Jacobi Polynomial Expansions of a Generalized Hypergeometric Function over a Semi-Infinite Ray

By Y. L. Luke and J. Wimp

1. Introduction. Suppose \( f(x) \) is continuous and has a piecewise continuous derivative for \( 0 \leq x/\lambda \leq 1 \). Then \( f(x) \) may be expanded into a uniformly convergent series of shifted Jacobi polynomials in the form

\[
f(x) = \sum_{n=0}^{\infty} a_n(\lambda) R_n^{(\alpha, \beta)}(x/\lambda),
\]

(1.1)

where \( R_n^{(\alpha, \beta)}(x) = P_n^{(\alpha, \beta)}(2x - 1) \) and the latter is the usual notation for the Jacobi polynomial \([1, \text{Ch. 10}]\). Various techniques are available for the determination of the coefficients \( a_n(\lambda) \). In this connection, see, for example, the references \([2, 3, 4, 5, 6, 7]\).

Suppose that \( f(x) \) satisfies the above conditions for \( 1 \leq x/\lambda \leq \infty \) where \( |\arg \lambda| < \varphi \). Then we may write

\[
f(x) = \sum_{n=0}^{\infty} b_n(\lambda) R_n^{(\alpha, \beta)}(\lambda/x),
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If \( f(x) \) has an asymptotic expansion of the form

\[
f(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \quad x \to \infty, \quad |\arg x| < \varphi,
\]

(1.3)

then (1.2) may be interpreted as a summability process which converts the generally divergent expansion (1.3) into a convergent expansion. If \( f(x) \) in (1.3) is of hypergeometric type, then the coefficients \( b_n(\lambda) \) may be found formally at least using the procedures \([5, 6]\). These yield for \( b_n(\lambda) \) an asymptotic series in \( \lambda \) which is also of hypergeometric type. The asymptotic representation for \( b_n(\lambda) \) in general is not suitable for computation. We are confronted with two problems: one is the interpretation of the asymptotic series for \( b_n(\lambda) \), and the other is the computation of \( b_n(\lambda) \).

In this paper, we show how both problems can be solved for a confluent hypergeometric function. Actually we derive a representation for \( b_n(\lambda) \) when \( f(x) \) is the G-function, which includes the confluent hypergeometric function as a special case. Our computational scheme for \( b_n(\lambda) \) is exhibited only when \( f(x) \) is a confluent hypergeometric function, although the ideas involved can be extended to cover other special cases of the G-function as well.

In Section II, we prove an expansion theorem of the form (1.2) when \( f(x) \) is the...
G-function and show how both convergent and asymptotic representations for $b_n(\lambda)$ may be derived. These results are specialized in Section III for the case when $f(x)$ is a confluent hypergeometric function, and in Section IV it is shown how $b_n(\lambda)$ may be computed by a recursion scheme. Finally, in Section V, we tabulate coefficients for the cases where $R_n^{(a,b)}(x)$ is the shifted Chebyshev polynomial and $f(x)$ is the error function, the exponential, sine and cosine integrals, and the Bessel functions $K_\ell(x)$ and $K_\ell(x)$.

2. Expansion of the G-Function. The G-function is given by the Mellin-Barnes integral

$$G_{p,q}(\lambda x | a^p, b^q) = \frac{1}{2\pi i} \int_L \prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=m+1}^{p} \Gamma(1 - b_j + s) \prod_{j=1}^{k} \Gamma(1 - a_j + s) \prod_{j=k+1}^{p} \Gamma(1 - a_j - s) \ (\lambda x)^s \ ds,$$

where an empty product is interpreted as $1$, $0 \leq m \leq q$, $0 \leq n \leq p$ and the parameters are such that no pole of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$ coincides with any pole of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$. We assume $x$ is real and the path $L$ runs parallel to the imaginary axis and is indented so that the poles of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$, are to the right, and all the poles of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$, to the left of $L$. The integral converges if $p + q < 2(m + k)$ and $| \arg \lambda | < (m + k - p/2 - q/2)\pi$. For a treatment of the G-function, see [1, Ch. 5].

Now from [1, 10.20(3)] we have the expansion

$$x^s = \frac{\Gamma(a + b + 1)(n + 1)}{\Gamma(n + a + b + 2)\Gamma(1 - s - n)} R_n^{(a,b)}(1/x), \ 1 < x < \infty,$$

uniformly for $Re(s) \leq \delta$, $\delta > 0$, $0 = \min(\beta + 1, \beta/2 + \frac{1}{4})$, $\alpha > -1$, $\beta > -1$. Put (2.2) in (2.1) and integrate along the path from $\lambda - \delta - i\infty$ to $\lambda - \delta + i\infty$. We then get

**Theorem I.** Let

1. $\alpha, \beta$ and $x$ be real, $\alpha > -1, \beta > -1, 1 < x < \infty$.

