 \cdot 1330177 \]
\[ N_{20} = 3^{19} \cdot 7^{18} \cdot 11^{16} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 503 \cdot 1389248989981 \]
\[ N_{21} = 3^{20} \cdot 7^{19} \cdot 11^{17} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 79 \cdot 22143032496967 \]
\[ N_{22} = 3^{21} \cdot 7^{20} \cdot 11^{18} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 79 \cdot 83 \cdot 6195097 \cdot 11133763 \]
\[ N_{23} = 3^{22} \cdot 7^{21} \cdot 11^{19} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 79 \cdot 83 \cdot 474230496549203 \]
\[ N_{24} = 3^{23} \cdot 7^{22} \cdot 11^{20} \cdot 19^2 \cdot 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot 59 \cdot 67 \cdot 71 \cdot 79 \cdot 83 \cdot 859 \cdot 3137 \cdot 7753 \cdot 70880471 \]

In every case the \(N_n\) has been factored completely.

It will be noted that \(N_{13} \equiv 0 \pmod{13}\) and \(N_{17} \equiv 0 \pmod{17}\). This suggests the following

**Theorem.** If \(p\) is a prime greater than 5, we have

\[ N_p \equiv 0 \pmod{p}. \]  

**(7)**

More generally if

\[ p' \mid m, \quad p - 1 \nmid 4m, \quad (-1)^m 2^m - 1 \neq 1 \pmod{p} \]

then

\[ N_m \equiv 0 \pmod{p'}. \]  

**(9)**

**Proof.** If we put

\[ \frac{u}{\phi(u)} = \sum_{n=0}^{\infty} \beta_{4n} \frac{x^{4n}}{(4n)!} \quad (\phi(u) = (\Phi(u))^{-1/2}) \]
then, as Hurwitz showed,

\[ \beta_{4m} = (1 + i)^{4m}(1 + i)^{4m} - 2E_m. \]

The writer [1, Theorem 9] has proved that if \( p \mid m \), \( p - 1 \not\equiv 4m \), then

\[ \beta_{4m} \equiv 0 \pmod{p'}. \]

Thus (10) yields

\[ \{ (1 + i)^{4m} - 2\}E_m \equiv 0 \pmod{p'}. \]

Since \( p - 1 \not\equiv 4m \), the denominator of \( E_m \) is not divisible by \( p \), so that (11) implies

\[ \{ (1 + i)^{4m} - 2\}N_m \equiv 0 \pmod{p'}. \]

In the next place, since

\( (1 + i)^{4m} = (-4)^m, \)

it is evident that

\( (1 + i)^{2m} - 2 \equiv 0 \pmod{p} \)

if and only if

\[ (-1)^{m_2^{2m-1}} \equiv 1 \pmod{p}. \]

If, therefore, (13) is not satisfied, it is clear from (12) that

\[ N_m \equiv 0 \pmod{p'}. \]

Finally we note that if \( p = m \), then (13) is not satisfied and (7) follows at once.

Duke University
Durham, North Carolina


**All Factors \( q < 10^8 \) in All Mersenne Numbers \( 2^p - 1, \ p \text{ Prime} < 10^4 \)**

By H. Riesel

During the year 1960 the author made additional investigations respecting factors of Mersenne numbers \( M_p = 2^p - 1 \), where \( p \) is a prime. The author has earlier examined the least factor \( q \) of \( M_p \), for \( p < 10^4 \), if \( q < 10 \cdot 2^{20} \approx 10^7 \). This inquiry was secondary to an effort to ascertain Mersenne primes [1].

The present examination, which resulted in 255 new factors of the numbers \( M_p \), was done in the following manner: All primes \( s < 10^3 \), for which \( s \equiv 1 \pmod{2p} \) and \( s \equiv 1 \pm \pmod{8} \) were tested as factors of \( M_p \) for \( 1200 < p < 10000 \). The range \( p < 1200 \) has been examined by Brillhart and Johnson [2].

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