then $D_n$ is also congruent to a diagonal matrix of the stated form. From the law of inertia ([1], p. 296–298) $D_n$ also has $p$ positive and $q$ negative elements and the proof is complete.

From this result Theorem 1 follows as a corollary since, if $w(x)$ is nonnegative, $C_n$ is positive definite and therefore congruent to a diagonal matrix with $n$ positive elements.

As a simple example consider a 2-point quadrature formula of the form

$$\int_1^1 (3 - 5 | x |) f(x) \, dx \simeq A_1 f(x_1) + A_2 f(x_2).$$

For this weight function the monomial integrals are $c_0 = 1, c_1 = 0, c_2 = -1/2, c_3 = 0$. There are no real values of $x_1, x_2$ for which (3) can be made exact for $f(x) = 1, x, x^2, x^3$. There are, however, an infinity of such formulas with real $x_1, x_2$ which are exact for $f(x) = 1, x, x^2$ and Theorem 2 still applies. One such formula is

$$x_1 = \frac{1}{2}, \quad x_2 = 1 \quad A_1 = 2, \quad A_2 = -1.$$
Proof. For each \(i\) the congruences
\[
\begin{cases}
    \{ a_i \neq 0 \pmod{p_i}, \\
    \{ a_i \neq m \pmod{p_i}, 
\end{cases}
\]
have a common solution. For if \(p_i = 2\) then \(a_i = 1\) is a solution; otherwise, \(p_i \geq 3\) and equations (1) eliminate at most 2 of the \(p\) congruence classes mod \(p_i\).

Let \(a = a_i \pmod{p_i}\); by the Chinese Remainder Theorem such an \(a\) exists (unique mod \(l\)). Clearly \(a\) satisfies the conclusions of the lemma.

**Lemma 2.** If \(p\) is a prime \(\geq 5\), then for any \(a, m, l\), with \((l, p) = 1\), there is a \(b\) such that
\[
\begin{align*}
    (2) \quad b &\equiv a \pmod{l}, \\
    (3) \quad 1 \leq b \leq 3lp/5, \\
    (4) \quad (b, p) = (m - b, p) = 1.
\end{align*}
\]

Proof. Let \(x_r = a + rl\). At least \(p - 2\) of the integers \(x_1, x_2, \ldots, x_p\) satisfy
\[
x_r \not\equiv 0 \pmod{p}
\]
and
\[
x_r \not\equiv m \pmod{p}.
\]

Let \(y_1, y_2, \ldots, y_{p-2}\) be the least positive residues \((\text{mod } lp)\) of any \(p - 2\) of such solutions. Let \(b = \text{Min} (y_i)\). Then clearly equations (2), (4) are satisfied and
\[
1 \leq b \leq 3l, \\
\leq 3lp/5, \quad \text{since } p \geq 5.
\]

**Lemma 3.** If \(t = \prod_{i=1}^s p_i\) where the \(p_i\) are distinct primes with \(p_r \geq 5\), and if \(m\) is even, then there is a \(b\) such that
\[
\begin{align*}
    (b, t) &= 1, \\
    (m - b, t) &= 1,
\end{align*}
\]
and
\[
1 \leq b \leq 3t/5.
\]

Proof. If \(s = 1\), the result follows from Lemma 2 with \(a = l = 1\). If \(s > 1\), the result follows from Lemmas 1 and 2 with
\[
l = \prod_{i=1}^{s-1} p_i.
\]

Definition. We define "a is \(P_r(b)\)" to mean "There exist \(a_1, a_2, \ldots, a_r\) such that \(a = \sum_{i=1}^r a_i\) and \((a_i, b) = 1 \quad (i = 1, \ldots, r)\)."

**Lemma 4.** If \(m\) is an even integer \(\geq 6\) and \(m \geq 3n/5\) then \(m\) is \(P_2(n)\).

Proof. Let \(n = \prod_{i=1}^r p_i^n\) where the \(p_i\) are distinct primes. Let \(t = \prod_{i=1}^r p_i\) and suppose, without loss of generality, \(p_r = \text{Max}(p_i)\). We consider two cases.