Let a real positive $\delta$ exist such that

2. (a) $Re(a_j - 1) < \mu - \delta, j = 1, 2, \cdots, k$; (b) $Re(b_j) > \mu - \delta, j = 1, 2, \cdots, m, \mu - \delta < 1, \mu = \min(\beta + 1, \beta/2 + \frac{1}{4})$.

3. $p + q < 2(m + k), | \arg \lambda | < (m + k - p/2 - q/2)\pi, \lambda \neq 0, 0 \leq n \leq p$.

Then

$$G_{p,q}(\lambda x^s | a^p, b^q) = \sum_{n=0}^{\infty} (2n + \alpha + \beta + 1)(n + \beta + 1) \times G_{p+2,q+2}(\lambda | a^{p}, b^{q} + 1, 1) R_n^{(a,b)}(1/x).$$

**Remark.** Assumptions 2 above insure the separation of poles and specify the regions in which they lie according to the remarks surrounding (2.1). Notice, however, that poles of $\Gamma(b_j - s)$ may lie to the left of the contour. They may be excluded.
by indentations since they lie in a region where the series for $x^a$ converges uniformly, provided they do not coincide with any of the poles of $\Gamma(1 - a + s)$. Hence, we may replace $2(b)$ by the weaker but more complicated condition

$$2(b) \quad 1 + \delta_{j-2} - \alpha_j \neq 0, -1, -2, \cdots,$$

$$j = 1, 2, \cdots m + 2, h = 1, 2, \cdots k, \delta_{-2} = 1, \delta_{-1} = \beta + 1, \delta_{j-2} = b_j, j > 1.$$  

Notice from the definition of the $G$-function that

$$G_{p+2,q+2}(\lambda |_{1,0}^{\beta+1,0} |_{1,0}^{\beta+1,0}) = (-)^n G_{p+2,q+2}(\lambda |_{1,0}^{\beta+1,0} |_{1,0}^{\beta+1,0}).$$

If $|\arg \lambda| < \frac{1}{2}(p - q + 1)\pi$, an asymptotic representation for the coefficients of $R_n(a,b)(1/x)$ in (2.3) follows by application of a result in [1, 5.3(6)]. An ascending series representation follows when [1, 5.3(5)] is applied to the right-hand side of (2.4).

3. Expansion of a Confluent Hypergeometric Function. We consider the function [1, Ch. 6],

$$G(a, c | x) = \frac{\Gamma(a) \Gamma(c)}{\Gamma(a+c)} = \sum_{n=0}^{\infty} C_n(a) T_n(x),$$  

Also, denote by $T_0(x)$ the shifted Chebyshev polynomial

$$T_n(x) = T_n(x - 1) = n! R_n^{-1/2,-1/2}(x).$$

From Theorem I, we get

**Theorem II.** Let

1. $1 \leq x \leq \infty$;
2. $a \neq 0, -1, -2, \cdots$; $\alpha \neq 0, -1, -2, \cdots$;
3. $|\arg \lambda| < 3\pi/2, \lambda \neq 0$.

Then

$$G(a, c | x) = \sum_{n=0}^{\infty} C_n(a) T_n(x),$$

where

$$C_n(a) = \frac{\epsilon_n}{\lambda^{1/2} \Gamma(a) \Gamma(c)} G_{2,4}(\lambda |_{0,0}^{1,0,0,0} |_{0,0}^{1,0,0,0}), \quad \epsilon_0 = 1, \epsilon_n = 2, n > 0,$$

or

$$C_n(a) = \frac{\epsilon_n(-)^n}{\lambda^{1/2} \Gamma(a) \Gamma(c)} G_{2,4}(\lambda |_{0,0}^{1,0,0,0} |_{0,0}^{1,0,0,0}).$$

