An Approximation to the Fermi Integral $F_{1/2}(x)$

By H. Werner and G. Raymann

The Fermi Integral as defined, for instance, in the Handbuch der Physik, Bd. XX, S. 58 [1], is given by

$$F_p(x) = \int_0^\infty \frac{t^p}{e^{tx} + 1} \, dt.$$ \hfill (1)

The function $F_{1/2}(x)$ has for negative values of $x$ an expansion of the form

$$F_{1/2}(x) = \frac{\sqrt{\pi}}{2} \sum_{r=1}^{\infty} (-1)^{r-1} e^{\frac{r^2}{4}} \left( \frac{\pi^2}{12 \cdot x^2} + \frac{\left( \frac{3}{8} \right) - \frac{7}{60} \cdot \frac{\pi^4}{x^4} + \cdots} {\binom{\left( \frac{3}{8} \right) - 1}{2n - 1} \frac{2^{2n-1}}{n} \cdot \frac{\pi^{2n}}{x^{2n}} + \cdots} \right);$$ \hfill (2)

and for large positive $x$ the asymptotic expansion

$$F_{1/2}(x) \sim x^{3/2} \left[ \frac{2}{3} + \frac{\pi^2}{12 \cdot x^2} + \frac{\left( \frac{3}{8} \right) - \frac{7}{60} \cdot \frac{\pi^4}{x^4} + \cdots} {\binom{\left( \frac{3}{8} \right) - 1}{2n - 1} \frac{2^{2n-1}}{n} \cdot \frac{\pi^{2n}}{x^{2n}} + \cdots} \right];$$ \hfill (3)

compare [2], formulas (10) and (12);

$B_{2n}$ are the Bernoulli numbers, given for example in [3], page 298. We obtained Chebyshev approximations to $F_{1/2}(x)$, based upon the table by McDougall and Stoner [4]. This table was subtabulated by interpolation with a fifth-degree polynomial. The approximations are

$$F_{1/2}^*(x) = e^x \sum_{r=0}^5 a_r e^{rx} \quad \text{for} \quad -\infty < x \leq 1;$$ \hfill (4)

$$F_{1/2}^*(x) = x^{3/2} \left[ \frac{2}{3} + \sum_{r=0}^5 b_r x^{2r+2} \right] \quad \text{for} \quad +1 < x < +\infty,$$
the coefficients

\[\begin{array}{ccc}
\nu & a_\nu & b_\nu \\
0 & +0.8860 7596 & +0.8435 00 \\
1 & -0.3087 1705 & +0.7108 09 \\
2 & +0.1463 8520 & -3.7124 56 \\
3 & -0.0584 3877 & +6.7056 28 \\
4 & +0.0143 1771 & -5.5948 77 \\
5 & -0.0015 0176 & +1.7777 87 \\
\end{array}\]

With these approximations, the relative error \(|F_{1/2}(x) - F^*_1(x)|/F_{1/2}(x)|\) is less than \(2 \times 10^{-4}\) and \(5 \times 10^{-4}\), respectively.

Another intensive table of \(F_p(x)\) has been given by G. A. Chisnall [5] who also discusses in [6] a method for the interpolation of the existing tables of \(F_{1/2}(x)\). It is not difficult to obtain analogous Chebyshev approximations to \(F_p(x)\) for any fixed values of \(p\) to a prescribed degree of accuracy if one is able to generate the function with this (or slightly more) accuracy.

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**On the Congruences \((p - 1)! \equiv -1\) and \(2^{p-1} \equiv 1 \pmod{p^2}\)**

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The results of computations to determine primes \(p\) such that one of the relations

\[\begin{align*}
(1) & \quad (p - 1)! \equiv -1 \pmod{p^2}, \\
(2) & \quad 2^{p-1} \equiv 1 \pmod{p^2}
\end{align*}\]

holds have been published previously [1-5]. The known Wilson primes (those satisfying (1)) are 5, 13, and 563, the last having been determined by Goldberg [3] in testing \(p < 10^4\). Froberg [4] tested \(10^4 < p < 30,000\) without finding additional Wilson primes.

Froberg [4] determined \(p = 1093\) and \(p = 3511\) to be the only primes less than

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