(a) \(P_r \geq 5\). Let \(b\) be defined as in Lemma 3. Then
A PARTITION PROBLEM

1 \leq b \leq 3t/5
\leq 3n/5, \text{ by definition of } t
\leq m, \quad \text{by hypothesis}
< m, \quad \text{since } (m-b, n) = 1.

Therefore, \( c = m - b \) is a positive integer; i.e., since \( m = b + c, \) \( m \) is \( P_2(n) \).

(b) \( P_r < 5 \). Now \( t = 1, 2, 3 \) or 6. It may be verified that at least one of the partitions \( m = 1 + (m - 1) \) or \( m = 5 + (m - 5) \) must have both parts prime to \( t \).

The result now follows.

\textbf{Lemma 5.} If \( m \geq 2, n \geq 1 \) and

either (a) \( m \geq 3n/5 + 1 \) \text{ and } \( (m, n) \neq (5, 6) \),
or (b) \( m \geq (3n + 8)/5 \),
or (c) \( m \geq n \),
then \( m \) is either \( P_2(n) \) or \( P_3(n) \).

\textit{Proof.} (a) If \( m \) is an even integer \( \geq 6 \), then \( m \) is \( P_2(n) \) by Lemma 4. If \( m \) is an odd integer \( \geq 7 \), then \( m - 1 \) is an even integer \( \geq 3n/5 \) and hence \( m - 1 \) is \( P_2(n) \) by Lemma 4; i.e., \( m \) is \( P_3(n) \). The cases \( m = 2, 3, 4, 5 \) may be settled by inspection of the following partitions:

\[ 2 = 1 + 1, \]
\[ 3 = 1 + 1 + 1, \]
\[ 4 = 2 + 2 = 3 - 1 \quad (n \leq 5/3(3) = 5), \]
\[ 5 = 3 + 1 + 1 = 2 + 2 + 1 \quad (n \leq 5/3(4) < 7). \]

(b) If \( (m, n) \neq (5, 6) \) the result follows from (a). If \( (m, n) = (5, 6) \) then \( m < (3n + 8)/5 \).

(c) If \( n \geq 3 \) then \( m \geq n > 3n/5 + 1 \) and the result follows from (a). If \( n = 1 \) or 2, the result follows since \( m = (m - 1) + 1 = (m - 2) + 1 + 1 \) and either \( m - 1 \) or \( m - 2 \) is odd.

\textbf{Lemma 6.} If \( u \geq w \geq 2 \) and \( u \geq w + (3v - 2)/5 \) then \( u \) is either \( P_u(v) \) or \( P_{u+1}(v) \).

\textit{Proof.}

\[ u = 1 + 1 + \cdots 1 + (u - w + 2) \]
\[ \underbrace{w - 2} \]

and \( (u - w + 2) \geq (3v + 8)/5 \), by hypothesis. Hence, by Lemma 5(b), \( (u - w + 2) \) is either \( P_u(v) \) or \( P_s(v) \). Therefore, \( u \) is either \( P_u(v) \) or \( P_{u+1}(v) \).

3. Main Theorem.

\textbf{Theorem 1.} If \( a, b, c, d \) satisfy

(5) \quad a, b, c, d, \geq 2

then for some \( m, n \) with \( |m - n| \leq 1 \), either \( a \) is \( P_m(b) \) and \( c \) is \( P_n(d) \), or \( b \) is \( P_m(a) \) and \( d \) is \( P_n(c) \).
Proof. Suppose the theorem is false. We shall deduce a contradiction. We consider two cases.

(i) Either \((a, b), (b, a), (c, d), \) or \((d, c) = (5, 6)\). Suppose, without loss of generality, \((a, b) = (6, 5)\). Now \(6 = 4 + 2\); hence \(6\) is \(P_4(5)\). Therefore \(c\) is not \(P_2(d)\) or \(P_3(d)\). By the converse of Lemma 5, \(d > c\); i.e.,
\[
d \geq c + 1.
\]
Now \(6 = 3 + 3 = 3 + 2 + 1 = 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1\); i.e., \(6\) is \(P_2(5), P_3(5), P_4(5), P_5(5), P_6(5)\). Now \(c\) is \(P_c(d)\), trivially. Therefore
\[
c \geq 8.
\]
Finally \(5\) is \(P_4(6)\), trivially; hence \(d\) is not \(P_4(c)\) or \(P_5(c)\). Therefore, by the converse of Lemma 6 with \(u = d\), \(v = c\), and \(w = 4\),
\[
d < (3c + 18)/5.
\]
From (6) and (8),
\[
c + 1 < (3c + 18)/5, \quad 2c < 13,
\]
which contradicts (7).

