A Note on Some Quadrature Formulas for the Interval \((-\infty, \infty)\)

By Seymour Haber

In a paper in this journal [1], W. M. Harper proposed a family of “Gaussian” quadrature formulas for \(\int_{-\infty}^{\infty} (1 + x^2)^{-k-1} f(x) \, dx\). It is the purpose of this note to re-derive some of his formulas from a different point of view, which suggests a different manner of using them and leads to a convergence theorem.

For the approximate evaluation of the integral over \((-\infty, \infty)\) of a rational or algebraic integrand—or any integrand which goes to zero as a negative power of \(|x|\) as \(x\) goes to infinity—it seems reasonable to want a formula based on a weight function with similar behavior, rather than on \(e^{-x^2}\) as in the Hermite-Gauss quadrature. If the integrand \(f\) goes to zero as \(|x|^{-p}\), a natural choice of weight function is \(w_n(x) = (1 + x^2)^{-\alpha}\), \(\alpha = p/2\). Setting \(f(x) = w_n(x)g(x)\), we are led to consider formulas of the form:

\[
\int_{-\infty}^{\infty} w_n(x)g(x) \, dx \sim \sum_{i=1}^{N} A_i^{(\alpha)} g(x_i^{(\alpha)});
\]

where \(g\) is a bounded function.

Since \(g\) is bounded on the whole line, we cannot base the formulas on consideration of polynomial approximation to \(g\); one choice that suggests itself is to consider approximation to \(g\) by functions of the form

\[
a_0 + a_1 + b_1x + \frac{a_2 + b_2x}{1 + x^2} + \cdots + \frac{a_m + b_mx}{(1 + x^2)^m}.
\]

We thus look to determine the abscissas and coefficients of the quadrature formula so as to make it exact for all functions of this form for \(m\) as high as possible.

Since \(w_n(x)\) is even, requiring that the formula be symmetric about zero, i.e. of the form

\[
2B_0^{(\alpha)} g(0) + \sum_{i=1}^{N} B_i^{(\alpha)} [g(x_i^{(\alpha)}) + g(-x_i^{(\alpha)})]
\]

insures its exactness for all term \(b_i x (1 + x^2)^{-r}\), and only the terms \(a_0, a_1 (1 + x^2)^{-1}, a_2 (1 + x^2)^{-2}, \cdots\) need further consideration. These are all even, and so it amounts to the same thing to consider the quadrature formula

\[
\int_{0}^{\infty} w_n(x)g(x) \, dx \sim B_0^{(\alpha)} g(0) + \sum_{i=1}^{N} B_i^{(\alpha)} g(x_i^{(\alpha)}).
\]

The \(B_i^{(\alpha)}\) and \(x_i^{(\alpha)}\) are to be determined so as to maximize the highest integer \(M\) such that (2) is exact whenever \(g = P((1 + x^2)^{-1})\) with \(P\) a polynomial of degree \(M\) or lower.

For such \(g\), setting \(y = (1 + x^2)^{-1}\) transforms the integral in (2) into

\[
\frac{1}{2} \int_{0}^{1} y^{-3/2}(1 - y)^{-1/2} P(y) \, dy;
\]

and the Jacobi-Gauss quadrature formula (see [2])

\[
\text{Received August 7, 1963.}
\]
for the exponents $\alpha - \frac{1}{2}$ and $-\frac{1}{2}$ and for $N$ abscissas evaluates this last integral exactly whenever the degree of $P$ is $\leq 2N - 1$, and that is the best that can be done. Thus our abscissas and coefficients are given by (since all the $y_i^{(a)}$ are less than 1):

\begin{equation}
B_0^{(a)} = 0; \quad B_i^{(a)} = \frac{1}{2} C_i^{(a)}, \quad x_i^{(a)} = (1 - y_i^{(a)})^{1/2}(y_i^{(a)})^{-1/2}, \quad i \geq 1
\end{equation}

where the $C_i^{(a)}$ and $y_i^{(a)}$ are the coefficients and abscissas of the Jacobi-Gauss formula.

Since the set of all functions of the form

\[(1 + x^2)^{-\alpha} \left[ a_0 + \frac{a_1 + b_1 x}{1 + x^2} + \cdots + \frac{a_{2N-1} + b_{2N-1} x}{(1 + x^2)^{2N-1}} \right]\]

is also that of all functions of the form $(1 + x^2)^{-2N-\alpha+1}Q(x)$ where $Q$ is a polynomial of degree $4N - 2$ or lower, the conditions determining the above formula for any $\alpha$ and $N$ are the same as those determining Harper's formula for (using "$k$" and "$n$" in the meaning given them in [1]) $k = \alpha + 2N - 2$, $n = 2N$. Thus we have just re-derived Harper's formulas for even $n$.

It follows from known properties of Jacobi-Gauss quadrature that the coefficients are non-negative; and if $f$ is continuous and $\alpha$ is chosen large enough to make $g$ bounded, it follows that the approximation obtained converges to the integral as $N$ increases.

National Bureau of Standards
Washington, D. C.


**Generalized Trigonometric Functions**

By F. D. Burgoyne

In an investigation into geometrical properties of the curves $x^n/a^n + y^n/b^n = 1$, use was made of the functions $s_n(u)$ where

\[ u = \int_0^{s_n(u)} (1 - t^n)^{1/n-1} \, dt \quad (0 \leq u \leq P_n) \]

and

\[ P_n = \int_0^1 (1 - t^n)^{1/n-1} \, dt = 2 \left( \frac{1}{n} \right)^2 \left\{ \frac{2}{n} \right\} ! \]

These functions may be called generalized trigonometric functions in view of the fact that $s_2(u) = \sin u$. Further, $s_n(u)$ is the Dixon function $s_{nm}$, considered by Dixon [1], Adams [2], and Laurent [3]. For $n = 4$ and 6 the functions are related to the Jacobian elliptic functions $sn(u)$ with moduli $2^{1/2}/2$, $(2 - 3^{1/2})^{1/2}/2$

Received April 22, 1963, revised May 28, 1963.