Also, if none of the quantities $\frac{1}{2}, a$ and $\sigma$ differ by an integer

$$C_n(a) = \frac{\epsilon_n(-)^n}{\lambda^{1/2}} \left\{ (a)_{-1/2}(\sigma)_{-1/2} \lambda^{1/2} \frac{\Gamma(a+1/2, -n/2, -n/2-\sigma)}{\Gamma(a+n/2+1)} \right\},$$  

$$+ \frac{\Gamma(\frac{1}{2} - a)(\gamma)_{-1/2} \lambda^{1/2} \frac{\Gamma(a+1/2, -n/2-\sigma)}{\Gamma(a+n/2+1)}},$$

$$+ \frac{\Gamma(\frac{1}{2} - \sigma)(\sigma)_{-1/2} \lambda^{1/2} \frac{\Gamma(\sigma)}{\Gamma(\sigma+1)}}{\Gamma(n - \sigma + 1)}.$$
and

\[ C_n(\lambda) \sim \frac{\epsilon_n(-)^n(a)\epsilon_n(\sigma)n!}{(4\lambda)^n} {}_3F_1 \left( \frac{n+1/2, n+a, n+\alpha}{2n+1} \left| -\frac{1}{\lambda} \right. \right), \quad |\lambda| \to \infty, \quad |\arg \lambda| < \pi. \]

Remark. Condition 1 of Theorem I is conservative. By an appeal to the convergence properties of expansions in Chebyshev polynomials [7], the range of \( x \) may be extended to give condition 1 above.

Since (3.3) converges,

\[ \lim_{n \to \infty} C_n(\lambda) = 0. \]

For later use, we record the fact that

\[ \lim_{x \to \infty} (\lambda x)^a \psi(a, c | \lambda x) = 1, \quad |\arg \lambda| < \frac{3\pi}{2}. \]

4. Calculation of the Coefficients \( C_n(\lambda) \). Let

\[ \varphi_{1,n}(\lambda) = \frac{(-)^n}{\epsilon_n} C_n(\lambda). \]

Following the method developed in [8], we can show from the representation (3.7) that \( \varphi_{1,n}(\lambda) \) satisfies the recursion relation

\[ \varphi_n(\lambda) + (A_n + B_n\lambda)\varphi_{n+1}(\lambda) + (C_n + D_n\lambda)\varphi_{n+2}(\lambda) + E_n\varphi_{n+3}(\lambda) = 0, \]

where

\[ A_n = (2n + 2) \left[ 1 - \frac{(n + \frac{3}{2})(n + a + 1)(n + \sigma + 1)}{(n + 2)(n + a)(n + \sigma)} \right], \]

\[ B_n = D_n = -4(n + 1)/(n + a)(n + \sigma), \]

\[ C_n = -1 + [2(n + 1)(2n + 3)/(n + a)(n + \sigma)], \]

\[ E_n = -(n + 1)(n - a + 3)(n - \sigma + 3)/(n + 2)(n + a)(n + \sigma). \]

We prove that the coefficients may be readily evaluated using (4.2) in the backward direction. This backward recursion technique has been treated by many authors [9], [10], [11], [12], [13]. The idea is as follows.

For fixed \( \lambda \), arbitrary \( \eta \) and \( \nu \) sufficiently large set

\[ \varphi_{\nu}(\lambda) = \varphi_{\nu-1}(\lambda) = 0, \]

\[ \varphi_{\nu-\frac{1}{2}}(\lambda) = \eta. \]

The sequence \( \varphi_{\nu-\frac{1}{2}}(\lambda), \ldots, \varphi_{\nu}(\lambda), \ldots, \varphi_1(\lambda), \varphi_0(\lambda) \) is generated from (4.2). Using (3.9) and

\[ T_n^*(0) = (-)^n \]

in (3.3) we would expect that if

\[ \omega_\nu = \sum_{\nu=0}^{\nu-2} \epsilon_\nu \varphi_\nu(\lambda), \]

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\[ C_n(\lambda) \sim (-)^n \epsilon_n \varphi_n^{(\nu)}(\lambda) / \omega_n, \]

with increasing accuracy as \( n \to \infty \). In fact if we define
\[ \varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,0}(\lambda) \varphi_n^{(\nu)}(\lambda) / \varphi_0^{(\nu)}(\lambda), \]
we have:

**Theorem III.** Let \( |\arg \lambda| < \pi, \lambda \neq 0 \), and neither \( \alpha \) nor \( \sigma \) be a negative integer or zero. Then
\[ \lim_{n \to \infty} \varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,1}(\lambda). \]