(ii) None of \((a, b), (b, a), (c, d), (d, c) = (5, 6)\). Without loss of generality, suppose \(a \geq b\). Then by Lemma 5(c) \(a\) is either \(P_2(b)\) or \(P_3(b)\). Hence \(c\) is neither \(P_2(d)\) nor \(P_3(d)\); by the converse of Lemma 5,
\[
c < d,
\]
\[
c < 3d/5 + 1.
\]
Similarly from (9) we deduce
\[
b < a,
\]
\[
b < 3a/5 + 1.
\]
Suppose without loss of generality, \(a \geq d\). In (10)
\[
c < 3a/5 + 1.
\]
From (12) and (13)
\[
3b + 5c < 3(3a/5 + 1) + 5(3a/5 + 1)
\]
\[
< \frac{3}{5}a + 8
\]
\[
< 5a + 8
\]
\[
\leq 5a + 7.
\]
i.e. \(a \geq (c - 1) + (3b - 2)/5\). Hence, by Lemma 6, \(a\) is either \(P_{c-1}(b)\) or \(P_c(b)\). Now \(c\) is \(P_c(d)\) trivially; hence we have the required contradiction.

4. Remark. The following theorem shows that Theorem 1 is best possible in that condition (5) cannot be relaxed.

Theorem 2. For arbitrary \(K\) there exist \(a, b, c, d, \) with
APPROXIMATIONS TO KELVIN FUNCTIONS

295

\[ a = 1, \]
\[ b, c, d > K \]

and such that the conclusion of Theorem 1 is false.

Proof. Let \( b = N, d = 2^M - 2, c = 2^{-r}(d!) \), where \( r \) is chosen to make \( c \) an odd integer. Clearly \( a \) is \( P_s(b) \) only for \( s = 1 \). Now \( c \) is not \( P_1(d) \), provided \( d > 3 \), and not \( P_s(d) \) since an odd integer cannot be the sum of two odd integers. Hence, we cannot find partitions of \( a, c \) satisfying the conclusions of Theorem 1.

Suppose \( d \) is \( P_s(c) \). Then

\[ d = d_1 + d_2 + \cdots + d_s \]

where each \( d_i \leq d \) and \( (d_i, c) = 1 \). Now \( c \) is divisible by all odd integers \( < d \); therefore \( d_i \) is a power of 2. I.e.,

\[ d = 2^{r_1} + 2^{r_2} + \cdots 2^{r_s}. \]

Since \( d = 2^M - 2 \) there are at least \( M - 1 \) summands in (14). I.e., if \( d = P_s(c) \), then \( s \geq M - 1 \). But clearly if \( b \) is \( P_s(a) \), then \( s \leq N \). If we now choose \( M - 1 > N + 1 \) and \( M, N \) large enough to ensure \( b, c, d > K \), the conclusion of Theorem 2 follows.

International Business Machines Corporation
Thomas J. Watson Research Center
Yorktown Heights, New York


Approximations to Kelvin Functions

By F. D. Burgoyne

While preparing a digital computer program to examine the behavior of large-taper hub flanges, it was found necessary to use approximations to the Kelvin functions \( \text{ber} \ x, \text{bei} \ x, \ker \ x, \text{kei} \ x \), and to their first derivatives. To obtain full machine accuracy, the approximations were required to be correct to nine significant figures. Several tabulations of these functions exist, but the only ones considered to be sufficiently accurate were those of Lowell [1] and Nosova [2]; however, limitations of internal memory in the computer used precluded the possibility of storing such tables and interpolating.

The functions actually required were \( Z_i(x) \) and \( Z_i'(x) \) (1 ≤ \( i \) ≤ 4), where

\[ Z_1(x) = \text{ber} \ x \]
\[ Z_2(x) = -\text{bei} \ x \]
\[ Z_3(x) = -\frac{2}{\pi} \text{kei} \ x \]
\[ Z_4(x) = -\frac{2}{\pi} \ker \ x; \]

Received February 5, 1962.