**Proof.** Denote by \( \varphi_{1,n}(\lambda), \varphi_{2,n}(\lambda) \) and \( \varphi_{3,n}(\lambda) \) the three linearly independent solutions of (4.2); \( \varphi_{1,n}(\lambda) \) is the solution we wish to calculate. We may write
\[ \varphi_n^{(\nu)} = \xi_1^{(\nu)} \varphi_{1,n} + \xi_2^{(\nu)} \varphi_{2,n} + \xi_3^{(\nu)} \varphi_{3,n}, \quad n < \nu - 2, \]
and the conditions (4.4) and (4.5) give
\[ 0 = \xi_1^{(\nu)} \varphi_{1,v} + \xi_2^{(\nu)} \varphi_{2,v} + \xi_3^{(\nu)} \varphi_{3,v}, \]
\[ 0 = \xi_1^{(\nu)} \varphi_{1,v-1} + \xi_2^{(\nu)} \varphi_{2,v-1} + \xi_3^{(\nu)} \varphi_{3,v-1}, \]
\[ \eta = \xi_1^{(\nu)} \varphi_{1,v-2} + \xi_2^{(\nu)} \varphi_{2,v-2} + \xi_3^{(\nu)} \varphi_{3,v-2}, \]
where \( \xi_1^{(\nu)}, \xi_2^{(\nu)} \) and \( \xi_3^{(\nu)} \) are independent of \( n \).

\[ \frac{\xi_2^{(\nu)}}{\xi_1^{(\nu)}} = \gamma_v, \quad \frac{\xi_3^{(\nu)}}{\xi_1^{(\nu)}} = \delta_v, \]
\[ \gamma_v = [-\varphi_{1,v-2} + \varphi_{1,v}, -\varphi_{2,v-1}, 1] / \tau_v, \]
\[ \delta_v = [-\varphi_{1,v-1} + \varphi_{1,v}, 1 + \varphi_{2,v}] / \tau_v, \]
\[ \tau_v = [\varphi_{1,v}, \varphi_{2,v}, 1]. \]

Thus
\[ \varphi_{1,n}^{(\nu)} = \frac{\varphi_{1,n} \left\{ 1 + (\gamma \varphi_{2,n} / \varphi_{1,n}) + (\delta \varphi_{3,n} / \varphi_{1,n}) \right\}}{1 + (\gamma \varphi_{2,n} / \varphi_{1,n}) + (\delta \varphi_{3,n} / \varphi_{1,n})}. \]

We will show that
\[ \lim_{n \to \infty} \gamma_v = \lim_{n \to \infty} \delta_v = 0. \]
Equation (3.8) gives
\[ \lim_{n \to \infty} \varphi_{1,v} = 0. \]

It may be directly verified that
\[ \varphi_{2,n} = C n^{2/3(\alpha+\sigma-2)} \exp \left[ \frac{\lambda}{n^{1/3}} \right] \cdot \left[ 1 + O \left( \frac{1}{n} \right) \right], \quad |\arg \lambda| < \pi, \]

* Henceforth we write, \( \xi_1^{(\nu)}(\lambda) = \xi_1^{(\nu)}, \varphi_{1,n}(\lambda) = \varphi_{1,n}, \) etc.
### Table I

Coefficients for the Series

\[
\begin{align*}
-Ei(-x) &= \int_x^\infty \frac{e^{-t}}{t} \, dt = \frac{e^{-x}}{x} \sum_{n=0}^{\infty} A_n T_n \left( \frac{4}{x} \right), \\
\text{Erfc}(x) &= \int_x^\infty e^{-t^2} \, dt = \frac{e^{-x^2}}{2x} \sum_{n=0}^{\infty} B_n T_{2n} \left( \frac{2}{x} \right),
\end{align*}
\]

\[4 \leq x \leq \infty, \quad 2 \leq x \leq \infty.\]

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Table II

Coefficients for the Series

\[ K_0(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{n=0}^{\infty} A_n T_n^* \left( \frac{1}{x} \right), \quad \frac{2}{x} \leq x \leq \infty, \]

\[ K_1(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{n=0}^{\infty} B_n T_n^* \left( \frac{1}{x} \right), \quad \frac{2}{x} \leq x \leq \infty. \]

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<td>0.98240 25732 25254 (-05)</td>
<td>0.21254 28930 87307 (-05)</td>
<td>27</td>
<td>-0.52547 (-15)</td>
<td>0.34480 (-15)</td>
</tr>
<tr>
<td>7</td>
<td>-0.18973 43014 87133 (-05)</td>
<td>0.13157 50436 91368 (-05)</td>
<td>28</td>
<td>0.11378 (-15)</td>
<td>-0.22564 (-15)</td>
</tr>
<tr>
<td>8</td>
<td>0.10063 43594 1568 (-06)</td>
<td>-0.55848 57495 6974 (-06)</td>
<td>29</td>
<td>0.512 (-17)</td>
<td>-0.10258 (-15)</td>
</tr>
<tr>
<td>9</td>
<td>0.80819 36482 224 (-07)</td>
<td>0.12353 72625 6629 (-06)</td>
<td>30</td>
<td>-0.2244 (-16)</td>
<td>-0.3571 (-16)</td>
</tr>
<tr>
<td>10</td>
<td>-0.38976 28287 529 (-07)</td>
<td>-0.10318 72179 187 (-07)</td>
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<td>0.1532 (-16)</td>
<td>0.845 (-17)</td>
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<tr>
<td>11</td>
<td>0.10335 65032 550 (-07)</td>
<td>-0.50159 03675 67 (-07)</td>
<td>32</td>
<td>-0.732 (-17)</td>
<td>-0.3 (-19)</td>
</tr>
<tr>
<td>12</td>
<td>-0.14104 34487 59 (-08)</td>
<td>0.36915 98898 01 (-07)</td>
<td>33</td>
<td>0.273 (-17)</td>
<td>-0.146 (-17)</td>
</tr>
<tr>
<td>13</td>
<td>-0.25232 07840 0 (-09)</td>
<td>-0.10980 57370 10 (-08)</td>
<td>34</td>
<td>-0.73 (-18)</td>
<td>0.110 (-17)</td>
</tr>
<tr>
<td>14</td>
<td>0.26099 83632 0 (-09)</td>
<td>0.20285 59643 1 (-09)</td>
<td>35</td>
<td>0.6 (-19)</td>
<td>-0.56 (-18)</td>
</tr>
<tr>
<td>15</td>
<td>-0.10597 88925 4 (-09)</td>
<td>-0.27237 6669 (-11)</td>
<td>36</td>
<td>0.9 (-19)</td>
<td>0.23 (-18)</td>
</tr>
<tr>
<td>16</td>
<td>0.28970 30157 (-10)</td>
<td>-0.19967 52281 (-10)</td>
<td>37</td>
<td>-0.8 (-19)</td>
<td>-0.7 (-19)</td>
</tr>
<tr>
<td>17</td>
<td>-0.41023 1426 (-11)</td>
<td>0.11219 38506 (-10)</td>
<td>38</td>
<td>-0.5 (-19)</td>
<td>0.1 (-19)</td>
</tr>
<tr>
<td>18</td>
<td>-0.10437 6957 (-11)</td>
<td>0.40081 1186 (-11)</td>
<td>39</td>
<td>-0.2 (-19)</td>
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<td>19</td>
<td>0.10904 1845 (-11)</td>
<td>0.96702 841 (-12)</td>
<td>40</td>
<td>0.1 (-19)</td>
<td>-0.1 (-19)</td>
</tr>
<tr>
<td>20</td>
<td>-0.52214 239 (-12)</td>
<td>-0.71861 28 (-13)</td>
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</table>
where $C_1$ is independent of $n$. The third linearly independent solution of (4.2) is the $L_{2,2}(-\lambda)$ term appearing in [15, 1.3.3(15)] which arises in the asymptotic expansion of (4.22) for large $\lambda$. A limit process, explained in [15, 1.3.4] is used to obtain $\varphi_{3,n}$, but our discussion here is necessarily brief. We need only the estimate

\begin{equation}
\varphi_{3,n} = C_2 \frac{\Gamma(n + a - 1) \Gamma(n + \sigma - 1)}{(4\lambda)^n n!} \left[ 1 + O\left(\frac{1}{n^2}\right)\right],
\end{equation}

where $C_2$ is independent of $n$. Thus

\begin{equation}
\lim_{r \to \infty} |\varphi_{2,r}| = \lim_{r \to \infty} |\varphi_{3,r}| = \infty.
\end{equation}

Also, from (4.23) and (4.24), we have

\begin{equation}
\tau_r = -\varphi_{2,r} \varphi_{3,r} \left[ 1 + O\left(\frac{1}{r}\right)\right].
\end{equation}

Hence (4.20) is easily shown and the statement (4.10) follows from (4.19).

5. Tables. Tables I–III contain coefficients to 20 D for the expansions of several important cases of the confluent hypergeometric function [1, 6.9]. Coefficients corresponding to different ranges of the independent variable as well as those for other functions, e.g., $J_\nu(x)$ and $Y_\nu(x)$, are under construction and the present tables are selected examples only. The expansions are readily evaluated using a nesting procedure described in [4], [7]. For similar expansions, see [7], and for many Chebyshev expansions of functions over a finite interval, see [2]–[6] and the references given there. The number in parenthesis after each entry in the tables is the power of 10 by which the entry is to be multiplied.

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